The Heston model with stochastic interest rates and pricing options with Fourier-cosine expansion

Kees Oosterlee [1,2]

[1]CWI, Center for Mathematics and Computer Science, Amsterdam,
[2]Delft University of Technology, Delft.

Joint Work with Fang Fang, Lech Grzelak

October 16th 2009

Centrum Wiskunde & Informatica

Contents

- COS option pricing method, based on Fourier-cosine expansions
  - Highly efficient for European and Bermudan options
  - Lévy processes and Heston stochastic volatility for asset prices
- Generalize to other derivative contracts
  - Swing options (buy and sell energy, commodity contracts)
  - Mean variance hedging under jump processes
- Generalize to hybrid products
  - Models with stochastic interest rate; stochastic volatility

Financial industry; Banks at Work

Front office ⇔ Back office

Pricing, selling products ⇔ Price validation, research into alternative models

- Pricing approach:
  1. Define some financial product
  2. Model asset prices involved (SDEs)
  3. Calibrate the model to market data (Numerics, Optimization)
  4. Model product price correspondingly (PDE, Integral)
  5. Price the product of interest (Numerics, MC)
  6. Set up hedge to remove the risk related to the product (Optimization)

Financial mathematics aspects

- Knowledge: What product are we dealing with?
  - Contract specification (contract function)
  - Early-exercise product, or not
  - Product's lifetime

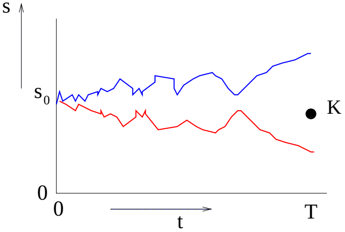
$\Rightarrow$ Determines the model for underlying asset (stochastic interest rate, ...)

- Financial sub-problem: Product pricing or parameter calibration

$\Rightarrow$ All this determines the choice of numerical method

Plain vanilla option

An option contract gives the holder the right to trade in the future at a previously agreed strike price, K, but takes away the obligation.



$$V^{call}(S,T) = \max(S_T - K, 0) =: E(S,T)$$

Early exercise:
$V(S,t) = \max\{E(S,t), V(S,t)\}.$

A pricing approach

$$V(S(t_0), t_0) = e^{-r(T-t_0)}\mathbb{E}^Q\{V(S(T), T)|S(t_0)\}$$

Quadrature:

$$V(S(t_0), t_0) = e^{-r(T-t_0)}\int_{\mathbb{R}} V(S(T), T)f(S(T)|S(t_0))dS$$

- Trans. PDF, $f(S(T)|S(t_0))$, typically not available, but its Fourier transform, called the characteristic function, ϕ, often is.

Geometric Brownian Motion

- Asset price, S, can be modeled by geometric Brownian motion:

$$dS_t = rS_t dt + \sigma S dW_t^Q,$$

with W_t Wiener process, r interest rate, σ volatility.

$\Rightarrow$ Itô's Lemma: Black-Scholes equation: (for a European option)

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \ .$$

Pricing: Feynman-Kac Theorem

Given the final condition problem

$$\begin{cases} \dfrac{\partial V}{\partial t} & + \quad \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} + rS\dfrac{\partial V}{\partial S} - rV = 0, \\ V(S,T) & = \quad \text{given} \end{cases}$$

Then the value, $V(S(t),t)$, is the unique solution of

$$V(S,t) = e^{-r(T-t)}\mathbb{E}^Q\{V(S(T),T)|S(t)\}$$

with the sum of the first derivatives of the option square integrable. and S satisfies the system of stochastic differential equations:

$$dS_t = rS_t dt + \sigma S_t dW_t^Q,$$

- Similar relations also hold for other SDEs in Finance

Centrum Wiskunde & Informatica

Numerical Pricing Approach

- One can apply several numerical techniques to calculate the option price:
  - Numerical integration,
  - Monte Carlo simulation,
  - Numerical solution of the partial-(integro) differential equation (P(I)DE)
- Each of these methods has its merits and demerits.
- Numerical challenges:
  - Speed of solution methods (for example, for calibration)
  - Early exercise feature (P(I)DE $\longrightarrow$ free boundary problem)
  - The problem's dimensionality (not treated here)

Motivation Fourier Methods

- Derive pricing methods that
  - are computationally fast
  - are not restricted to Gaussian-based models
  - should work as long as we have the characteristic function,

$$\phi(u) = \mathbb{E}\left(e^{iuX}\right) = \int_{-\infty}^{\infty} e^{iux} f(x) dx;$$

    (available for Lévy processes and also for Heston's model).
  - In probability theory a characteristic function of a continuous random variable X, equals the Fourier transform of the density of X.

