GARCH option pricing via local risk minimization.

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Plan of the talk

1. Modeling of the underlying: the GARCH approach.
2. Derivatives pricing in incomplete markets. Replication and utility optimization approaches.
3. GARCH option pricing via LRM.
4. Martingale measures, "closed-form" prices, and a new interpretation of the Duan and Heston-Nandi pricing formulas.

Time series

- More general than strong Euler discretizations of SDEs.
- Possibility of incorporating in the pricing pure time series theoretical concepts: cointegration (Duan, Pliska (2001))
- Need to go beyond linear models. They are all homoscedastic.
- No stochastic volatility. The number of innovations remains constant.
- ARCH models: Engle (1982); Nobel prize 2003. Originally introduced to model the variance in the UK inflation.
- GARCH: Bollerslev (1986); more parsimonious.
- Asymmetric GARCH: Ding, Granger, and Engle (1993). It captures the different impact that positive and negative shocks have on volatility.

Generalized Autorregressive Conditional Heteroscedasticity

- The original Ding, Granger, and Engle (1993)-He and Terasvirta (1999) model:

$$\begin{aligned}\log\left(S_n/S_{n-1}\right) &=: \quad r_n = \mu + \sigma_n \epsilon_n, \quad \mu \in \mathbb{R},\\ \sigma_n^\delta &= \quad \omega + \sum_{i=1}^{p} \beta_i \sigma_{n-i}^\delta + \sum_{i=1}^{q} \alpha_i (|\overline{r}_{n-i}| - \gamma_i \overline{r}_{n-i})^\delta.\end{aligned}$$

- The Heston-Nandi model (2000). Picked because it yields a closed-form option pricing formula:

$$\begin{aligned}\log\left(S_n/S_{n-1}\right) &=: \quad r_n = r + \lambda\sigma_n^2 + \sigma_n \epsilon_n, \quad \mu \in \mathbb{R},\\ \sigma_n^2 &= \quad \omega + \sum_{i=1}^{p} \beta_i \sigma_{n-i}^2 + \sum_{i=1}^{q} \alpha_i (\epsilon_{n-i} - \gamma_i \sigma_{n-i})^2.\end{aligned}$$

GARCH: main features

- Innovations could be multinomial, Gaussian, *t*-Student, or empirical (Barone-Adesi, Engle, Mancini (2007)).
- Successful in capturing both leptokurticity and volatility clustering. For all these models leptokurticity and heteroscedasticity are linked. For example, for Gaussian innovations:

$$\kappa = \frac{E[\sigma_n^4 \epsilon_n^4]}{(E[\sigma_n^2 \epsilon_n^2])^2} = 3 + 3\frac{\mathrm{var}\left(\sigma_n^2\right)}{(E[\sigma_n^2])^2}.$$

- GARCH processes are white noises. The squared process is an ARMA process whenever the kurtosis of the GARCH process is finite. Always use Ljung-Box statistics as portmanteau test of randomness.

- Manageable characterization of weak stationarity: if $q \geq 1$, $p \geq 0$, $\omega > 0$, $\alpha_i, \beta_i \geq 0$, and

$$\sum_{i=1}^{q} \alpha_i \gamma_i^2 + \sum_{i=1}^{p} \beta_i < 1,$$

then the model has a unique stationary solution for the log-returns such that

$$\mathrm{var}\,(r_n) = E[\sigma_n^2] = \frac{\omega + \sum_{i=1}^{q} \alpha_i}{1 - \sum_{i=1}^{p} \beta_i - \sum_{i=1}^{q} \alpha_i \gamma_i^2}.$$

- Kurtosis: Ling and McAleer (2002) contains a characterization for the existence of the fourth moment for asymmetric GARCH. This characterization is much needed in the optimization problem in the next point and for pricing via local risk minimization.

