Mean Field Games: numerical methods

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Introduction

- Consider a game leading to **Nash equilibria** with **a very large number** N **of rational agents** (or players). The players have a **limited global information** on the game.
- In symmetric situations, Lasry and Lions have made a theory allowing for **passing to the limit when** $N \to \infty$**.**
  They have introduced, partially justified and analyzed models of **mean field games**. Analogy with statistical physics and mechanics.
- **New systems of PDEs with uniqueness, stability...**

Applications

- Economics
  - Economical equilibrium with anticipation.
  - Example: Economic models for the behaviour of customers in front of a technological breakthrough, for instance heat insulation of houses. (Lachapelle, Salomon,Turinici).
- Finance
  - Self-consistent models for volatility
  - Price formation

I. A short review

Mean field games: infinite horizon

The mean field game system of PDE : Find $u \in \mathcal{C}^2(\mathbb{T})$, $m \in W^{1,p}(\mathbb{T})$ and $\lambda \in \mathbb{R}$ s.t.

$$\begin{cases} -\nu\Delta u + H(x, \nabla u) + \lambda = V[m], \\ -\nu\Delta m - \operatorname{div}\left(m\dfrac{\partial H}{\partial p}(x, \nabla u)\right) = 0, \\ \displaystyle\int_{\mathbb{T}} u dx = 0, \qquad \int_{\mathbb{T}} m dx = 1, \quad \text{and} \quad m > 0 \quad \text{in } \mathbb{T}. \end{cases} \qquad (*)$$

- $\mathbb{T}$ unit torus of $\mathbb{R}^d$
- $\nu > 0$
- H is a $\mathcal{C}^1$ Hamiltonian (convex):

$$H(x,p) = \sup_{\alpha\in\mathbb{R}^d} \left(p\cdot\alpha - L(x,\alpha)\right), \quad \text{with} \quad \lim_{|\alpha|\to\infty} \inf_x \frac{L(x,\alpha)}{|\alpha|} = +\infty$$

- V is an operator **from the space of probability measures on $\mathbb{T}$ into a bounded set of Lipschitz functions on $\mathbb{T}$** such that

  $V[m_n]$ converges uniformly on $\mathbb{T}$ to $V[m]$ if m_n weakly converges to m.

  Typical examples for V include nonlocal smoothing operators.

- Local operators

$$V[m](x) = f(m(x), x)$$

  may be considered as well.

$(*)$ has been obtained by J-M. Lasry and P-L. Lions by passing to the limit **in stochastic differential games involving a very large number N of identical rational agents** (or players) with a **(limited) global information**

- Dynamics:

$$dX_t^i = \sqrt{2\nu}\, dW_t^i - \alpha^i dt, \quad X_0^i = x^i \in \mathbb{R}^d$$

  For simplicity, the control α^i is adapted to W_t^i.

- Cost:

$$J^i(\alpha^1, \ldots, \alpha^N) = \liminf_{T\to\infty} \frac{1}{T}\mathbb{E}\left(\int_0^T \left(L(X_t^i, \alpha_t^i) + V\left[\frac{1}{N-1}\sum_{j\neq i}\delta_{X_t^j}\right](X_t^i)\right) dt\right)$$

- $(\bar\alpha^1, \ldots, \bar\alpha^N)$ is a Nash point if $\quad \forall i = 1, \ldots, N,$

$$\bar\alpha^i = \operatorname*{Argmin}_{\alpha^i} J^i(\bar\alpha^1, \ldots, \bar\alpha^{i-1}, \alpha^i, \bar\alpha^{i+1}, \ldots, \bar\alpha^N).$$

- Structure assumption on H: there exists $\theta \in (0,1)$ such that for $|p|$ large,

$$\inf_{x\in\mathbb{T}} \left(\frac{\partial H}{\partial x} \cdot p + \frac{\theta}{d\nu} H^2 \right) > 0$$

- Thanks to ergodicity, there is a system of $2N$ PDEs
  - N ergodic HJB equations
  - N Kolmogorov equations for the stationary measures of $(X_t^i)_i$

  whose solutions yield Nash equilibria.

- $N \to \infty$, pass to the limit and obtain (*).

for all i, $1 \le i \le N$,

$$\begin{cases} -\nu\Delta v_i + H(x, \nabla v_i) + \lambda_i = \displaystyle\int_{\mathbb{T}^{N-1}} V \left[\frac{1}{N-1} \sum_{j \neq i} \delta_{x^j} \right] (x_i) \prod_{j \neq i} m_j(x^j) dx^j, \\ -\nu\Delta m_i - \operatorname{div}\left(m_i \dfrac{\partial H}{\partial p}(x, \nabla v_i) \right) = 0, \\ \displaystyle\int_{\mathbb{T}} v_i dx = 0, \qquad \int_{\mathbb{T}} m_i dx = 1 \quad \text{and} \quad m_i > 0 \quad \text{in } \mathbb{T}. \end{cases}$$

The feedback $\bar{\alpha}^i(\cdot) = \dfrac{\partial H}{\partial p}(\cdot, \nabla v_i(\cdot))$ yields a Nash point

$$\lambda_i = J^i(\bar{\alpha}^1, \ldots, \bar{\alpha}^N).$$

Uniqueness for the mean field problem

By contrast with the system of PDEs for N players, the mean field system (*) is well posed under some assumptions:

Theorem (Lasry-Lions) If V is strictly monotone, i.e.

$$\int_{\mathbb{T}} (V[m_1] - V[m_2])(m_1 - m_2)dx \leq 0 \Rightarrow m_1 = m_2,$$

then the solution of the mean field system (*) is unique.

Remark The economical interpretation of this assumption on V is: *any given player does not like to have the same policy as other players (there is no fashion phenomenon)*

Proof

Consider two solutions of (*): (λ_1, u_1, m_1) and (λ_2, u_2, m_2):

• multiply the first equation by $m_1 - m_2$

$$\int_{\mathbb{T}} \Big(\nu \nabla (u_1 - u_2) \cdot \nabla (m_1 - m_2) + (H(x, \nabla u_1) - H(x, \nabla u_2))(m_1 - m_2) \Big) dx$$

$$= \int_{\mathbb{T}} (V[m_1] - V[m_2])(m_1 - m_2) dx$$

• multiply the second equation by $u_1 - u_2$

$$0 = \int_{\mathbb{T}} \nu \nabla (u_1 - u_2) \cdot \nabla (m_1 - m_2) dx$$

$$+ \int_{\mathbb{T}} \left(m_1 \frac{\partial H}{\partial p}(x, \nabla u_1) - m_2 \frac{\partial H}{\partial p}(x, \nabla u_2) \right) \cdot \nabla (u_1 - u_2) dx.$$

• subtract:

$$0 = \begin{cases} \int_{\mathbb{T}} m_1 \left(H(x, \nabla u_1) - H(x, \nabla u_2) - \frac{\partial H}{\partial p}(x, \nabla u_1) \cdot \nabla(u_1 - u_2) \right) dx \\ + \int_{\mathbb{T}} m_2 \left(H(x, \nabla u_2) - H(x, \nabla u_1) - \frac{\partial H}{\partial p}(x, \nabla u_2) \cdot \nabla(u_2 - u_1) \right) dx \\ + \int_{\mathbb{T}} (V[m_1] - V[m_2])(m_1 - m_2) dx \end{cases}$$

Since H is convex and V is monotone, the 3 terms vanish.