Class of Affine Diffusion (AJD) processes

Duffie, Pan, Singleton (2000): The following system of SDEs:

$$d\mathbf{X}_t = \mu(\mathbf{X}_t)dt + \sigma(\mathbf{X}_t)d\mathbf{W}_t,$$

is of the affine form, if the drift, volatility, jump intensity and interest rate satisfy:

$$\begin{aligned} \mu(\mathbf{X}_t) &= a_0 + a_1\mathbf{X}_t \text{ for } (a_0, a_1) \in \mathbb{R}^n \times \mathbb{R}^{n\times n}, \\ \sigma(\mathbf{X}_t)\sigma(\mathbf{X}_t)^T &= (c_0)_{ij} + (c_1)_{ij}^T\mathbf{X}_t, (c_0, c_1) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n\times n}, \end{aligned}$$

The discounted characteristic function then has the following form:

$$\phi(\mathbf{X_t}, \mathbf{t}, \mathbf{T}, \mathbf{u}) = e^{A(\mathbf{u},t,T)+\mathbf{B}(\mathbf{u},t,T)^T\mathbf{X_t}},$$

The coefficients $A(\mathbf{u}, t, T)$ and $\mathbf{B}(\mathbf{u}, \mathbf{t}, \mathbf{T})^{\mathbf{T}}$ satisfy a system of Riccati-type ODEs.

CWI Centrum Wiskunde & Informatica

The COS option pricing method, based on Fourier Cosine Expansions

Series Coefficients of the Density and the Ch.F.

- Fourier-Cosine expansion of density function on interval $[a, b]$:

$$f(x) = {\sum_{n=0}^{\infty}}' F_n \cos\left(n\pi \frac{x-a}{b-a}\right),$$

with $x \in [a, b] \subset \mathbb{R}$ and the coefficients defined as

$$F_n := \frac{2}{b-a} \int_a^b f(x) \cos\left(n\pi \frac{x-a}{b-a}\right) dx.$$

- F_n has direct relation to ch.f., $\phi(\omega) := \int_{\mathbb{R}} f(x) e^{i\omega x} dx$ ($\int_{\mathbb{R}\backslash[a,b]} f(x) \approx 0$),

$$\begin{aligned} F_n \approx A_n \quad &:= \quad \frac{2}{b-a} \int_{\mathbb{R}} f(x) \cos\left(n\pi \frac{x-a}{b-a}\right) dx \\ &= \quad \frac{2}{b-a} \mathrm{Re}\left\{\phi\left(\frac{n\pi}{b-a}\right) \exp\left(-i\frac{na\pi}{b-a}\right)\right\}. \end{aligned}$$

CWI Centrum Wiskunde & Informatica

Recovering Densities

- Replace F_n by A_n, and truncate the summation:

$$f(x) \approx \frac{2}{b-a} {\sum_{n=0}^{N-1}}' \mathrm{Re}\left\{ \phi\left(\frac{n\pi}{b-a}; x\right) \exp\left(in\pi\frac{-a}{b-a}\right)\right\} \cos\left(n\pi\frac{x-a}{b-a}\right),$$

- Example: $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$, $[a, b] = [-10, 10]$ and $x = \{-5, -4, \cdots, 4, 5\}$.

N	4	8	16	32	64
error	0.2538	0.1075	0.0072	4.04e-07	3.33e-16
cpu time (sec.)	0.0025	0.0028	0.0025	0.0031	0.0032

Exponential error convergence in N.

CWI Centrum Wiskunde & Informatica

Pricing European Options

- Start from the risk-neutral valuation formula:

$$v(x,t_0) = e^{-r\Delta t}\mathbb{E}^{\mathbb{Q}}\left[v(y,T)|x\right] = e^{-r\Delta t}\int_{\mathbb{R}} v(y,T)f(y|x)dy.$$

- Truncate the integration range:

$$v(x,t_0) = e^{-r\Delta t}\int_{[a,b]} v(y,T)f(y|x)dy + \varepsilon.$$

- Replace the density by the COS approximation, and interchange summation and integration:

$$\hat{v}(x,t_0) = e^{-r\Delta t}{\sum_{n=0}^{N-1}}{}' \mathrm{Re}\left\{\phi\left(\frac{n\pi}{b-a};x\right)e^{-in\pi\frac{a}{b-a}}\right\}V_n,$$

where the series coefficients of the payoff, V_n, are analytic.

Pricing European Options

- Log-asset prices: $x := \ln(S_0/K)$ and $y := \ln(S_T/K)$,
- The payoff for European options reads

$$v(y,T) \equiv [\alpha \cdot K(e^y - 1)]^+.$$

- For a call option, we obtain

$$\begin{aligned} V_k^{call} &= \frac{2}{b-a}\int_0^b K(e^y - 1)\cos\left(k\pi\frac{y-a}{b-a}\right)dy \\ &= \frac{2}{b-a}K\left(\chi_k(0,b) - \psi_k(0,b)\right), \end{aligned}$$

- For a vanilla put, we find

$$V_k^{put} = \frac{2}{b-a}K\left(-\chi_k(a,0) + \psi_k(a,0)\right).$$

CWI Centrum Wiskunde & Informatica

Heston model

- The Heston stochastic volatility model can be expressed by the following 2D system of SDEs

$$\begin{cases} dS_t &= r_t S_t dt + \sqrt{\sigma_t} S_t dW_t^S, \\ d\sigma_t &= -\kappa(\sigma_t - \overline{\sigma})dt + \gamma\sqrt{\sigma_t} dW_t^{\sigma}, \end{cases}$$

- With $x_t = \log S_t$ this system is in the affine form.

