The value of purchasing with discount

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... a rather silly question, ins't it?

- Consider an asset with current market value $100.
- If the discount is 8%, the value of the right to purchase the asset today at a discount is, obviously, $8 (we can buy the asset for 92 and sell it inmediatly for 100).
- A more interesting problem arises when the right can only be exercised in the future, when the discount changes over time, or when the amount of the asset that can be bought is stochastic.

Outline of the talk

- We shall present two examples of purchasing with discount:

1. Proportional-strike options
2. Variable purchase options

1. Proportional-strike options

- Proportional-strike options are options with an exercise price that is a fraction of the market price of the underlying asset. That is, they give the right to buy at a discount one unit of the asset.
- The option is always in-the-money at maturity and its valuation is straightforward.
- More complex situations occur when the option is American, the underlying asset pays a continuous dividend or the exercise price changes over time.
- This is the case of a residential real estate program in China (See Gu, 2002).

1. Proportional-strike options

- Under this program, a state employee can buy her house at a proportion that declines over time.
- The employee can also qualify for a subsidized mortgage. The homeowner has the option, but not the obligation, of taking the subsidy.
- The value of this program to the employee is the value of a proportional-strike option plus the value of the state subsidy.
- To find the optimal time to buy the house we simply maximize the total value.

1. Proportional-strike options

- Let S_t be the market price of the house that the employee is entitled to.
- The state employee can buy the house at a decreasing proportion X_t of the market price.
- Let $X_t = 1 - d_t = Le^{-gt}$, where d_t is the discount, and L and g are a positive constants.
- The house provides a continuous dividend yield (in the form of rental rate) of q.

1. Proportional-strike options

- Value of the option to buy at the time of the purchase of the house

$$C_T = (1 - X_T)S_T$$

- Current value of this option

$$C_0 = (1 - X_T)S_0 e^{-qT}$$

where S_0 is the current market price of the house.

1. Proportional-strike options

- At time T, the employee is also offered a subsidized mortgage (with maturity τ) at an interest rate (r_s) below the market rate (r_m).
- It is easy to see that, at time T, this subsidy has a value of $\gamma(X_T S_T - J_T)$, where

$$\begin{aligned}\gamma &= 1 - \frac{r_s}{r_m}\left(\frac{1-\left(1+\frac{r_m}{12}\right)^{-12\tau}}{1-\left(1+\frac{r_s}{12}\right)^{-12\tau}}\right)\\ J_T &= J_0 e^{rT}\left(\frac{1-e^{(w-r)(T+1)}}{1-e^{w-r}}\right).\end{aligned}$$

Here, J_T denotes value of the savings of the employee to buy the house. The savings started at time 0 with a total contribution of J_0 and grows at rate w. They have been capitalized at the risk-free interest rate r.

1. Proportional-strike options

- The value today of the housing program (V) is the value to buy the house at a discount plus the value of the subsidized mortgage:

$$
\begin{aligned}
V &= \left(1 - Le^{-gT}\right) S_0 e^{-qT} + \\
&\max\left\{0; \gamma \left[Le^{-(g+q)T} S_0 - J_0 \left(\frac{1 - e^{(w-r)(T+1)}}{1 - e^{w-r}}\right)\right]\right\}.
\end{aligned}
$$

- Since the employee can choose the time at which she buys the house, she must solve a maximization problem to obtain the optimal time (T^*).