The strict monotonicity of V implies that $m_1 = m_2$.

The identities $u_1 = u_2$ and $\lambda_1 = \lambda_2$ come from the uniqueness for the HJB equation:

$$-\nu \Delta u + H(x, \nabla u) + \lambda = f \quad \text{with} \quad \int_{\mathbb{T}} u = 0.$$

Finite horizon Nash equilibrium with N players

The N players initial conditions are random, independent, with the same probability distribution m^0.

Cost of the player i at time t:

$$\mathbb{E}\left(\int_t^T \left(L(X_s^i, \alpha_s^i) + V\left[\frac{1}{N-1}\sum_{j\neq i}\delta_{X_s^j}\right](X_s^i)\right)ds + V_0\left[\frac{1}{N-1}\sum_{j\neq i}\delta_{X_T^j}\right](X_T^i)\right)$$

$N \to \infty$: with the change of variable $t \to T - t$,

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu\Delta u + H(x, \nabla u) = V[m], \\ \dfrac{\partial m}{\partial t} + \nu\Delta m + \operatorname{div}\left(m\dfrac{\partial H}{\partial p}(x, \nabla u)\right) = 0, \\ \displaystyle\int_{\mathbb{T}} m dx = 1, \quad \text{and} \quad m > 0 \quad \text{in } \mathbb{T}, \\ u(t=0) = V_0[m(t=0)], \qquad m(t=T) = m_\circ. \end{cases} \qquad (**)$$

Existence for (**) (Lasry-Lions)

If

• same kind of assumptions on V and V_0 as in the stationary case (V and V_0 are nonlocal smoothing operators).

• H is smooth on $\mathbb{T} \times \mathbb{R}^d$ and $\left|\frac{\partial H}{\partial x}(x,p)\right| \leq C(1+|p|)$

then (**) has at least a smooth solution.

Uniqueness for (**) (Lasry-Lions)

If the operators V and V_0 are strictly monotone, i.e.

$$\int_{\mathbb{T}} (V[m] - V[\tilde{m}])(m - \tilde{m}) \leq 0 \Rightarrow V[m] = V[\tilde{m}],$$
$$\int_{\mathbb{T}} (V_0[m] - V_0[\tilde{m}])(m - \tilde{m}) \leq 0 \Rightarrow V_0[m] = V_0[\tilde{m}],$$

then (**) has at most a solution.

II. Finite Horizon: A numerical method

Take $d = 2$:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu \Delta u + H(x, \nabla u) = V[m], \\ \dfrac{\partial m}{\partial t} + \nu \Delta m + \mathrm{div}\left(m \dfrac{\partial H}{\partial p}(x, \nabla u) \right) = 0, \\ \displaystyle\int_{\mathbb{T}} m dx = 1, \qquad m > 0 \quad \text{in } \mathbb{T}, \\ u(t=0) = V_0[m(t=0)], \quad m(t=T) = m_\circ. \end{cases} \tag{**}$$

- Let $\mathbb{T}_h$ be a uniform grid on the torus with mesh step h, and x_{ij} be a generic point in $\mathbb{T}_h$.
- Uniform time grid: $\Delta t = T/N_T$, $t_n = n\Delta t$.
- The values of u and m at $(x_{i,j}, t_n)$ are approximated by $U^n_{i,j}$ and $M^n_{i,j}$.

Goal: propose a fully implicit scheme, robust when $\nu \to 0$, which guarantees existence, and possibly uniform bounds and uniqueness.

Notation:

- The discrete Laplace operator:

$$(\Delta_h W)_{i,j} = -\frac{1}{h^2}(4W_{i,j} - W_{i+1,j} - W_{i-1,j} - W_{i,j+1} - W_{i,j-1}).$$

- Right-sided finite difference formulas for $\partial_1 w(x_{i,j})$ and $\partial_2 w(x_{i,j})$:

$$(D_1^+ W)_{i,j} = \frac{W_{i+1,j} - W_{i,j}}{h}, \quad \text{and} \quad (D_2^+ W)_{i,j} = \frac{W_{i,j+1} - W_{i,j}}{h}.$$

- The set of 4 finite difference formulas at $x_{i,j}$:

$$[D_h W]_{i,j} = \Big((D_1^+ W)_{i,j}, (D_1^+ W)_{i-1,j}, (D_2^+ W)_{i,j}, (D_2^+ W)_{i,j-1} \Big).$$

Discrete HJB equation

$$\frac{\partial u}{\partial t} - \nu \Delta u + H(x, \nabla u) = V[m]$$

$$\downarrow$$

$$\frac{U_{i,j}^{n+1} - U_{i,j}^{n}}{\Delta t} - \nu (\Delta_h U^{n+1})_{i,j} + g(x_{i,j}, [D_h U^{n+1}]_{i,j}) = \big(V_h[M^{n+1}]\big)_{i,j}$$

where

- $$\begin{aligned} & g(x_{i,j}, [D_h U^{n+1}]_{i,j}) \\ =& g\Big(x_{i,j}, (D_1^+ U^{n+1})_{i,j}, (D_1^+ U^{n+1})_{i-1,j}, (D_2^+ U^{n+1})_{i,j}, (D_2^+ U^{n+1})_{i,j-1}\Big), \end{aligned}$$
- for instance,
$$(V_h[M])_{i,j} = V[m_h](x_{i,j}),$$
calling m_h the piecewise constant function on $\mathbb{T}$ taking the value $M_{i,j}$ in the square $|x - x_{i,j}|_\infty \leq h/2$.

Classical assumptions on the discrete Hamiltonian g

$$(q_1, q_2, q_3, q_4) \to g\left(x, q_1, q_2, q_3, q_4\right).$$

- **Monotonicity:** g is nonincreasing with respect to q_1 and q_3 and nondecreasing with respect to to q_2 and q_4.
- **Consistency:**

$$g\left(x, q_1, q_1, q_3, q_3\right) = H(x, q), \quad \forall x \in \mathbb{T}, \forall q = (q_1, q_3) \in \mathbb{R}^2.$$

- **Differentiability:** g is of class $\mathcal{C}^1$, and

$$\left|\frac{\partial g}{\partial x}\Big(x, (q_1, q_2, q_3, q_4)\Big)\right| \leq C(1 + |q_1| + |q_2| + |q_3| + |q_4|).$$

- **Convexity:** $(q_1, q_2, q_3, q_4) \to g\left(x, q_1, q_2, q_3, q_4\right)$ is convex.