$\Rightarrow$ Itô's Lemma: multi-D partial differential equation

Characteristic Functions Heston Model

- For Lévy and Heston models, the ChF can be represented by

$$\begin{aligned}\phi(\omega;\mathbf{x}) &= \varphi_{levy}(\omega)\cdot e^{i\omega\mathbf{x}} \quad \text{with} \quad \varphi_{levy}(\omega) := \phi(\omega;0),\\ \phi(\omega;\mathbf{x},u_0) &= \varphi_{hes}(\omega;u_0)\cdot e^{i\omega\mathbf{x}},\end{aligned}$$

- The ChF of the log-asset price for Heston's model:

$$\begin{aligned}\varphi_{hes}(\omega;\sigma_0) &= \exp\left(i\omega r\Delta t + \frac{\sigma_0}{\gamma^2}\left(\frac{1-e^{-D\Delta t}}{1-Ge^{-D\Delta t}}\right)(\kappa - i\rho\gamma\omega - D)\right)\cdot\\ &\quad \exp\left(\frac{\kappa\bar{\sigma}}{\gamma^2}\left(\Delta t(\kappa - i\rho\gamma\omega - D) - 2\log\left(\frac{1-Ge^{-D\Delta t}}{1-G}\right)\right)\right),\end{aligned}$$

with $D = \sqrt{(\kappa - i\rho\gamma\omega)^2 + (\omega^2 + i\omega)\gamma^2}$ and $G = \frac{\kappa - i\rho\gamma\omega - D}{\kappa - i\rho\gamma\omega + D}$.

Heston Model

- We can present the V_k as $\mathbf{V}_k = U_k \mathbf{K}$, where

$$U_k = \begin{cases} \frac{2}{b-a}\left(\chi_k(0,b) - \psi_k(0,b)\right) & \text{for a call} \\ \frac{2}{b-a}\left(-\chi_k(a,0) + \psi_k(a,0)\right) & \text{for a put.} \end{cases}$$

- The pricing formula simplifies for Heston and Lévy processes:

$$v(\mathbf{x}, t_0) \approx \mathbf{K} e^{-r\Delta t} \cdot \mathrm{Re}\left\{ {\sum_{n=0}^{N-1}}' \varphi\left(\frac{n\pi}{b-a}\right) U_n \cdot e^{in\pi \frac{\mathbf{x}-a}{b-a}} \right\},$$

where $\varphi(\omega) := \phi(\omega; 0)$

Numerical Results

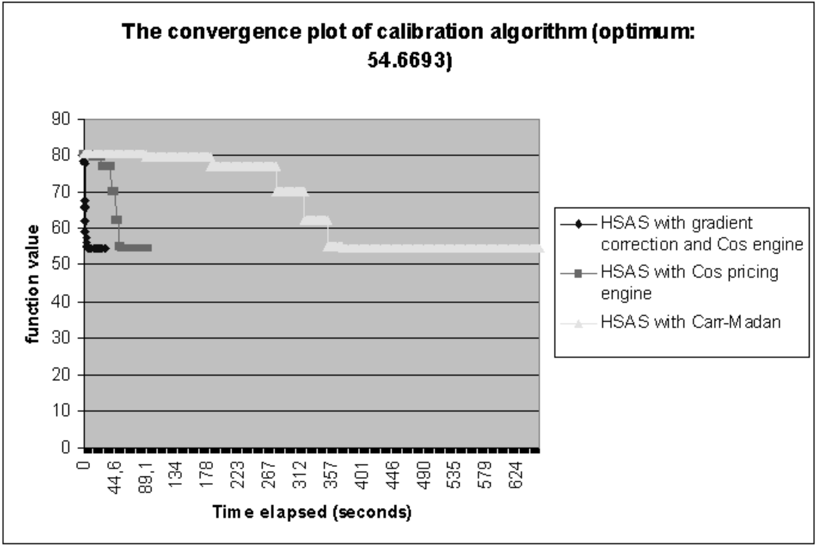
Pricing 21 strikes $K = 50, 55, 60, \cdots, 150$ simultaneously under Heston's model. Other parameters: $S_0 = 100, r = 0, q = 0, T = 1, \kappa = 1.5768, \gamma = 0.5751, \bar{\sigma} = 0.0398, \sigma_0 = 0.0175, \rho = -0.5711$.

COS	N	96	128	160
	(msec.)	2.039	2.641	3.220
	max. abs. err.	4.52e-04	2.61e-05	$4.40e-06$
Carr-Madan	N	2048	4096	8192
	(msec.)	20.36	37.69	76.02
	max. abs. error	2.61e-01	$2.15e-03$	2.08e-07

Error analysis for the COS method is provided in the paper.

Numerical Results within Calibration

- Calibration for Heston's model: Around 10 times faster than Carr-Madan.