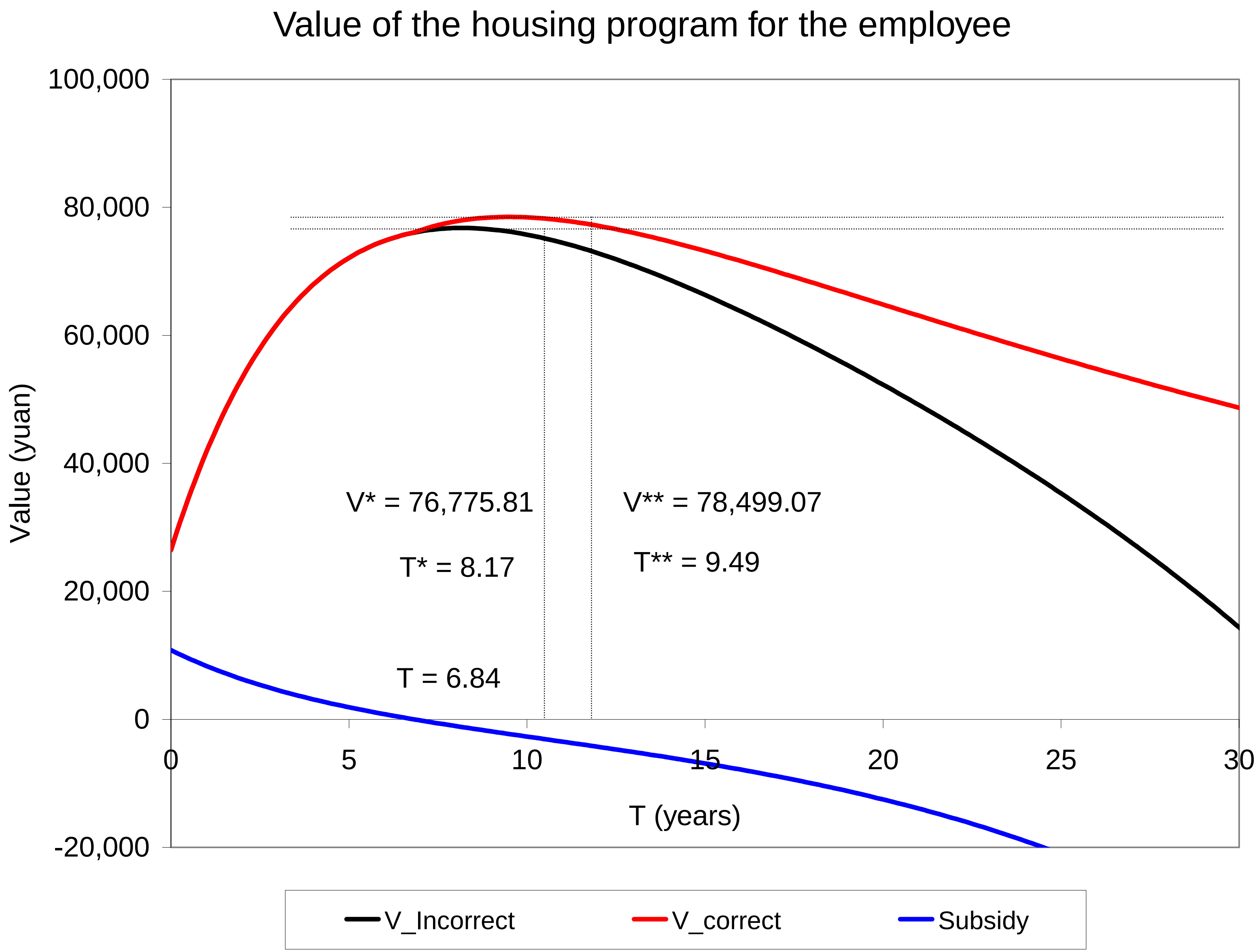
- Note that this function is not differentiable, so that it has to be maximized numerically.

1. Proportional-strike options – Example

- Current market value of the house $(S_0): 120,000$ yuan
- Current discount $(1 - L): 0.13$
- Growth rate of (1 - discount) $(g): 0.2$ (in 5 years, discount $= 0.68$)
- Rental rate $(q): 0.03$
- Initial contribution to the housing savings account $(J_0): 2,000$ yuan
- Savings growth rate $(w): 0.15$ per year
- Subsidized 25-year loan (τ), at 5.742 percent (r_s)
- Market rate for the same loan $(r_m): 6.94$ percent
- Risk-free interest rate $(r): 6$ percent

  - Optimal exercise time $(T^{**}): 9.49$ years
  - Value of the housing program $(V^{**}): 78,499.97$ yuan

1. Proportional-strike options – Example



2. Variable Purchase Options (VPOs)

- VPOs are securities issued in Australia by some firms that can be used to guarantee the success of a future equity offering.
- The VPO gives, at maturity, the right to buy at a discount a stochastic number of shares that depends on the terminal stock price. This number decreases with the stock price.
- The VPO is designed to be in-the-money at maturity. For example, if the discount is 8%, the optionholder will have the right to buy $108.69 of shares of the issuer for $100.