Necessary and sufficient condition for the existence of the moment of order $2m$ is that

$$\rho\left[E\left[A^{\otimes m}\right]\right] < 1, \qquad (1)$$

where $\rho(B) = \max\{|\text{eigenvalues of the matrix } B|\}$, A is given by

$$A = \left(\begin{array}{ccc|ccc} \alpha_1 Z_t & \cdots & \alpha_p Z_t & \beta_1 Z_t & \cdots & \beta_q Z_t \\ & I_{(p-1)\times(p-1)} & 0_{(p-1)\times 1} & & 0_{(p-1)\times q} & \\ \hline \alpha_1 & \cdots & \alpha_p & \beta_1 & \cdots & \beta_q \\ & 0_{(q-1)\times p} & & & I_{(q-1)\times(q-1)} & 0_{(q-1)\times 1} \end{array}\right),$$

and $Z_t := (|\epsilon_t| - \gamma\epsilon_t)^2$. For $m = 1$, the condition (1) is the same as before. The kurtosis is finite whenever (1) holds with $m = 2$.

	Condition for finite kurtosis
GARCH(1,0)	$(1+\gamma^2)\alpha < 1.$
GARCH(2,0)	$1/2\left(1+\gamma^2\right)\left(\gamma^2{\alpha_1}^2+{\alpha_1}^2+2\,\alpha_2+\sqrt{\left(1+\gamma^2\right){\alpha_1}^2\left(\gamma^2{\alpha_1}^2+4\,\alpha_2+{\alpha_1}^2\right)}\right) < 1.$
GARCH(2,1)	$1/2\left(\gamma^4+2\,\gamma^2+1\right){\alpha_1}^2+1/2\left(2\,\beta+2\,\beta\,\gamma^2\right)\alpha_1+\gamma^2\alpha_2+1/2\,\beta^2+\alpha_2$ $+1/2\sqrt{\left(\left(1+\gamma^2\right)^2{\alpha_1}^2+\left(2\,\beta+2\,\beta\,\gamma^2\right)\alpha_1+4\,\alpha_2+4\,\gamma^2\alpha_2+\beta^2\right)\left(\left(1+\gamma^2\right)\alpha_1+\beta\right)^2}$
GARCH(2,2)	$1/2\left(\gamma^4+2\,\gamma^2+1\right){\alpha_1}^2+1/2\left(2\,\beta_1+2\,\beta_1\gamma^2\right)\alpha_1+1/2\,{\beta_1}^2+\alpha_2+\beta_2+\gamma^2\alpha_2$ $+1/2\sqrt{\left(\left(1+\gamma^2\right)\alpha_1+\beta_1\right)^2\left(\left(1+\gamma^2\right)^2{\alpha_1}^2+\left(2\,\beta_1+2\,\beta_1\gamma^2\right)\alpha_1+4\,\alpha_2+4\,\beta_2+4\,\gamma^2\alpha_2+\right.}$

- Model selection and parameter estimation using historical information:
  - Preliminary ARMA estimation for the squared process: Yule-Walker, Burg, Hannan-Rissanen, Innovations.
  - Constrained maximum quasi-likelihood, BIC, AICC.
  - Useful in calibrations to the market as preliminary estimation tool.

Incompleteness: the problem

A market model with a single risky asset modeled using a GARCH process driven by non-binomial innovations is incomplete, that is, not every payoff can be replicated via a self-financing portfolio.

The incompleteness comes from reasons different to those in SV modeling or in multi-excited geometric Brownian motions. Here the problem has to do with the poor cohabitation between discreet time modeling and the infinite number of states of the innovations.

First solutions to the problem

- Find continuous time or diffusion limits: Duan (1997) for the symmetric GARCH(p,q) process and Kallsen, Taqqu (1998) for ARCH. Extensions are not unique; which one should we use?
- Utility maximization approach: Duan (1995). Assumptions on the preferences of the buying agent (i.e. constant relative risk aversion and normally distributed logarithmic aggregate consumption under the physical measure) legitimize the formulation of the so called **locally risk-neutral valuation principle**.