Generalization

- COS option pricing method, based on Fourier-cosine expansions
  - Highly efficient for European and Bermudan options
  - Lévy processes and Heston stochastic volatility for asset prices
- Generalize to other derivative contracts
  - Swing options (buy and sell energy, commodity contracts)
  - Mean variance hedging under jump processes
- Generalize to hybrid products
  - Models with stochastic interest rate; stochastic volatility

An exotic contract: A hybrid product

- Based on sets of assets with different expected returns and risk levels.
- Proper construction may give reduced risk and an expected return greater than that of the least risky asset.
- A simple example is a portfolio with a stock with a high risk and return and a bond with a low risk and return.
- Example:

$$V(S, t_0) = \mathbb{E}^Q \left(e^{-\int_0^T r_s ds} \max \left(0, \frac{1}{2}\frac{S_T}{S_0} + \frac{1}{2}\frac{B_T}{B_0} \right) \right)$$

Heston-Hull-White hybrid model

- The Heston-Hull-White hybrid model can be expressed by the following 3D system of SDEs

$$\begin{cases} dS_t &= r_t S_t dt + \sqrt{\sigma_t} S_t dW_t^S, \\ dr_t &= \lambda(\theta_t - r_t)\, dt + \eta r_t^p dW_t^r, \\ d\sigma_t &= -\kappa(\sigma_t - \overline{\sigma}) dt + \gamma \sqrt{\sigma_t} dW_t^\sigma, \end{cases}$$

- Full correlation matrix
- System is not in the affine form. The symmetric instantaneous covariance matrix is given by:

$$\sigma(\mathbf{X}_t)\sigma(\mathbf{X}_t)^T = \begin{bmatrix} \sigma_t & \rho_{x,\sigma}\gamma\sigma_t & \rho_{x,r}\eta r_t^p \sqrt{\sigma_t} \\ * & \gamma^2\sigma_t & \rho_{r,\sigma}\gamma\eta r_t^p \sqrt{\sigma_t} \\ * & * & \eta^2 r_t^{2p} \end{bmatrix}.$$

$\Rightarrow$ Itô's Lemma: multi-D partial differential equation

Reformulated HHW Model

- A well-defined Heston hybrid model with *indirectly imposed correlation*, $\rho_{x,r}$:

$$
\begin{aligned}
dS_t &= r_t S_t dt + \sqrt{\sigma_t} S_t dW_t^x + \Omega_t r_t^p S_t dW_t^r + \Delta \sqrt{\sigma_t} S_t dW_t^\sigma, \qquad S_0 > 0, \\
dr_t &= \lambda(\theta_t - r_t)dt + \eta r_t^p dW_t^r, \qquad r_0 > 0, \\
d\sigma_t &= \kappa(\bar{\sigma} - \sigma_t)dt + \gamma\sqrt{\sigma_t} dW_t^\sigma, \qquad \sigma_0 > 0,
\end{aligned}
$$

with

$$
\begin{aligned}
dW_t^x dW_t^\sigma &= \hat{\rho}_{x,\sigma}, \\
dW_t^x dW_t^r &= 0, \\
dW_t^\sigma dW_t^r &= 0,
\end{aligned}
$$

- We have included a time-dependent function, Ω_t, and a parameter, Δ.

Basics

- Decompose a given general symmetric correlation matrix, $\mathbf{C}$, as $\mathbf{C} = \mathbf{L}\mathbf{L}^T$, where $\mathbf{L}$ is a lower triangular matrix with strictly positive entries.
- Rewrite a system of SDEs in terms of the independent Brownian motions with the help of the lower triangular matrix $\mathbf{L}$.

“Equivalence”

- By exchanging the order of the state variables $\mathbf{X}_t = [x_t, \sigma_t, r_t]$ to $\mathbf{X}_t^* = [r_t, \sigma_t, x_t]$, the HHW and HCIR models have $\rho_{r,\sigma} = 0$, $\rho_{x,r} \neq 0$ and $\rho_{x,\sigma} \neq 0$ and read: $d\mathbf{X}_t^* = [\ldots]dt +$

$$\begin{bmatrix} \eta r_t^p & 0 & 0 \\ 0 & \gamma\sqrt{\sigma_t} & 0 \\ \rho_{x,r}\sqrt{\sigma_t}S_t & \rho_{x,\sigma}\sqrt{\sigma_t}S_t & \sqrt{\sigma_t}S_t\sqrt{1-\rho_{x,\sigma}^2-\rho_{x,r}^2} \end{bmatrix} \begin{bmatrix} d\widetilde{W}_t^r \\ d\widetilde{W}_t^\sigma \\ d\widetilde{W}_t^x \end{bmatrix}. \quad (1)$$

- The reformulated hybrid model is given, in terms of the independent Brownian motions, by: $d\mathbf{X}_t^* = [\ldots]dt +$

$$\begin{bmatrix} \eta r_t^p & 0 & 0 \\ 0 & \gamma\sqrt{\sigma_t} & 0 \\ \Omega_t r_t^p S_t & \sqrt{\sigma_t}S_t\left(\hat{\rho}_{x,\sigma}+\Delta\right) & \sqrt{\sigma_t}S_t\sqrt{1-\hat{\rho}_{x,\sigma}^2} \end{bmatrix} \begin{bmatrix} d\widetilde{W}_t^r \\ d\widetilde{W}_t^\sigma \\ d\widetilde{W}_t^x \end{bmatrix},$$