The discrete version of

$$\frac{\partial m}{\partial t} + \nu \Delta m + \operatorname{div}\left(m \frac{\partial H}{\partial p}(x, \nabla v) \right) = 0. \qquad (\dagger)$$

It is chosen so that

- each time step leads to a linear system for M^n with a matrix
  - whose diagonal coefficients are negative,
  - whose off-diagonal coefficients are nonnegative,

  in order to hopefully use some **discrete maximum principle.**

- The argument for uniqueness should hold in the discrete case, so **the discrete Hamiltonian g should be used for $(\dagger)$ as well.**

Principle: Discretize

$$-\int_{\mathbb{T}} \operatorname{div}\left(m \frac{\partial H}{\partial p}(x, \nabla u)\right) w = \int_{\mathbb{T}} m \frac{\partial H}{\partial p}(x, \nabla u) \cdot \nabla w$$

by

$$h^2 \sum_{i,j} M_{i,j} \nabla_q g(x_{i,j}, [D_h U]_{i,j}) \cdot [D_h W]_{i,j}.$$

Resulting scheme:

$$\left.\begin{array}{l}
\frac{1}{\Delta t}\left(M_{i,j}^{n+1} - M_{i,j}^{n}\right) + \nu(\Delta_h M^n)_{i,j} \\
+\frac{1}{h}\left(\begin{array}{l} M_{i,j}^{n} \frac{\partial g}{\partial q_1}(x_{i,j}, [D_h U^n]_{i,j}) - M_{i-1,j}^{n} \frac{\partial g}{\partial q_1}(x_{i-1,j}, [D_h U^n]_{i-1,j}) \\ +M_{i+1,j}^{n} \frac{\partial g}{\partial q_2}(x_{i+1,j}, [D_h U^n]_{i+1,j}) - M_{i,j}^{n} \frac{\partial g}{\partial q_2}(x_{i,j}, [D_h U^n]_{i,j}) \end{array}\right) \\
-\frac{1}{h}\left(\begin{array}{l} M_{i,j}^{n} \frac{\partial g}{\partial q_3}(x_{i,j}, [D_h U^n]_{i,j}) - M_{i,j-1}^{n} \frac{\partial g}{\partial q_3}(x_{i,j-1}, [D_h U^n]_{i,j-1}) \\ +M_{i,j+1}^{n} \frac{\partial g}{\partial q_4}(x_{i,j+1}, [D_h U^n]_{i,j+1}) - M_{i,j}^{n} \frac{\partial g}{\partial q_4}(x_{i,j}, [D_h U^n]_{i,j}) \end{array}\right)
\end{array}\right\} = 0,$$

Classical discrete Hamiltonians g can be chosen.

For example, if the Hamiltonian is of the form

$$H(x, \nabla u) = \psi(x, |\nabla u|),$$

a possible choice is the **Godunov scheme**

$$g(x, q_1, q_2, q_3, q_4) =$$
$$\psi\left(x, \sqrt{\min(q_1, 0)^2 + \max(q_2, 0)^2 + \min(q_3, 0)^2 + \max(q_4, 0)^2}\right).$$

If $\psi(x, w)$ is convex and nondecreasing w.r.t. w, then g is a convex function of (q_1, q_2, q_3, q_4); g is nonincreasing w.r.t. q_1 and q_3 and nondecreasing w.r.t. q_2 and q_4.

Finally, it can be proven that the global scheme is consistent if H is smooth enough.

Existence and estimates for the discrete problem

Theorem Assume that $M^{N_T} \geq 0$ and that $h^2 \sum_{i,j} M_{i,j}^{N_T} = 1$. Under the assumptions above on V, V_0 and g, **the discrete problem has a solution and there is a Lipschitz estimate on U_h^n uniform in n, h and Δt.**

Strategy of proof

$$\mathcal{K} = \left\{ (M_{i,j})_{0\leq i,j<N} : h^2 \sum_{i,j} M_{i,j} = 1, M_{i,j} \geq 0 \right\}.$$

Apply Brouwer fixed point theorem to a well chosen mapping

$$\begin{array}{llll} \chi : & \mathcal{K}^{N_T} & \longrightarrow & \mathcal{K}^{N_T}, \\ & (M^n)_n \to & (U^n)_n \to & (M^n)_n. \end{array}$$

Uniqueness

Theorem Same assumptions as above on V, V_0, H and g. Assume also that the operators V_h and $V_{0,h}$ are strictly monotone, i.e.

$$\begin{aligned} \Big(V_h[M] - V_h[\widetilde{M}], M - \widetilde{M}\Big)_2 \le 0 \quad &\Rightarrow \quad V_h[M] = V_h[\widetilde{M}], \\ \Big(V_{0,h}[M] - V_{0,h}[\widetilde{M}], M - \widetilde{M}\Big)_2 \le 0 \quad &\Rightarrow \quad V_{0,h}[M] = V_{0,h}[\widetilde{M}]. \end{aligned}$$

Then the discrete problem (slightly modified) has a unique solution.

Proof The choice of the scheme makes it possible to mimic the proof used in the continuous case: uses the convexity assumption on g.

III. Infinite Horizon: A numerical method

$$
\left\{
\begin{array}{l}
-\nu(\Delta_h U)_{i,j} + g(x_{i,j}, [D_h U]_{i,j}) + \lambda = (V_h[M])_{i,j}\ ,\\
\left(
\begin{array}{l}
-\nu(\Delta_h M)_{i,j}\\
-\dfrac{1}{h}\left(
\begin{array}{l}
M_{i,j}\dfrac{\partial g}{\partial q_1}(x_{i,j}, [D_h U]_{i,j}) - M_{i-1,j}\dfrac{\partial g}{\partial q_1}(x_{i-1,j}, [D_h U]_{i-1,j})\\
+M_{i+1,j}\dfrac{\partial g}{\partial q_2}(x_{i+1,j}, [D_h U]_{i+1,j}) - M_{i,j}\dfrac{\partial g}{\partial q_2}(x_{i,j}, [D_h U]_{i,j})
\end{array}
\right)\\
-\dfrac{1}{h}\left(
\begin{array}{l}
M_{i,j}\dfrac{\partial g}{\partial q_3}(x_{i,j}, [D_h U]_{i,j}) - M_{i,j-1}\dfrac{\partial g}{\partial q_3}(x_{i,j-1}, [D_h U]_{i,j-1})\\
+M_{i,j+1}\dfrac{\partial g}{\partial q_4}(x_{i,j+1}, [D_h U]_{i,j+1}) - M_{i,j}\dfrac{\partial g}{\partial q_4}(x_{i,j}, [D_h U]_{i,j})
\end{array}
\right)
\end{array}
\right) = 0,\\
M_{i,j} \geq 0,
\end{array}
\right.
$$

and

$$
h^2 \sum_{i,j} M_{i,j} = 1, \quad \text{and} \sum_{i,j} U_{i,j} = 0.
$$

Existence for the discrete problem: strategy of proof

- Use Brouwer fixed point theorem in the set $\mathcal{K}$ of discrete probability measures for a mapping $\chi : \quad M \to U \to M$.
- The map $\Phi : \quad M \to U$ consists of solving

$$\begin{cases} -\nu(\Delta_h U)_{i,j} + g(x_{i,j}, [D_h U]_{i,j}) + \lambda = (V_h[M])_{i,j} \,, \\ \sum_{i,j} U_{i,j} = 0 \end{cases}$$