2. Variable Purchase Options

- Handley (2000) describes and prices standard VPOs.
- There are also Asian VPOs, where the number of shares that can be bought depends on the average stock price.
- These options are equivalent to options on the ratio of the stock price to its average. Moreno and Navas (2008) refer to them as Australian options.
- With an alternative definition, the reciprocal ratio shows up.

2. Variable Purchase Options

Payoffs

- Standard call option

$$C_T = \max\{S_T - K, 0\}$$

$$C_0 = e^{-rT} \int_{-\infty}^{\infty} C_T \, dF(S_T)$$

- VPO

$$C_T = \max\{N_T S_T - K, 0\}$$

$$N_T = N_T(S_T, K, d)$$

$$N_T = \frac{K}{S_T(1-d)} \Rightarrow C_0 = \frac{Kd}{1-d} e^{-rT}$$

2. Variable Purchase Options

Payoffs

- Asian VPO $\equiv$ VPO with $N_T = N_T(A_T, K, d)$

$$C_T = \frac{K}{1-d} \max\left\{\frac{S_T}{A_T} - (1-d), 0\right\}$$

- Australian option

$$C_T = \max\left\{\frac{S_T}{A_T} - K, 0\right\}$$

2. Variable Purchase Options

Valuation framework

- Assume that the asset price Z_t follows a GBM process under the risk-neutral probability measure

$$dZ_t = \alpha_Z Z_t dt + \sigma_Z Z_t dW_t$$

where α_Z is the risk-neutral drift of the process, and σ_Z^2 is the logarithmic variance parameter.

- Then, we can write

$$Z_t = Z_0 \exp\left\{\left(\alpha_Z - \frac{1}{2}\sigma_Z^2\right)t + \sigma_Z W_t\right\}$$

- Thus, Z_t follows a lognormal process. For $u > t$

$$[\ln Z_u] \mid Z_t \sim N\left(\ln(Z_t) + \left(\alpha_Z - \frac{1}{2}\sigma_Z^2\right)(u-t), \sigma_Z^2(u-t)\right)$$

2. Variable Purchase Options

Valuation framework

- Moments of Z_t under the risk-neutral measure

$$\begin{aligned} E(Z_t) &= Z_0 e^{\alpha_Z t} \\ V(Z_t) &= [E(Z_t)]^2 \left[e^{\sigma_Z^2 t} - 1\right] \\ \text{Cov}(Z_t, Z_s) &= Z_0^2 e^{\alpha_Z (t+s)} \left[e^{\sigma_Z^2 s} - 1\right], \; s < t \end{aligned}$$

2. Variable Purchase Options

Black-Scholes formula (generalized)

The price at time 0 of an European call option on Z that matures at T and with strike price K is given by

(1) $$C(Z,0,T,K) = e^{-rT}E(Z_T)N(d_1) - Ke^{-rT}N(d_2)$$

where

$$d_1 = \frac{\ln\left(e^{-\alpha_Z T}E(Z_T)/K\right) + \left(\alpha_Z + \frac{1}{2}\sigma_Z^2\right)T}{\sigma_Z\sqrt{T}}$$

$$d_2 = d_1 - \sigma_Z\sqrt{T}$$

2. Variable Purchase Options – Geometric Average

Discrete monitoring

Consider n monitoring dates in $[0, T]$; then we can write

$$G_n = (S_1 \cdots S_n)^{\frac{1}{n}} = \left(\prod_{i=1}^{n} S_i \right)^{\frac{1}{n}}, \quad G_0 \equiv S_0$$

2. Variable Purchase Options – Geometric Average

Discrete monitoring

From the stock price process and using $T = n\Delta t$, we have

$$
\begin{aligned}
G_n &= S_0 \exp\left\{ \left(r - q - \frac{1}{2}\sigma^2 \right) \frac{n+1}{2} \Delta t + \frac{\sigma}{n} \sum_{i=1}^{n} W_{t_i} \right\} \\
\frac{S_n}{G_n} &= \exp\left\{ \left(r - q - \frac{1}{2}\sigma^2 \right) \frac{n-1}{2} \Delta t + \frac{\sigma}{n} \left[n\, W_{t_n} - \sum_{i=1}^{n} W_{t_i} \right] \right\} \\
\frac{G_n}{S_n} &= \exp\left\{ - \left(r - q - \frac{1}{2}\sigma^2 \right) \frac{n-1}{2} \Delta t - \frac{\sigma}{n} \left[n\, W_{t_n} - \sum_{i=1}^{n} W_{t_i} \right] \right\}
\end{aligned}
$$