CWI Centrum Wiskunde & Informatica

"Equivalence"

- The reformulated HHW model is a well-defined Heston hybrid model with non-zero correlation, $\rho_{x,r}$, for:

$$\begin{aligned}\Omega_t &= \rho_{x,r} r_t^{-p} \sqrt{\sigma_t},\\ \hat{\rho}_{x,\sigma}^2 &= \rho_{x,\sigma}^2 + \rho_{x,r}^2,\\ \Delta &= \rho_{x,\sigma} - \hat{\rho}_{x,\sigma},\end{aligned}$$

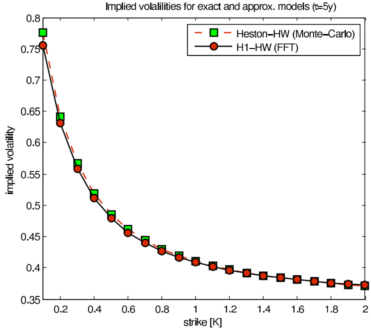
- In order to satisfy the affinity constraints, we *approximate* Ω_t by a deterministic time-dependent function:

$$\Omega_t \approx \rho_{x,r}\mathbb{E}\left(r_t^{-p}\sqrt{\sigma_t}\right) = \rho_{x,r}\mathbb{E}\left(r_t^{-p}\right)\mathbb{E}\left(\sqrt{\sigma_t}\right),$$

  assuming independence between r_t and σ_t.
- The model is in the affine class
  $\Rightarrow$ Fast pricing of options with the COS method

Numerical Experiment; Implied vol

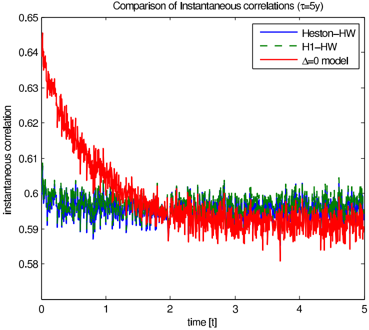
- Implied volatilities for the HHW (obtained by Monte Carlo) and the approximate (obtained by COS) models.
- For short and long maturity experiments, we obtain a very good fit of the approximate to the full-scale HHW model.
- The parameters are $\theta = 0.03$, $\kappa = 1.2$, $\bar{\sigma} = 0.08$, $\gamma = 0.09$, $\lambda = 1.1$, $\eta = 0.1$, $\rho_{x,v} = -0.7$, $\rho_{x,r} = 0.6$, $S_0 = 1$, $r_0 = 0.08$, $v_0 = 0.0625$, $a = 0.2813$, $b = -0.0311$ and $c = 1.1347$.



- $\tau = 5y$.

Numerical Experiment; Instantaneous Correlation

- We check, by means of Monte Carlo simulation, the behavior of the instantaneous correlations between stock S_t and the interest rate r_t.
- Three models: The HHW model, the model with $\Delta = 0$, and constant $\overline{\Omega}$, and the approximate HHW model .
- For the HHW and the H1-HW model, the instantaneous correlations are stable and oscillate around the exact value, chosen to be 0.6.



Conclusions

- We presented the COS method, based on Fourier-cosine series expansions, for European options.
- The method also works efficiently for Bermudan and discretely monitored barrier options.
- COS method can be applied to affine approximations of HHW hybrid models.
- Generalized to full set of correlations, to Heston-CIR, and Heston-multi-factor models
- Papers available: http://ta.twi.tudelft.nl/mf/users/oosterle/oosterlee/ http://ta.twi.tudelft.nl/mf/users/oosterle/oosterlee/oosterleerecent.html

Pricing European Options

- Log-asset prices: $x := \ln(S_0/K)$ and $y := \ln(S_T/K)$,
- The payoff for European options reads

$$v(y, T) \equiv [\alpha \cdot K(e^y - 1)]^+.$$

- For a call option, we obtain

$$\begin{aligned} V_k^{call} &= \frac{2}{b-a}\int_0^b K(e^y - 1)\cos\left(k\pi\frac{y-a}{b-a}\right)dy \\ &= \frac{2}{b-a}K\left(\chi_k(0,b) - \psi_k(0,b)\right), \end{aligned}$$

- For a vanilla put, we find

$$V_k^{put} = \frac{2}{b-a}K\left(-\chi_k(a,0) + \psi_k(a,0)\right).$$

CWI Centrum Wiskunde & Informatica

Market modeled by alternative processes

$$
\begin{aligned}
dS_t &= rS_t dt + \sigma S_t dW_t^Q \\
\Rightarrow S_t &= S_0 e^{X_t},\ X_t = (r - \frac{\sigma^2}{2})t + \sigma W_t^Q.
\end{aligned}
$$

- Compound Poisson (jump diffusion model)

$$
X_t = (\mu - \frac{\sigma^2}{2})t + \sigma W_t + \sum_{i=1}^{N_t} Y_i,
$$

where N_t is Poisson: $P(N_t = n) = e^{-\lambda t}(\lambda t)^n / n!$, with intensity λ, Y_i i.i.d. wi th law F, for example, normally distributed (mean μ_J, variance σ_J^2).