- (U, λ) is obtained by considering the ergodic approximation:

$$-\nu(\Delta_h U^{(\rho)})_{i,j} + g(x_{i,j}, [D_h U^{(\rho)}]_{i,j}) + \rho U^{(\rho)}_{i,j} = (V_h[M])_{i,j} \,,$$

and passing to the limit as $\rho \to 0$.

- We need Hölder or Lipschitz estimates on $U^{(\rho)} - U^{(\rho)}_{0,0}$ uniform in ρ and h.

Difficulty

The proof of existence for the continuous problem used the estimate $\|\nabla u\|_\infty \leq C$, which was obtained with the Bernstein method and the assumption: there exists $\theta \in (0,1)$ such that for $|p|$ large,

$$\inf_{x\in\mathbb{T}} \left(\frac{\partial H}{\partial x} \cdot p + \frac{\theta}{2\nu} H^2 \right) > 0.$$

Discrete case: this argument seems difficult to reproduce.

We had to make more restrictive assumptions on H and g to obtain bounds on $\|D_h u\|_\infty$ uniform in h.

Assumptions on the Hamiltonian

$$H(x,p) = \max_{\alpha \in \mathcal{A}} \Big(p \cdot \alpha - L(x,\alpha) \Big),$$

where

- $\mathcal{A}$ is a compact subset of $\mathbb{R}^2$,
- L is a $\mathcal{C}^0$ function on $\mathbb{T} \times \mathcal{A}$,

For the discrete Hamiltonian $g(x,q)$

- monotonicity, consistency.
- continuous with respect to x, $\mathcal{C}^1$ with respect to q
- sublinear with respect to q,
- there exists $g^\infty : \mathbb{R}^4 \to \mathbb{R}$ monotonous and sublinear s.t. $\lim_{\epsilon \to 0} \sup_x \left| \epsilon g(x, \frac{q}{\epsilon}) - g^\infty(q) \right| = 0.$

Existence and uniqueness for the stationary problem

Theorem Under the above assumptions on V and g, the discrete stationary problem has at least a solution. There is a uniform Hölder estimate on u_h. With stronger assumptions on the continuous and discrete Hamiltonians, uniform Lipschitz estimates (using the recent theory of Krylov).

Uniqueness: Ok if

$$\left(V_h[M] - V_h[\widetilde{M}], M - \widetilde{M}\right)_2 \le 0 \Rightarrow M = \widetilde{M}.$$

Remark Existence is still OK if for $\gamma > 1$,

$$g(x, q_1, q_2, q_3, q_4) \ge \alpha((q_1)_-^2 + (q_2)_+^2 + (q_3)_-^2 + (q_4)_+^2)^{\gamma/2} - C,$$

but no bounds on u_h uniform in h.

Convergence

The same method used for uniqueness can be used for proving convergence of the discrete scheme under some assumptions on consistency and stronger assumptions on V_h.

Example

If there exist $s > 0$ such that

$$h^2 \left(V_h[M] - V_h[\widetilde{M}], M - \widetilde{M} \right)_2 \geq c \| V_h[M] - V_h[\widetilde{M}] \|_\infty^s,$$

then uniform convergence for u, convergence of λ and a convergence related to V for m.

Uses the Hölder or Lipschitz estimates on U_h uniform w.r.t. h.

IV. Infinite Horizon: long time approximation

Long time approximation (Eductive strategy, see Guéant-Lasry)

$$\left\{\begin{array}{rcl}\dfrac{\partial \tilde{u}}{\partial t} - \nu\Delta\tilde{u} + H(x,\nabla\tilde{u}) & = & V[\tilde{m}],\\ \dfrac{\partial \tilde{m}}{\partial t} - \nu\Delta\tilde{m} - \mathrm{div}\left(\tilde{m}\dfrac{\partial H}{\partial p}(x,\nabla\tilde{u})\right) & = & 0,\\ \tilde{u}(0,x) = \tilde{u}_0(x), \quad \tilde{m}(0,x) & = & \tilde{m}_0(x),\end{array}\right.$$

with $\int_{\mathbb{T}} \tilde{m}_0 = 1 \quad \text{and} \quad \tilde{m}_0 \geq 0.$

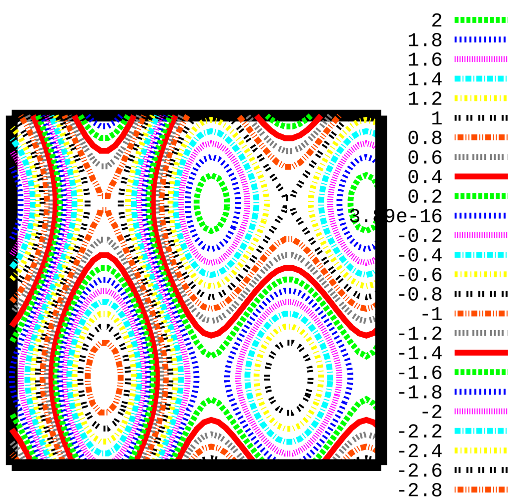
We expect that

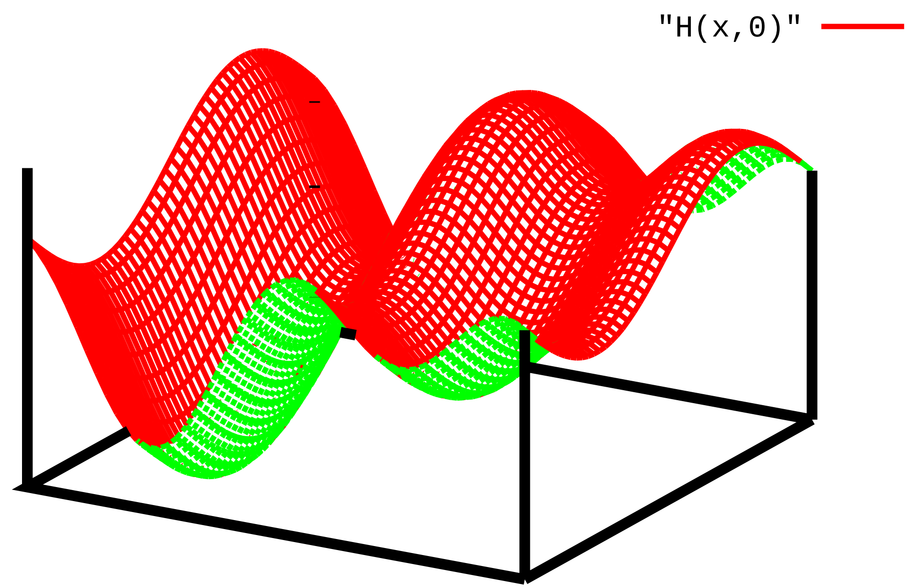
$$\lim_{t\to\infty} (\tilde{u}(t,x) - \lambda t) = u(x), \qquad \lim_{t\to\infty} \tilde{m}(t,x) = m(x),$$

Same thing at the discrete level.