2. Variable Purchase Options – Geometric Average

Discrete monitoring

Consider European call options on S_n/G_n and G_n/S_n that mature at time T and with strike price K. Moreno and Navas (2008) show that the prices at time 0 of these options are given by expression (1), where the expected values and the logarithmic variances of the assets at maturity are shown in the following table:

Z_n	$E(Z_n)$	$\sigma_Z^2 T$
G_n	$S_0 \exp\left\{\left(r - q - \frac{n-1}{6n}\sigma^2\right)\frac{n+1}{2n}T\right\}$	$\frac{(n+1)\left(n+\frac{1}{2}\right)}{3n^2}\,\sigma^2 T$
S_n/G_n	$\exp\left\{\left(r - q - \frac{n+1}{6n}\sigma^2\right)\frac{n-1}{2n}T\right\}$	$\frac{(n-1)\left(n-\frac{1}{2}\right)}{3n^2}\,\sigma^2 T$
G_n/S_n	$\exp\left\{-\left(r - q - \frac{5n-1}{6n}\sigma^2\right)\frac{n-1}{2n}T\right\}$	$\frac{(n-1)\left(n-\frac{1}{2}\right)}{3n^2}\,\sigma^2 T$

2. Variable Purchase Options – Geometric Average

Continuous monitoring

By definition, the continuos geometric mean is as follows

$$G_T = \exp\left\{\frac{1}{T}\int_0^T \ln(S_t)\, dt\right\}$$

Given the stock price process, we can write

$$\begin{aligned}
G_T &= S_0 \exp\left\{\frac{1}{2}\left(r - q - \frac{1}{2}\sigma^2\right)T + \frac{\sigma}{T}\int_0^T W_t dt\right\} \\
\frac{S_T}{G_T} &= \exp\left\{\frac{1}{2}\left(r - q - \frac{1}{2}\sigma^2\right)T + \frac{\sigma}{T}\left[T\, W_T - \int_0^T W_t dt\right]\right\} \\
\frac{G_T}{S_T} &= \exp\left\{-\frac{1}{2}\left(r - q - \frac{1}{2}\sigma^2\right)T - \frac{\sigma}{T}\left[T\, W_T - \int_0^T W_t dt\right]\right\}
\end{aligned}$$

2. Variable Purchase Options – Geometric Average

Continuous monitoring

Consider European call options on S_T/G_T and G_T/S_T that mature at time T and with strike price K. MN(2008) shows that the prices at time 0 of these options are given by expression (1), where the expected values and the logarithmic variances of the assets at maturity are given in the following table:

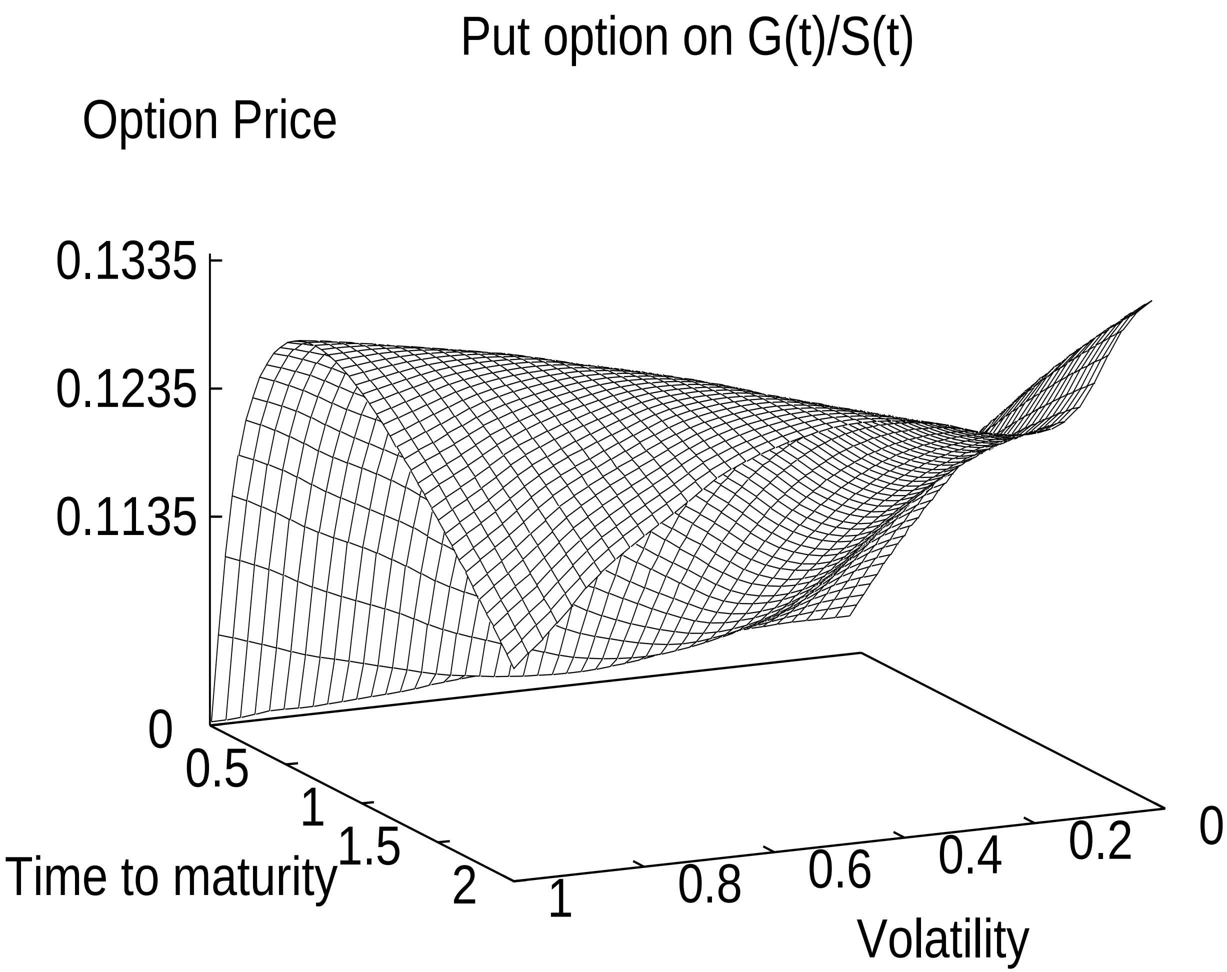
Z_T	$E(Z_T)$	$\sigma_Z^2 T$
S_T	$S_0 \exp\{(r-q)T\}$	$\sigma^2 T$
G_T	$S_0 \exp\left\{\frac{1}{2}\left(r-q-\frac{1}{6}\sigma^2\right)T\right\}$	$\frac{\sigma^2}{3}T$
S_T/G_T	$\exp\left\{\frac{1}{2}\left(r-q-\frac{1}{6}\sigma^2\right)T\right\}$	$\frac{\sigma^2}{3}T$
G_T/S_T	$\exp\left\{-\frac{1}{2}\left(r-q-\frac{5}{6}\sigma^2\right)T\right\}$	$\frac{\sigma^2}{3}T$

2. Variable Purchase Options – Geometric Average

Geometric call option prices

Parameters				n				
σ	T	K	Z_n	1	10	100	$1,000$	∞
$.2$	$.5$	$.8$	S_n	22.576	22.576	22.576	22.576	22.576
			G_n	22.576	20.696	20.561	20.548	20.546
			S_n/G_n	19.025	20.400	20.531	20.545	20.546
			G_n/S_n	19.025	18.133	18.161	18.166	18.166
$.2$	1	$.8$	S_n	25.187	25.187	25.187	25.187	25.187
			G_n	25.187	21.361	21.084	21.056	21.053
			S_n/G_n	18.097	20.756	21.023	21.050	21.053
			G_n/S_n	18.097	16.491	16.580	16.593	16.594
$.4$	$.5$	$.8$	S_n	24.801	24.801	24.801	24.801	24.801
			G_n	24.801	20.766	20.558	20.538	20.536
			S_n/G_n	19.025	20.332	20.514	20.534	20.536
			G_n/S_n	19.025	20.266	20.908	20.979	20.987