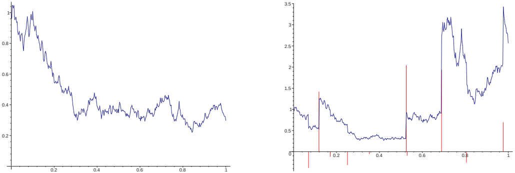
Lévy Processes

- Lévy process $\{X_t\}_{t\geq 0}$: process with stationary, independent increments.
- Brownian motion and Poisson processes belong to this class
- Combinations of these give Jump-Diffusion processes
- Replace deterministic time by a random business time given by a Gamma process: the Variance Gamma process [Carr, Madan, Chang 1998]. Infinite activity jumps:
  - small jumps describe the day-to-day "noise" that causes minor fluctuations in stock prices;
  - big jumps describe large stock price movements caused by major market upsets

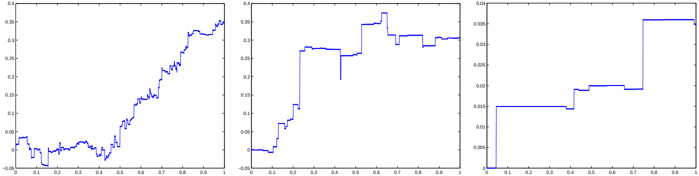
CWI Centrum Wiskunde & Informatica

SDE Simulation

- GBM, JD



- Variance Gamma process with gamma distributed times, positive drift



Truncation Range

$$[a, b] := \left[(c_1 + x_0) - L\sqrt{c_2 + \sqrt{c_4}}, \quad (c_1 + x_0) + L\sqrt{c_2 + \sqrt{c_4}} \right],$$

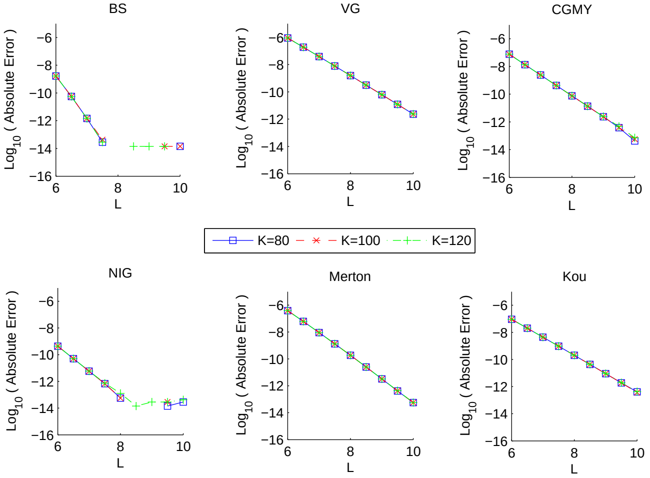


Table: Cumulants of $\ln(S_t/K)$ for various models.

BS	$c_1 = (\mu - \frac{1}{2}\sigma^2)t, \quad c_2 = \sigma^2 t, \quad c_4 = 0$	
NIG	$c_1 = (\mu - \frac{1}{2}\sigma^2 + w)t + \delta t\beta/\sqrt{\alpha^2 - \beta^2}$ $c_2 = \delta t\alpha^2(\alpha^2 - \beta^2)^{-3/2}$ $c_4 = 3\delta t\alpha^2(\alpha^2 + 4\beta^2)(\alpha^2 - \beta^2)^{-7/2}$ $w = -\delta(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta+1)^2})$	
Kou	$c_1 = t\left(\mu + \frac{\lambda p}{\eta_1} + \frac{\lambda(1-p)}{\eta_2}\right)$ $c_4 = 24t\lambda\left(\frac{p}{\eta_1^4} + \frac{1-p}{\eta_2^4}\right)$	$c_2 = t\left(\sigma^2 + 2\frac{\lambda p}{\eta_1^2} + 2\frac{\lambda(1-p)}{\eta_2^2}\right)$ $w = \lambda\left(\frac{p}{\eta_1+1} - \frac{1-p}{\eta_2-1}\right)$
Merton	$c_1 = t(\mu + \lambda\bar{\mu})$ $c_4 = t\lambda\left(\bar{\mu}^4 + 6\bar{\sigma}^2\bar{\mu}^2 + 3\bar{\sigma}^4\lambda\right)$	$c_2 = t\left(\sigma^2 + \lambda\bar{\mu}^2 + \bar{\sigma}^2\lambda\right)$
VG	$c_1 = (\mu + \theta)t$ $c_4 = 3(\sigma^4\nu + 2\theta^4\nu^3 + 4\sigma^2\theta^2\nu^2)t$	$c_2 = (\sigma^2 + \nu\theta^2)t$ $w = \frac{1}{\nu}\ln(1 - \theta\nu - \sigma^2\nu/2)$
CGMY	$c_1 = \mu t + Ct\Gamma(1-Y)\left(M^{Y-1} - G^{Y-1}\right)$ $c_2 = \sigma^2 t + Ct\Gamma(2-Y)\left(M^{Y-2} + G^{Y-2}\right)$ $c_4 = Ct\Gamma(4-Y)\left(M^{Y-4} + G^{Y-4}\right)$ $w = -C\Gamma(-Y)[(M-1)^Y - M^Y + (G+1)^Y - G^Y]$	