Locally risk-neutral valuation principle (LRNVP)

- One starts with the modified symmetric GARCH process:

$$\log\left(S_n/S_{n-1}\right) \quad =: \quad r_n = r + \lambda\sigma_n - \frac{1}{2}\sigma_n^2 + \sigma_n\epsilon_n,$$
$$\sigma_n^2 \quad = \quad \omega + \sum_{i=1}^{p}\beta_i\sigma_{n-i}^2 + \sum_{i=1}^{q}\alpha_i\epsilon_{n-i}^2.$$

- One plus the conditionally expected rate of return equals $\exp(r + \lambda\sigma_n)$ when computed with respect to the physical probability. Hence, λ is interpreted as a unit risk premium.
- The price of an option is given by the expectation of its discounted payoff with respect to an equivalent risk neutral probability Q that satisfies the LRNVP:

$$E_{n-1}^{Q}\left[\frac{S_n}{S_{n-1}}\right] = e^r, \quad \mathrm{Var}_{n-1}^{Q}\left[\log\left(\frac{S_n}{S_{n-1}}\right)\right] = \mathrm{Var}_{n-1}^{P}\left[\log\left(\frac{S_n}{S_{n-1}}\right.\right.$$

- Prices are obtained via Monte Carlo using ad hoc variance reduction techniques (Duan, Gauthier, Simonato (1998, 2001)), analytical approximations (Duan, Gauthier, Simonato (1999)), recombining tree approximations (Ritchken, Trevor (1999), Lyuu, Wu (2005)).
- Poor understanding of the seller's side of the contract. No agreement in deltas: see Garcia, Renault (1999) for a discussion.
- Heston and Nandi (2000) provide a closed-form pricing formula for an asymmetric version of the Duan model. It is not clear that his formula has the same utility maximization legitimacy: "motivated by previous lognormal option formulas..."

Pricing by local risk minimization

We abandon the use of self-financing portfolios we introduce the notion of **generalized trading strategy**, in which the possibility of additional investment in the numéraire asset throughout the trading periods up to expiry time T is allowed.

Definition

A **generalized trading strategy** is a pair of stochastic processes (ξ^0, ξ) such that $\{\xi_n^0\}_{n\in\{0,\ldots,T\}}$ is adapted and $\{\xi_n\}_{n\in\{1,\ldots,T\}}$ is predictible. The **value process** V of (ξ^0, ξ) is defined as

$$V_0 := \xi_0, \quad \text{and} \quad V_n := \xi_n^0 + \xi_n \cdot S_n, \quad n \geq 1.$$

Definition

The **gains process** G:

$$G_0 := 0 \quad \text{and} \quad G_n := \sum_{k=1}^{n} \xi_k \cdot (S_k - S_{k-1}), \quad n = 0, \ldots, T.$$

The **cost process** C: $C_n := V_n - G_n$, $n = 0, \ldots, T$.

It is easy to check that the strategy (ξ^0, ξ) is self-financing if and only if the value process takes the form

$$V_0 = \xi_1^0 + \xi_1 \cdot S_0 \quad \text{and} \quad V_n = V_0 + \sum_{k=1}^{n} \xi_k \cdot (S_k - S_{k-1}) = V_0 + G_n, \quad n = 1, .$$

or, equivalently, if $V_0 = C_0 = C_1 = \ldots = C_T$.

Definition

Assume that both H and the $\{S_n\}_{n\in\{0,\ldots,T\}}$ are $L^2(\Omega, P)$. A generalized trading strategy is called **admissible** for H whenever it is in $L^2(\Omega, P)$ and its associated value process is such that

$$V_T = H, \quad \mathrm{P} \ a.s. \quad \text{and} \quad V_t \in L^2(\Omega, P), \quad \text{for each } t,$$

and its gain process $G_t \in L^2(\Omega, P)$, for each t.

Remark: since they are not self-financing these strategies may be available even for non-attainable payoffs!

Local risk minimizing strategies

Definition

The **local risk process** of an admissible strategy (ξ^0, ξ) is the process

$$R_t(\xi^0, \xi) := E_t[(C_{t+1} - C_t)^2], \quad t = 0, \ldots, T-1.$$

The admissible strategy $(\widehat{\xi}^0, \widehat{\xi})$ is called **local risk-minimizing** if

$$R_t(\widehat{\xi}^0, \widehat{\xi}) \leq R_t(\xi^0, \xi), \quad \mathrm{P} \ a.s.$$

for all t and each admissible strategy (ξ^0, ξ).

Theorem (Föllmer, Schweizer, Sondermann)

An admissible strategy is local risk-minimizing if and only if the cost process is a P-martingale and it is strongly orthogonal to S, in the sense that $\mathrm{cov}_n(S_{n+1} - S_n, C_{n+1} - C_n) = 0$, P-a.s., for any $t = 0, \ldots, T-1$.