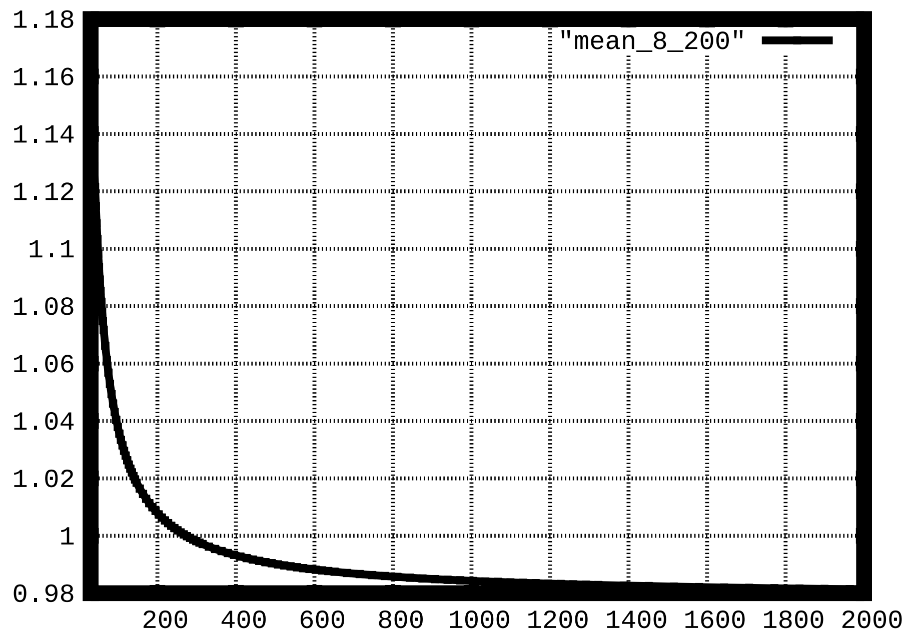
We use a semi-implicit linearized scheme. It requires the numerical solution of a linearized problem. Linearizing must be done carefully and is not always possible. In such cases, an explicit method can be used.

$$\nu = 1, \quad H(x,p) = \sin(2\pi x_2) + \sin(2\pi x_1) + \cos(4\pi x_1) + |p|^2, \quad F(x,m) = m^2$$



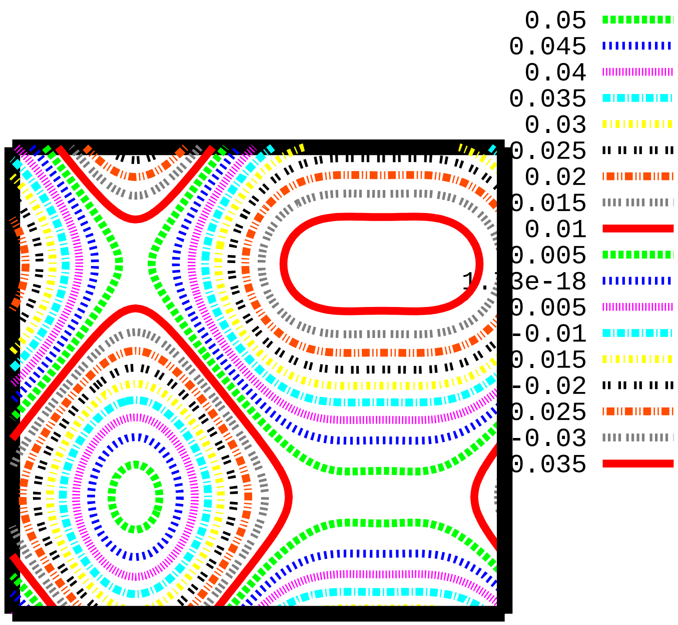


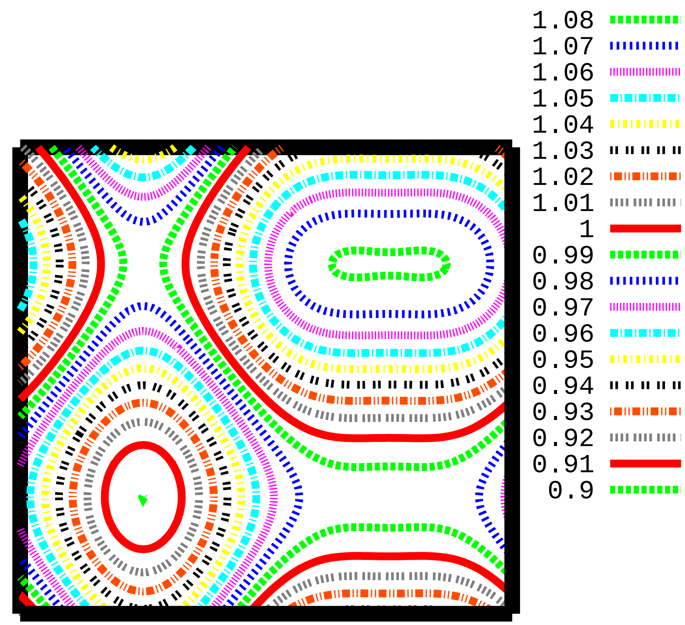
$$H(x,0) = \sin(2\pi x_2) + \sin(2\pi x_1) + \cos(4\pi x_1).$$



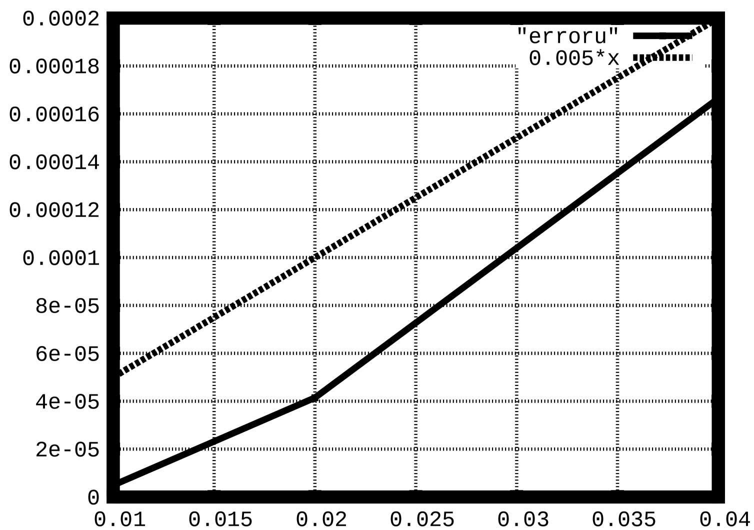
$\nu = 1,$ Convergence $\frac{1}{t}\int_{\mathbb{T}} \tilde{u}(x,t)dx \to \lambda$ as $t \to \infty$.