2. Variable Purchase Options – Geometric Average



2. Variable Purchase Options – Arithmetic Average

- Discrete monitoring

$$A_n = \frac{1}{n}(S_1 + \cdots + S_n) = \frac{1}{n}\sum_{i=1}^{n} S_i, \quad A_0 \equiv S_0$$

- Continuous monitoring

$$A_T = \frac{1}{T}\int_0^T S_t \, dt, \quad A_0 \equiv S_0$$

- Problem: To price options on A_n (arithmetic Asian options) we need the distribution of the sum of lognormal variables, which is unknown.

2. Variable Purchase Options – Arithmetic Average

Some ways of pricing arithmetic Asian options

1. PDE + numerical solution: Kemna and Vorst (1990), Vazquez-Abad and Dufresne (1998)

2. Monte Carlo simulation: Carverhill and Clewlow (1990), Ju (1997)

3. Edgeworth series expansions (approximate the true distribution of A_n with an alternative one): Jarrow and Rudd (1982), Turnbull and Wakeman (1991), Hansen and Jorgensen (2000)

4. Reciprocal gamma distribution (approximate the true distribution of A_n with that of A_T): Merton (1975), Majumdar and Radner (1991), Milevsky and Posner (1998)

2. Variable Purchase Options – Arithmetic Average

Edgeworth / Wilkinson Approximations

True value of the option

$$C(F) = e^{-rT} \int_{-\infty}^{\infty} \max\{S_T - K, 0\} dF(S_T)$$

- Approximated value of the option using the first four cumulants (Jarrow and Rudd, 1982)

$$\begin{aligned} C(F) &= C(A) + e^{-rT} \frac{k_2(F) - k_2(A)}{2!} a(K) - e^{-rT} \frac{k_3(F) - k_3(A)}{3!} \frac{da(K)}{dS_T} \\ &+ e^{-rT} \frac{k_4(F) - k_4(A) + 3(k_2(F) - k_2(A))^2}{4!} \frac{d^2 a(K)}{dS_T^2} + \varepsilon \end{aligned}$$

$$C(A) = e^{-rT} \int_{-\infty}^{\infty} \max\{S_T - K, 0\} dA(S_T)$$

- Second order Edgeworth expansion $\equiv$ Wilkinson expansion

2. Variable Purchase Options – Arithmetic Average

Pricing Options with the Gamma Distribution

- It is well kown that

$$\text{Infinite sum of lognormals} \sim \text{Reciprocal gamma}$$

- Thus, we can value arithmetic Asian options using the reciprocal gamma as the state-price density function
    - For continuous monitoring, the solution will be correct.
    - For discrete monitoring, the solution will be an approximation.

Pricing Options with the Gamma Distribution

Gamma distribution: $X \sim \Gamma(\alpha, \beta)$

- Density function

$$g(x) = \frac{\beta^{-\alpha} x^{\alpha-1} \exp\left\{-\frac{x}{\beta}\right\}}{\Gamma(\alpha)}, \; x > 0, \quad \Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$$

- First two central moments:

$$E(X) = \alpha\beta, \quad V(X) = \alpha\beta^2$$

2. Variable Purchase Options – Arithmetic Average

Pricing Options with the Gamma Distribution

- $Y = \frac{1}{X}$ follows a reciprocal gamma distribution
- First two non-central moments are

$$
\begin{aligned}
M_1 &= E(Y) = \frac{1}{\beta(\alpha - 1)} \\
M_2 &= E(Y^2) = \frac{1}{\beta^2(\alpha - 1)(\alpha - 2)}
\end{aligned}
$$

- It is easy to see that

$$
V(Y) = M_2 - M_1^2 = \frac{1}{\beta^2(\alpha - 1)^2(\alpha - 2)}
$$

$$
\alpha = \frac{2M_2 - M_1^2}{M_2 - M_1^2}, \quad \beta = \frac{M_2 - M_1^2}{M_1 M_2}
$$