- An admissible strategy is local risk-minimizing for a fixed probability measure P. This measure is not necessarily risk-neutral.
- This approach does not make the difference between shorfall and windfall.
- Once P has been fixed, the local risk-minimizing strategy, if it exists, is unique and the payoff H can be decomposed as

$$H = V_0 + G_T + L_T, \tag{2}$$

G_n gains process and $L_n := C_n - C_0$ the **global risk process**.

GARCH pricing by local risk minimization

We will carry out this pricing program for any GARCH model

$$\begin{aligned} \log\left(\frac{S_n}{S_{n-1}}\right) &= s_n - s_{n-1} = \mu + \sigma_n \epsilon_n, \quad \mu \in \mathbb{R}, \\ \sigma_n^2 &= \sigma_n^2(\sigma_{n-1}, \ldots, \sigma_{n-\max(p,q)}, \epsilon_{n-1}, \ldots, \epsilon_{n-q}), \end{aligned}$$

where the function $\sigma_n^2(\sigma_{n-1}, \ldots, \sigma_{n-\max(p,q)}, \epsilon_n, \ldots, \epsilon_{n-q})$ is constructed so that the following two conditions hold:

- (GARCH1) There exists a constant $\omega > 0$ such that $\sigma_n^2 \geq \omega$.
- (GARCH2) The process $\{\sigma_n \epsilon_n\}_{n \in \mathbb{N}}$ is weakly (covariance) stationary.

We reformulate the problem by finding a local risk-minimizing strategy in which we take the log-prices s_n as the risky asset and $h(s_T) := H(\exp(s_T))$ as the payoff function.

Pricing with respect to the physical measure

$h \in L^2(\Omega, P, \mathcal{F}_T)$ a contingent product on $s = \log(S)$. There exists a unique local risk-minimizing strategy for h, with respect to the physical measure P, determined by:

$$\widehat{\xi}_k = \frac{1}{\sigma_k} E_{k-1}\left[h\left(1-\frac{\mu}{\sigma_T}\epsilon_T\right)\left(1-\frac{\mu}{\sigma_{T-1}}\epsilon_{T-1}\right)\cdots\left(1-\frac{\mu}{\sigma_{k+1}}\epsilon_{k+1}\right)\epsilon\right.$$

$$\widehat{\xi}_T = \frac{1}{\sigma_T} E_{T-1}\left[h\epsilon_T\right],$$

$$V_k = E_k\left[h\left(1-\frac{\mu}{\sigma_T}\epsilon_T\right)\left(1-\frac{\mu}{\sigma_{T-1}}\epsilon_{T-1}\right)\cdots\left(1-\frac{\mu}{\sigma_{k+1}}\epsilon_{k+1}\right)\right],$$

$$V_T = h.$$

The position on the riskless asset is given by $\widehat{\xi}_k^0 := V_k - \widehat{\xi}_k s_k$.
Very heavy to evaluate!!

The minimal martingale measure

It is an equivalent martingale measure for which the value process of the local risk-minimizing strategy *with respect to the physical measure* can be interpreted as an arbitrage free price for h.

Minimal martingale measure: martingale measure $\widehat{\mathbb{P}}$ equivalent to the physical probability $\mathbb{P}$ that satisfies the following two conditions: $E\left[\left(d\widehat{\mathbb{P}}/d\mathbb{P}\right)^2\right] < \infty$ and every $\mathbb{P}$-martingale $M \in L^2(\Omega, \mathbb{P})$ that is strongly orthogonal to the price process s, is also a $\widehat{\mathbb{P}}$-martingale.

It satisfies an entropy minimizing property (Schweizer 2001) and if $\widehat{E}$ denotes the expectation with respect to $\widehat{\mathbb{P}}$, then the value process V_k can be expressed as

$$V_k = \widehat{E}_k[h],$$

which obviously yields the interpretation that we are looking for.

Minimal martingale measures exist in the GARCH context only when the innovations are bounded (for example, when the innovations are multinomial) and certain inequalities among the model parameters are respected.