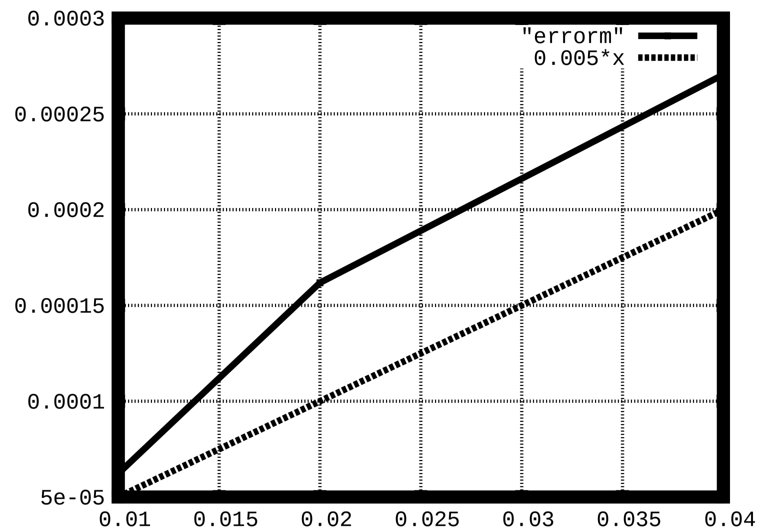
Very long time steps are used near convergence.





$\nu = 1$, left: u, right m.

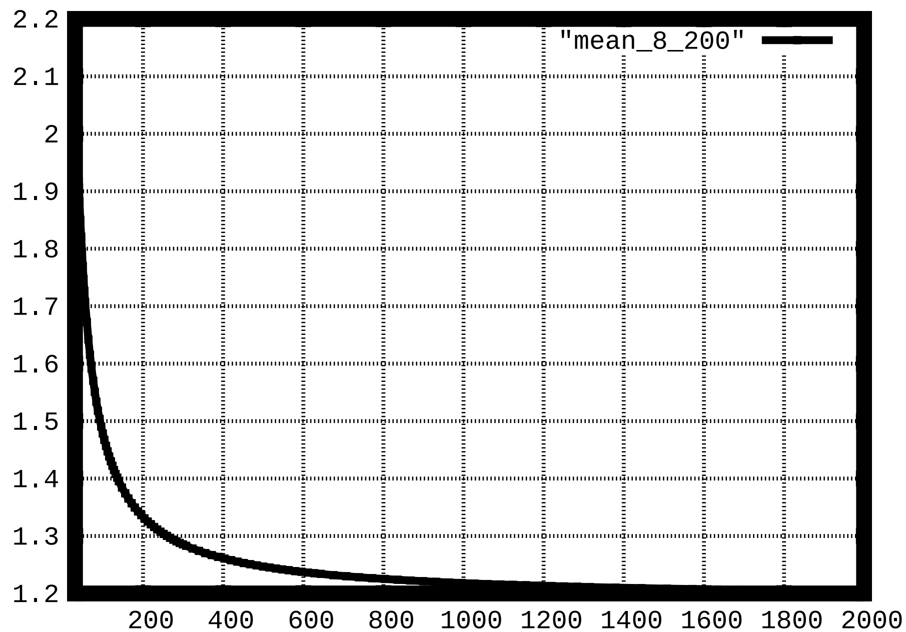




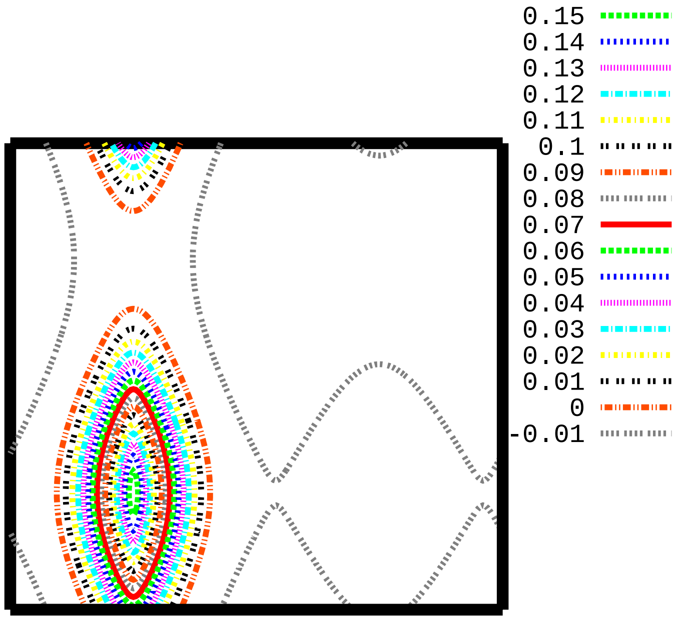
Convergence as $h \longrightarrow 0$

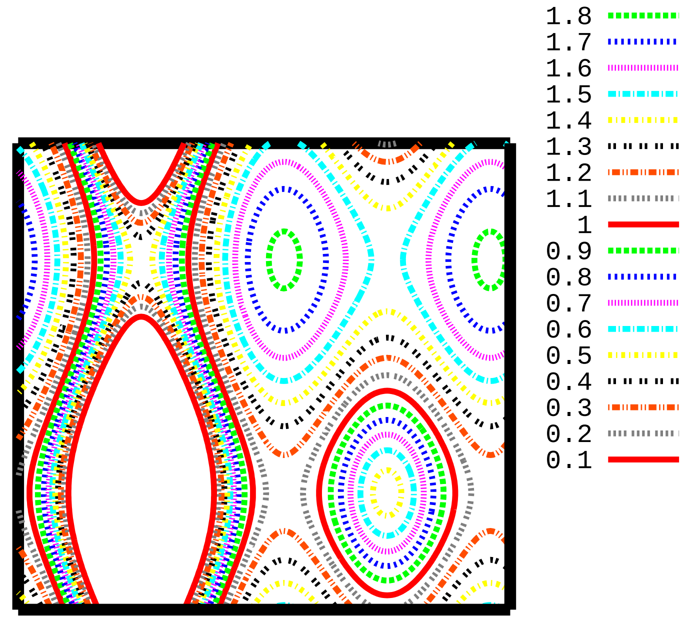
$\nu = 0.01,$

$$H(x,p) = \sin(2\pi x_2) + \sin(2\pi x_1) + \cos(4\pi x_1) + |p|^2, \quad F(x,m) = m^2.$$