where w is the drift correction term that satisfies $\exp(-wt) = \varphi(-i, t)$

American Options and Extrapolation

Let $v(M)$ denote the value of a Bermudan option with M early exercise dates, then we can rewrite the 3-times repeated Richardson extrapolation scheme as

$$v_{AM}(d) = \frac{1}{12}\left(64v(2^{d+3}) - 56v(2^{d+2}) + 14v(2^{d+1}) - v(2^{d})\right), \qquad (2)$$

where $v_{AM}(d)$ denotes the approximated value of the American option.

Centrum Wiskunde & Informatica

Numerical Results

Table: Parameters for American put options under the CGMY model

Test No.	S_0	K	T	r	Other Parameters
3	1	1	1	0.1	$C = 1, G = 5, M = 5, Y = 0.5$
4	90	98	0.25	0.06	$C = 0.42, G = 4.37, M = 191.2, Y = 1.0102$

d in Eq. (2)	Test No. 3		Test No. 4	
	error	time (milli-sec.)	error	time (milli-sec.)
0	4.41e-05	56.1	-2.80e-03	57.0
1	7.69e-06	111.6	-7.42e-04	112.1
2	9.23e-07	223.9	-2.49e-04	223.3
3	3.04e-07	446.5	-1.62e-04	444.7

Deficiencies of the Black-Scholes Model

- Suppose we solve the 1D Black-Scholes equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

  for σ, since V is known from the market.

- We then find that the volatility, σ, obtained for different K and dates T on the same asset is not constant.

$\Rightarrow$ Does not fit in Black-Scholes model, so look for market consistent asset price models.

Market modeled by Lévy processes

$$
\begin{aligned}
dS(t) &= rS(t)dt + \sigma S dW(t) \\
\Rightarrow S(t) &= S(0)e^{L(t)},\; L(t) = (\mu - \frac{\sigma^2}{2})t + \sigma W(t).
\end{aligned}
$$

- Lévy process $\{X_t\}_{t\geq 0}$: process with stationary, independent increments.
- Example: Replace deterministic time by a random business time given by a Gamma process: the Variance Gamma process [Carr, Madan, Chang 1998].

CWI Centrum Wiskunde & Informatica

CGMY Parameters for “ABN AMRO Bank”

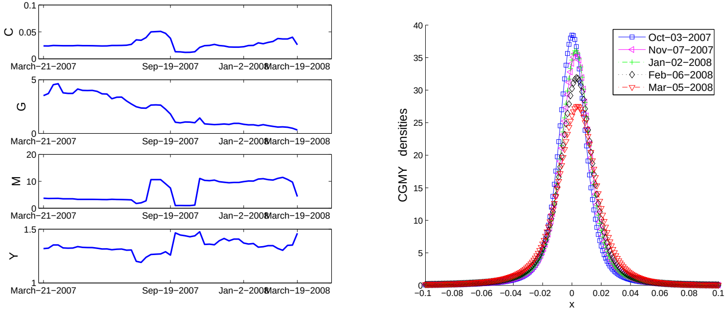


Figure: Evolution of the CGMY parameters and densities of “ABN AMRO Bank”

CGMY Parameters for “DSG International PLC”

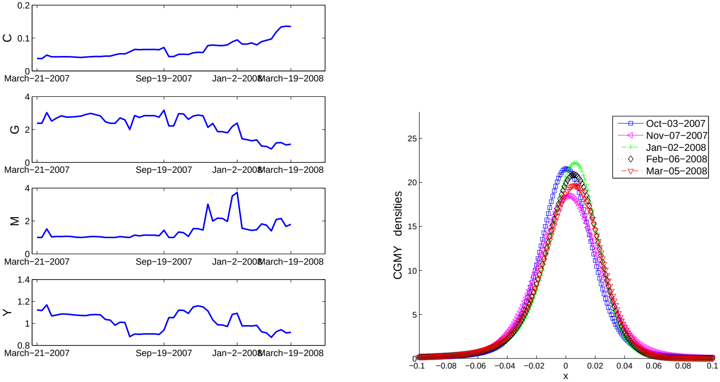


Figure: Evolution of the CGMY parameters and densities of “DSG International PLC”

Pricing: Feynman-Kac Theorem

Given the system of stochastic differential equations:

$$dS_t = rS_t dt + \sigma S_t dW_t^Q,$$

and an option, V, such that

$$V(S,t) = e^{-r(T-t)}\mathbb{E}^Q\{V(S(T),T)|S(t)\}$$

with the sum of the first derivatives of the option square integrable.