Proposition: suppose that the innovations in the GARCH model are bounded, that is, there exists $K > 0$ such that $\epsilon_k < K$, for all $k = 1, \ldots, T$, and that $K < \sigma^2 \sqrt{\omega}/\mu$. Then, there exists a unique minimal martingale measure $\widehat{\mathbb{P}}$ with respect to $\mathbb{P}$. Conversely, if there exists a minimal martingale measure then the innovations in the model are necessarily bounded.
Whenever the minimal martingale measure exists, its Radon-Nikodym derivative is given by

$$\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}} = \prod_{k=1}^{T} \left(1 - \frac{\mu \epsilon_k}{\sigma^2 \sigma_k} \right). \tag{3}$$

LRM with respect to an equivalent martingale measure

Reasons to do it:

- The drift terms are usually very small. More explicit justification later.
- The expressions for the values and the hedging rations of the replicating strategy will be far simpler. The values directly admit an interpretation as arbitrage-free prices for the derivative.
- It can be shown that local risk-minimizing trading strategies computed with respect to a martingale measure also minimize the so called **remaining conditional risk**: $R_t^R(\xi^0, \xi) := E_t[(C_T - C_t)^2]$, $t = 0, \ldots, T$; this is in general not true outside the martingale setup.

GARCH with Gaussian innovations

Theorem: Let $\{s_0, s_1, \ldots, s_T\}$ be a GARCH driven by innovations $\{\epsilon_i\}_{i\in\{1,\ldots,T\}} \sim \mathrm{IIDN}(0,1)$. Then,

(i) The process

$$Z_n := \prod_{k=1}^{n} \exp\left(-\frac{\mu}{\sigma_k}\epsilon_k\right)\exp\left(-\frac{1}{2}\frac{\mu^2}{\sigma_k^2}\right), \quad n = 1, \ldots, T,$$

is a square integrable P-martingale.

(ii) Z_T defines an equivalent measure Q such that $Z_T = \frac{dQ}{dP}$.

(iii) The process

$$\widetilde{\epsilon}_n := \epsilon_n + \frac{\mu}{\sigma_n}, \quad n = 1, \ldots, T, \tag{4}$$

forms a $\mathrm{IIDN}(0,1)$ noise with respect to the new Q.

(iv) The log-prices $\{s_0, s_1, \ldots, s_T\}$ form a martingale with respect to Q and they are fully determined by the relations

$$\begin{aligned} s_n &= s_0 + \sigma_1 \widetilde{\epsilon}_1 + \cdots + \sigma_n \widetilde{\epsilon}_n, \\ \sigma_n^2 &= \widetilde{\sigma}_n^2(\sigma_{n-1}, \ldots, \sigma_{n-\max(p,q)}, \widetilde{\epsilon}_{n-1}, \ldots, \widetilde{\epsilon}_{n-q}). \end{aligned}$$

(v) If the process $\{\sigma_n \epsilon_n\}_{n \in \{1,\ldots,T\}}$ is chosen so that it has finite kurtosis with respect to P, then the martingale $\{s_0, s_1, \ldots, s_T\}$ is square integrable with respect to Q. Very important for LRM pricing!

(vi) The random variables in the process $\{\sigma_i \widetilde{\epsilon}_i\}_{i \in \{1,\ldots,T\}}$ are zero mean and uncorrelated with respect to Q.

Prices with respect to the martingale measure

$$
\begin{aligned}
V_k &= \widetilde{E}_k[h(s_T)], \quad k = 0, \ldots, T, \\
\widehat{\xi}_k &= \frac{1}{\sigma_k} \widetilde{E}_{k-1}\left[\widetilde{\epsilon}_k h(s_T)\right], \quad k = 1, \ldots, T, \\
L_T &= h(s_T) - \widetilde{E}[h(s_T)] - \sum_{k=1}^{T} \widetilde{\epsilon}_k \widetilde{E}_{k-1}\left[\widetilde{\epsilon}_k h(s_T)\right].
\end{aligned}
$$

The position on the riskless asset is given by $\widehat{\xi}_k^0 := V_k - \widehat{\xi}_k s_k$.