$\nu = 0.01,$ Convergence $\frac{1}{t}\int_{\mathbb{T}} \tilde{u}(x,t)dx \to \lambda$ as $t \to \infty$



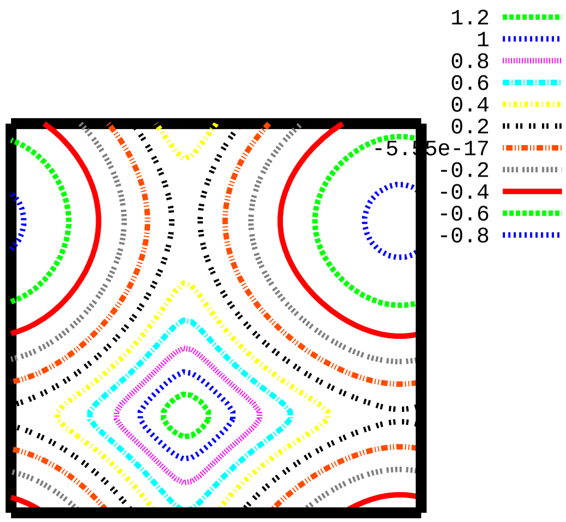


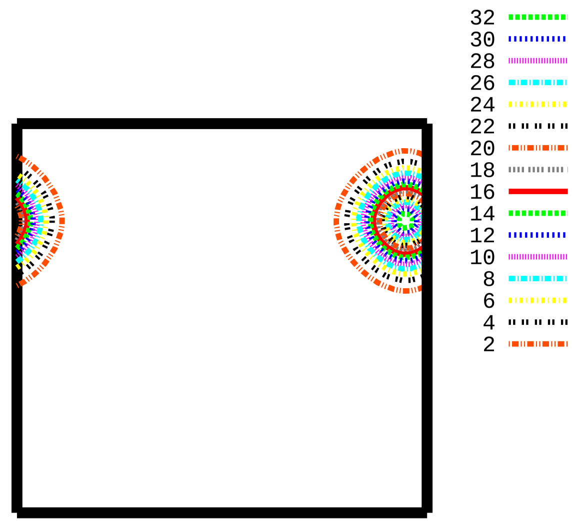
$\nu = 0.01,$ left: $u,$ right $m.$

Note that the supports of ∇u and of m tend to be disjoint as $\nu \to 0$.

$$V[m](x) = F(m(x)) = -\log(m(x)).$$

Same Hamiltonian as before. We now take $\nu = 0.1$.





left: u, right m.

The measure m_h concentrates near the minimum of u_h.

Deterministic limit $\nu \to 0$

Theorem (Lasry-Lions)
If

- $H(x,p) \geq H(x,0) = 0$,
- $V[m] = F(m) + f_0(x)$ where $F' > 0$,

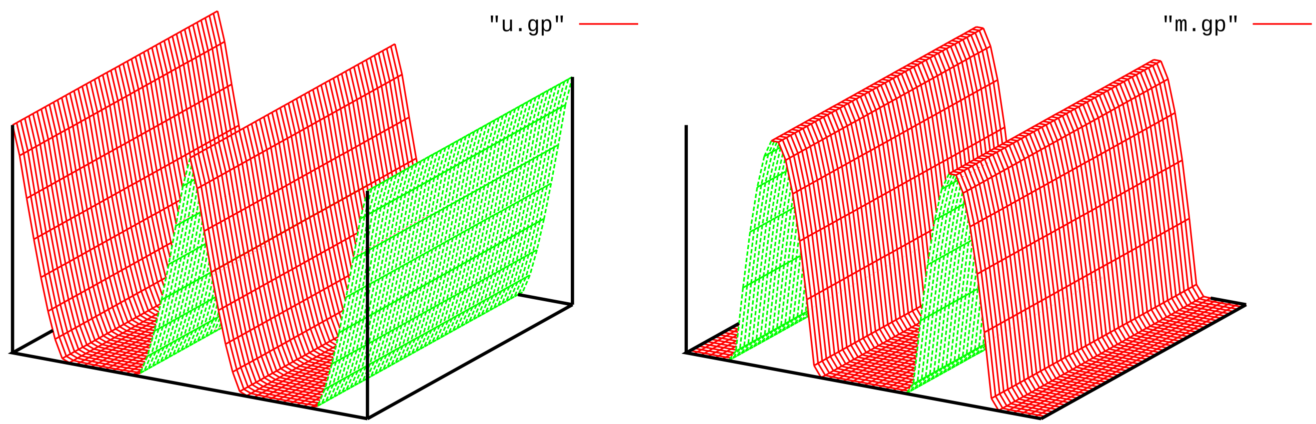
then

$$\lim_{\nu \to 0} (\lambda_\nu, m_\nu) = (\lambda, m),$$

where

$$m(x) = \big(F^{-1}(\lambda - f_0(x))\big)^+ \qquad \text{and} \qquad \int_{\mathbb{T}} m\,dx = 1.$$

$$\nu = 0.001,$$
$$H(x,p) = |p|^2,$$
$$V[m](x) = 4\cos(4\pi x) + m(x)$$

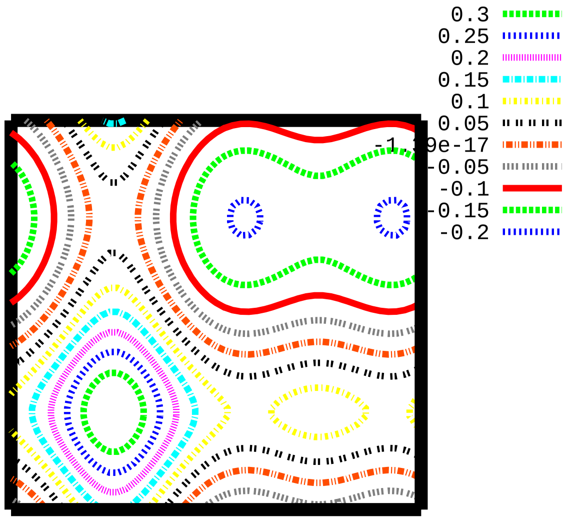


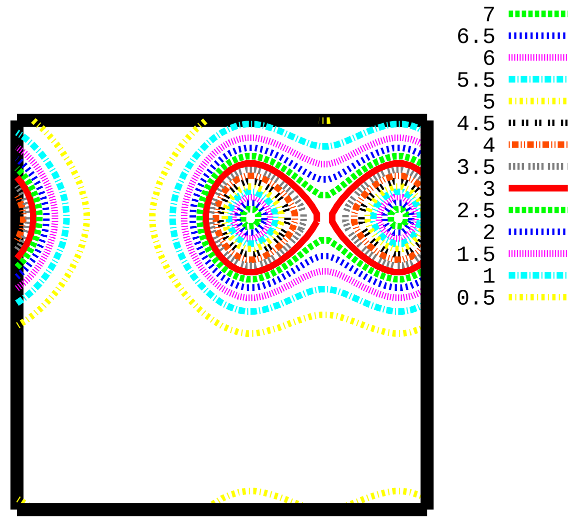
left: u, right m.