2. Variable Purchase Options – Arithmetic Average

Steps to price options with the gamma distribution

1. Compute the first two risk-neutral moments of the underlying asset at maturity (M_1, M_2)

2. Obtain α and β using those moments

3. Use the cumulative density function of the gamma distribution as $N(d)$ in the Black-Scholes formula

2. Variable Purchase Options – Arithmetic Average

Pricing Asian VPOs with gamma distribution (discrete)
MN08 – Lemma 7

1. The moments of the ratio $S_n/A_n,\ n \geq 2$ can be approximated by (Mood *et al*, 1974)

$$E\left(\frac{S_n}{A_n}\right) \simeq \frac{E(S_n)}{E(A_n)} - \frac{1}{(E(A_n))^2}\text{Cov}(A_n, S_n) + \frac{E(S_n)}{(E(A_n))^3}V(A_n)$$

$$V\left(\frac{S_n}{A_n}\right) \simeq \left(\frac{E(S_n)}{E(A_n)}\right)^2\left(\frac{V(S_n)}{(E(S_n))^2} + \frac{V(A_n)}{(E(A_n))^2} - 2\frac{\text{Cov}(A_n, S_n)}{E(S_n)E(A_n)}\right)$$

2. Variable Purchase Options – Arithmetic Average

2. The moments of the ratio $A_n/S_n, \ n \geq 2$ are given by

$$E\left(\frac{A_n}{S_n}\right) = \frac{1}{n}\exp\{-n(r-q-\sigma^2)\Delta t\}\ h_1(r-q-\sigma^2)$$

$$\begin{aligned} V\left(\frac{A_n}{S_n}\right) &= \left(\frac{1}{n}\right)^2 \exp\{-n(2(r-q)-3\sigma^2)\Delta t\} \\ &\times[2f_1(r-q-\sigma^2)(h_1(2(r-q)-3\sigma^2)-h_1(r-q-2\sigma^2)) \\ &-h_1(2(r-q)-3\sigma^2)-\exp\{-n\sigma^2\Delta t\}h_1^2(r-q-\sigma^2)] \end{aligned}$$

2. Variable Purchase Options – Arithmetic Average

Pricing Asian VPOs with gamma distribution (continuous)
MN08 – Lemma 9

1. The moments of S_T/A_T can be approximated by

$$E\left(\frac{S_T}{A_T}\right) \simeq \frac{E(S_T)}{E(A_T)} - \frac{1}{(E(A_T))^2}\text{Cov}(A_T, S_T) + \frac{E(S_T)}{(E(A_T))^3}V(A_T)$$

$$V\left(\frac{S_T}{A_T}\right) \simeq \left(\frac{E(S_T)}{E(A_T)}\right)^2\left(\frac{V(S_T)}{(E(S_T))^2} + \frac{V(A_T)}{(E(A_T))^2} - 2\frac{\text{Cov}(A_T, S_T)}{E(S_T)E(A_T)}\right)$$

2. Variable Purchase Options – Arithmetic Average

2. The moments of the ratio A_T/S_T are given by

$$E\left(\frac{A_T}{S_T}\right) = \frac{1}{T}\Phi(\sigma^2 - (r-q))$$

$$V\left(\frac{A_T}{S_T}\right) = \left(\frac{1}{T}\right)^2 \exp\{-(2(r-q) - 3\sigma^2)T\} \times \left[2\frac{\Phi(2(r-q) - 3\sigma^2) - \Phi(r-q-2\sigma^2)}{r-q-\sigma^2} - \exp\{-\sigma^2 T\}\Phi^2(r-q-\sigma^2)\right]$$