Proposition. Let V_k be the value process of the local risk minimizing strategy computed with the physical probability and $\widetilde{V}_k$ computed with respect to the martingale measure introduced above. The linear Taylor expansions of V_k and $\widetilde{V}_k$ in the drift term μ coincide.

GARCH with multinomial innovations

Theorem:

$\{s_0, s_1, \ldots, s_T\}$ a GARCH process driven by multinomial innovations $\{\epsilon_i\}_{i\in\{1,\ldots,T\}} \sim \mathrm{IID}(0,1)$ with a pdf

$$p(x) = \sum_{i=1}^{m} \delta(x - x_i)p_i.$$

(i) The process

$$Z_n := \prod_{k=1}^{n} \frac{f(\epsilon_k, \sigma_k)}{E_{k-1}\left[f(\epsilon_k, \sigma_k)\right]}, \quad n = 1, \ldots, T, \tag{5}$$

is a square integrable P-martingale. $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ positive measurable function that satisfies:

$$\frac{\sum_{i=1}^{m} f(x_i, \sigma_k)p_i x_i}{\sum_{i=1}^{m} f(x_i, \sigma_k)p_i} + \frac{\mu}{\sigma_k} = 0, \quad k \in \{1, \ldots, T\}.$$

(ii) Z_T defines an equivalent measure Q such that $Z_T = \frac{dQ}{dP}$.

(iii) The process

$$\widetilde{\epsilon}_n := \epsilon_n + \frac{\mu}{\sigma_n}, \quad n = 1, \ldots, T, \tag{6}$$

forms a sequence of mean zero, Q-uncorrelated multinomial variables with conditional densities

$$p_{\widetilde{\epsilon}_n | \mathcal{F}_{n-1}}(x) = \frac{\sum_{i=1}^{m} f(x_i, \sigma_n) p_i \delta\left(x - x_i - \frac{\mu}{\sigma_n}\right)}{\sum_{j=1}^{m} f(x_j, \sigma_n) p_j},$$

and conditional variance

$$\widetilde{\mathrm{var}}_{n-1}(\widetilde{\epsilon}_n) = \frac{\sum_{i=1}^{m} f(x_i, \sigma_n) p_i x_i^2}{\sum_{j=1}^{m} f(x_j, \sigma_n) p_j} - \frac{\mu^2}{\sigma_n^2}.$$

(iv) The log-prices $\{s_0, s_1, \ldots, s_T\}$ form a square integrable martingale with respect to Q.

(v) The random variables in the process $\{\sigma_i \widetilde{\epsilon}_i\}_{i\in\{1,\ldots,T\}}$ are zero mean and uncorrelated with respect to Q.

Prices:

$$V_k = \widetilde{E}_k[h(s_T)], \quad k = 0, \ldots, T,$$

$$\widehat{\xi}_k = \frac{1}{\sigma_k \Sigma_k^2} \widetilde{E}_{k-1}\left[\widetilde{\epsilon}_k h(s_T)\right], \quad \Sigma_n^2 := \widetilde{\mathrm{var}}_{n-1}(\widetilde{\epsilon}_n) \quad k = 1, \ldots, T,$$

$$L_T = h(s_T) - \widetilde{E}[h(s_T)] - \sum_{k=1}^{T} \frac{\widetilde{\epsilon}_k}{\Sigma_k^2} \widetilde{E}_{k-1}\left[\widetilde{\epsilon}_k h(s_T)\right].$$

The position on the riskless asset is given by $\widehat{\xi}_k^0 := V_k - \widehat{\xi}_k s_k$.

Generalization to predictable drifts and the price representation

- The same theorem is valid for processes

$$\log\left(\frac{S_n}{S_{n-1}}\right) = s_n - s_{n-1} = \mu_n + \sigma_n \epsilon_n, \text{ where } \mu_n \text{ is } \mathcal{F}_{n-1}\text{-measurable.}$$

- Rewrite $s_n = s_{n-1} + \widetilde{\mu}_n - \frac{1}{2}\sigma_n^2 + \sigma_n \epsilon_n$ with

$$\widetilde{\mu}_n := \mu_n + \frac{1}{2}\sigma_n^2 \quad \text{and let} \quad \widetilde{\epsilon}_n := \epsilon_n + \frac{\widetilde{\mu}_n}{\sigma_n}$$