The supports of ∇u and of m tend to be disjoint.

$$m(x) \approx (\lambda - 4\cos(4\pi x))^+$$

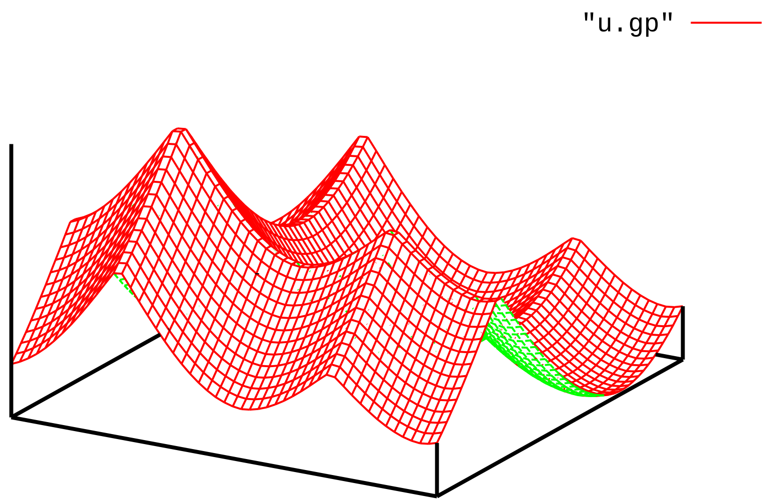
A nonlocal operator V

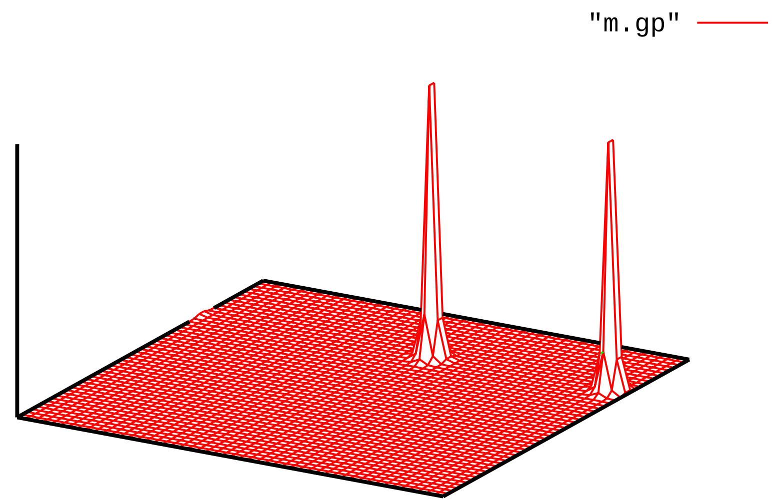




$$\nu = 0.1,$$
$$H(x,p) = \sin(2\pi x_2) + \sin(2\pi x_1) + \cos(4\pi x_1) + |p|^{3/2},$$
$$F(x,m) = 200(1-\Delta)^{-1}(1-\Delta)^{-1}m$$

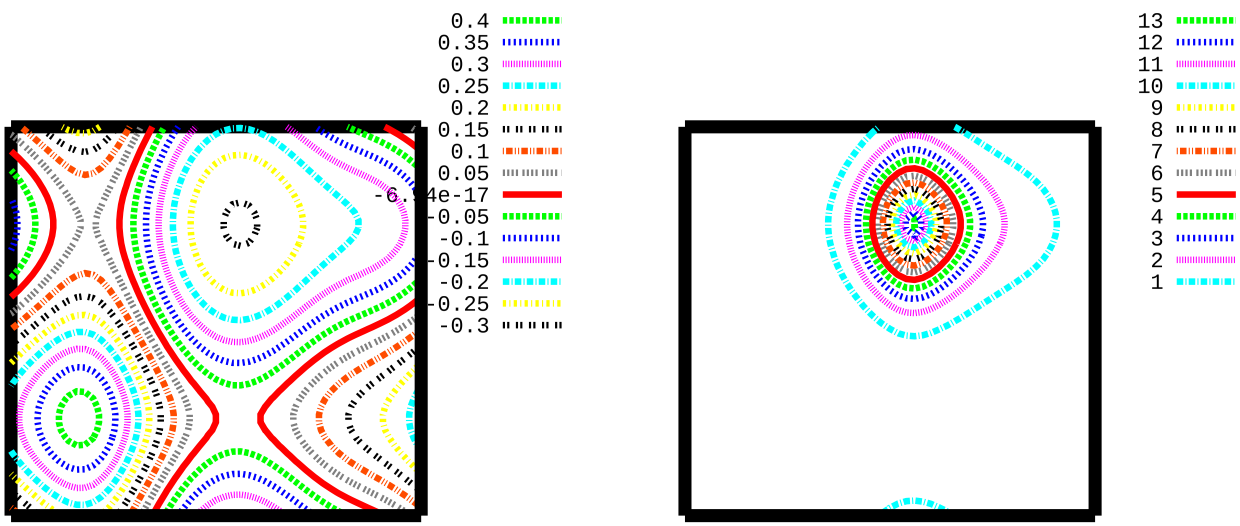
left: u, right m.





$$\nu = 0.001,$$
$$H(x,p) = \sin(2\pi x_2) + \sin(2\pi x_1) + \cos(4\pi x_1) + |p|^2,$$
$$F(x,m) = (1-\Delta)^{-1}(1-\Delta)^{-1}m$$

left: u, right m.



$$\nu = 0.1,$$
$$H(x,p) = \sin(2\pi x_2) + \sin(2\pi x_1) + \cos(4\pi x_1) + (0.6 + 0.59\cos(2\pi x))|p|^{3/2},$$
$$F(x,m) = 200(1-\Delta)^{-1}(1-\Delta)^{-1}m$$

left: u, right m.

Conclusion

- A promising method for doing some quantitative economics.
- Difficult open problems in the mathematical theory
  - Justification of the passage to the limit as $N \to \infty$.
  - Planification problem: drive the measure m from m_0 to m_T.
- Numerical methods can be designed and analyzed.
- Open problems
  - Hölder or Lipschitz estimates with more general assumptions
  - error estimates
- Recent work on planification