with

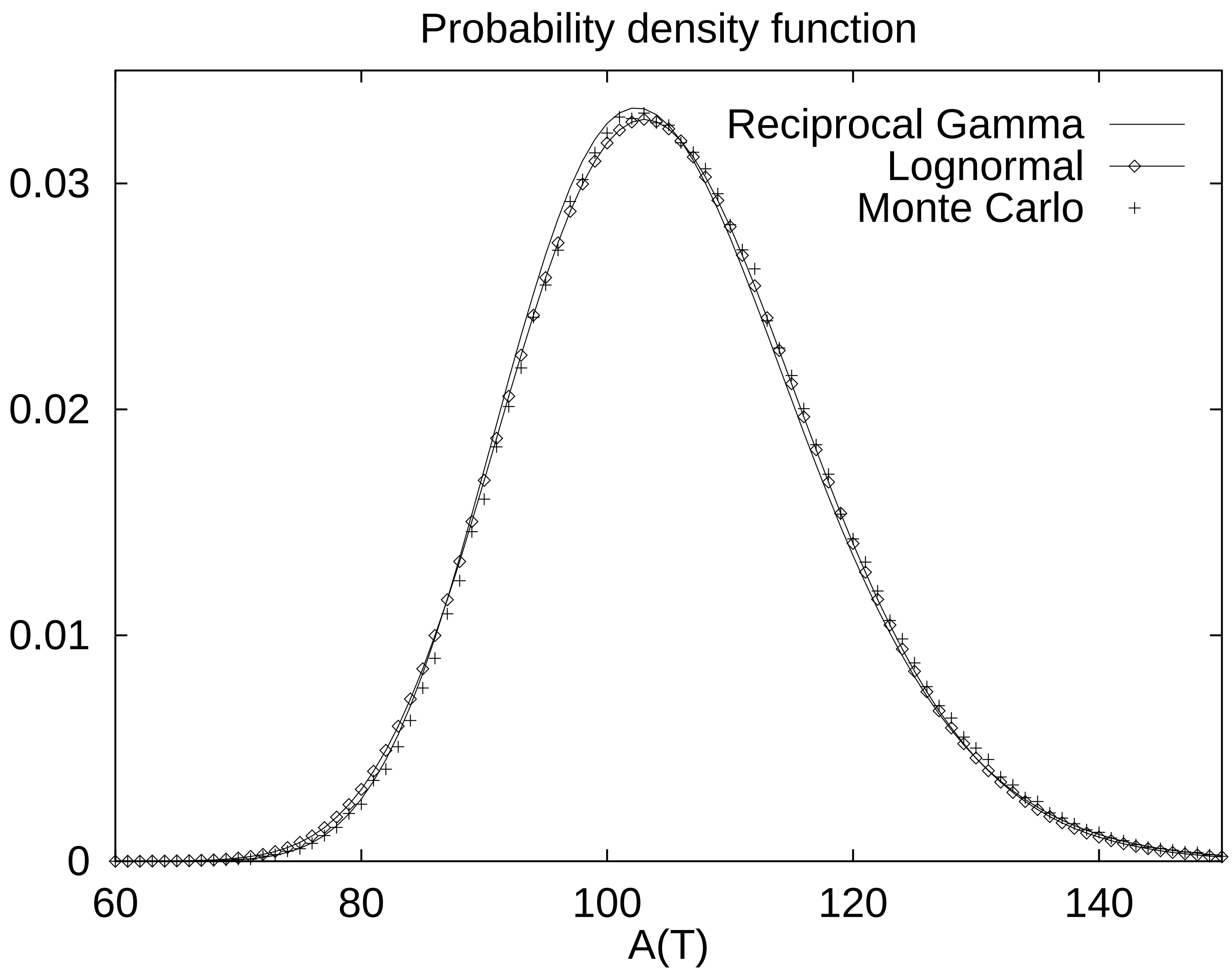
$$\Phi(x) = \frac{\exp\{xT\} - 1}{x},\ x \neq 0, \quad \Phi(0) = T$$

2. Variable Purchase Options – Arithmetic Average

Arithmetic call option prices

Parameters				n				
σ	T	K	Z_n	1	10	100	1,000	∞
.2	.5	.8	A_n (MC)	22.551	20.737	20.731	20.723	–
			A_n (GD)	22.535	20.883	20.728	20.711	20.711
			A_n (W)	22.576	20.885	20.729	20.714	20.712
			S_n/A_n (MC)	19.025	20.329	20.381	20.377	–
			S_n/A_n (GD)	19.025	20.208	20.358	20.374	20.375
			S_n/A_n (W)	19.025	20.209	20.360	20.375	20.377
			A_n/S_n (MC)	19.025	18.332	18.317	18.319	–
			A_n/S_n (GD)	19.025	18.388	18.327	18.321	18.321
			A_n/S_n (W)	19.025	18.389	18.330	18.324	18.324

2. Variable Purchase Options – Arithmetic Average



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