- The Q measure constructed with the $\{\widetilde{\mu}_n\}$ is such that the innovations $\{\widetilde{\epsilon}_n\}$ form an $\mathrm{IIDN}(0,1)$ process with respect to which the price process (not the log-prices!) form a martingale:

$$S_n = S_{n-1} \exp\left(-\frac{1}{2}\sigma_n^2 + \sigma_n \widetilde{\epsilon}_n\right).$$

- The LRM associated strategy strategy:

$$\widehat{V}_k = \widetilde{E}_k[H(S_T)], \quad k = 0, \ldots, T,$$

$$\widehat{\xi}_k = \frac{S_{k-1}}{\Sigma_k^2} \widetilde{E}_{k-1}\left[H\left(\exp\left(-\frac{1}{2}\sigma_n^2 + \sigma_n \widetilde{\epsilon}_n\right) - 1\right)\right], \quad k = 1, \ldots, T,$$

$$L_T = H(S_T) - \widetilde{V}_0 - \sum_{k=1}^{T} \widehat{\xi}_k (S_k - S_{k-1}),$$

where $\Sigma_k^2 := S_{k-1}^2 \left(\mathrm{e}^{\sigma_k^2} - 1\right)$.

Duan's model

$$
\begin{aligned}
s_n &= s_{n-1} + \lambda\sigma_n - \frac{1}{2}\sigma_n^2 + \sigma_n\epsilon_n, \\
\sigma_n^2 &= \alpha_0 + \sum_{i=1}^{q} \alpha_i \sigma_{n-i}^2 \epsilon_{n-i}^2 + \sum_{i=1}^{p} \beta_i \sigma_{n-i}^2.
\end{aligned}
$$

- We risk neutralize using the method just introduced with $\mu_n = \lambda\sigma_n - \frac{1}{2}\sigma_n^2$ and $\widetilde{\mu}_n = \lambda\sigma_n$. In terms of the martingale measure we obtain:

$$
\begin{aligned}
s_n &= s_{n-1} - \frac{1}{2}\sigma_n^2 + \sigma_n \widetilde{\epsilon}_n, \\
\sigma_n^2 &= \alpha_0 + \sum_{i=1}^{q} \alpha_i \sigma_{n-i}^2 (\widetilde{\epsilon}_{n-i} - \lambda)^2 + \sum_{i=1}^{p} \beta_i \sigma_{n-i}^2
\end{aligned}
$$

- Coincides with the expression obtained by Duan via the LRNV (local risk neutral valuation) principle.
- The associated LRM pricing formula coincides with Duan's but not the hedging strategy. LRM hedging ratio provides a variance-optimal self-financing hedging strategy (by theorem!!) and hence the hedging square-error is smaller than in Duan.

Heston-Nandi model

$$
\begin{aligned}
s_n &= s_{n-1} + \lambda \sigma_n^2 + \sigma_n \epsilon_n, \\
\sigma_n^2 &= \alpha_0 + \sum_{i=1}^{q} \alpha_i (\epsilon_{n-i} - \gamma_i \sigma_{n-i})^2 + \sum_{i=1}^{p} \beta_i \sigma_{n-i}^2 .
\end{aligned}
$$

- We risk neutralize using $\mu_n = \lambda \sigma_n^2$ and $\widetilde{\mu}_n = \lambda \sigma_n^2 + 1/2 \sigma_n^2 = \left(\lambda + \frac{1}{2}\right) \sigma_n^2$. In terms of the martingale measure we obtain:

$$
\begin{aligned}
s_n &= s_{n-1} - \frac{1}{2}\sigma_n^2 + \sigma_n \widetilde{\epsilon}_n, \\
\sigma_n^2 &= \alpha_0 + \sum_{i=1}^{q} \alpha_i \left(\widetilde{\epsilon}_{n-i} - \left(\lambda + \gamma_i + \frac{1}{2} \right) \sigma_{n-i} \right)^2 + \sum_{i=1}^{p} \beta_i \sigma_{n-i}^2
\end{aligned}
$$

- Coincides with the expression obtained by Heston-Nandi based on "previous pricing formulas..."
- LRM provides a hedging strategy.