Then the value, $V(S(t),t)$, is the unique solution of the final condition problem

$$\begin{cases} \frac{\partial V}{\partial t} & + \quad \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \\ V(S,T) & = \quad \text{given} \end{cases}$$

- Similar relations also hold for other SDEs in Finance

Characteristic Functions Heston Model

- For Lévy and Heston models, the ChF can be represented by

$$\begin{aligned}\phi(\omega;\mathbf{x}) &= \varphi_{levy}(\omega)\cdot e^{i\omega\mathbf{x}} \quad \text{with} \quad \varphi_{levy}(\omega) := \phi(\omega;0),\\ \phi(\omega;\mathbf{x},u_0) &= \varphi_{hes}(\omega;u_0)\cdot e^{i\omega\mathbf{x}},\end{aligned}$$

- The ChF of the log-asset price for Heston's model:

$$\begin{aligned}\varphi_{hes}(\omega;u_0) = \; & \exp\left(i\omega\mu\Delta t + \frac{u_0}{\eta^2}\left(\frac{1-e^{-D\Delta t}}{1-Ge^{-D\Delta t}}\right)(\lambda - i\rho\eta\omega - D)\right)\cdot\\ & \exp\left(\frac{\lambda\bar{u}}{\eta^2}\left(\Delta t(\lambda - i\rho\eta\omega - D) - 2\log\left(\frac{1-Ge^{-D\Delta t}}{1-G}\right)\right)\right),\end{aligned}$$

with $D = \sqrt{(\lambda - i\rho\eta\omega)^2 + (\omega^2 + i\omega)\eta^2}$ and $G = \frac{\lambda - i\rho\eta\omega - D}{\lambda - i\rho\eta\omega + D}$.

Heston Model

- We can present the V_k as $\mathbf{V}_k = U_k \mathbf{K}$, where

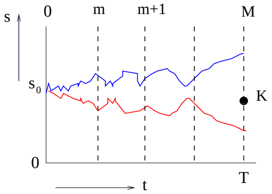
$$U_k = \begin{array}{ll} \frac{2}{b-a}\left(\chi_k(0,b) - \psi_k(0,b)\right) & \text{for a call} \\ \frac{2}{b-a}\left(-\chi_k(a,0) + \psi_k(a,0)\right) & \text{for a put.} \end{array}$$

- The pricing formula simplifies for Heston and Lévy processes:

$$v(\mathbf{x}, t_0) \approx \mathbf{K} e^{-r\Delta t} \cdot \mathrm{Re}\left\{ {\sum_{n=0}^{N-1}}' \varphi\left(\frac{n\pi}{b-a}\right) U_n \cdot e^{in\pi \frac{\mathbf{x}-a}{b-a}} \right\},$$

where $\varphi(\omega) := \phi(\omega; 0)$

Pricing Bermudan Options



- The pricing formulae

$$\begin{cases} c(x,t_m) &= e^{-r\Delta t}\int_{\mathbb{R}} v(y,t_{m+1})f(y|x)dy \\ v(x,t_m) &= \max\left(g(x,t_m), c(x,t_m)\right) \end{cases}$$

and $v(x,t_0) = e^{-r\Delta t}\int_{\mathbb{R}} v(y,t_1)f(y|x)dy$.

  - Use Newton's method to locate the early exercise point x_m^*, which is the root of $g(x,t_m) - c(x,t_m) = 0$.
  - Recover $V_n(t_1)$ recursively from $V_n(t_M)$, $V_n(t_{M-1}), \cdots, V_n(t_2)$.
  - Use the COS formula for $v(x,t_0)$.

V_k-Coefficients

- Once we have x_m^*, we split the integral, which defines $V_k(t_m)$:

$$V_k(t_m) = \begin{cases} C_k(a, x_m^*, t_m) + G_k(x_m^*, b), & \text{for a call,} \\ G_k(a, x_m^*) + C_k(x_m^*, b, t_m), & \text{for a put,} \end{cases}$$

for $m = M-1, M-2, \cdots, 1$. whereby

$$G_k(x_1, x_2) := \frac{2}{b-a} \int_{x_1}^{x_2} g(x, t_m) \cos\left(k\pi \frac{x-a}{b-a}\right) dx.$$

and

$$C_k(x_1, x_2, t_m) := \frac{2}{b-a} \int_{x_1}^{x_2} \hat{c}(x, t_m) \cos\left(k\pi \frac{x-a}{b-a}\right) dx.$$

Theorem

The $G_k(x_1, x_2)$ are known analytically and the $C_k(x_1, x_2, t_m)$ can be computed in $O(N \log_2(N))$ operations with the Fast Fourier Transform.

Bermudan Details

- Formula for the coefficients $C_k(x_1, x_2, t_m)$:

$$C_k(x_1, x_2, t_m) = e^{-r\Delta t}\mathrm{Re}\left\{\sum_{j=0}^{\prime N-1} \varphi_{levy}\left(\frac{j\pi}{b-a}\right) V_j(t_{m+1}) \cdot M_{k,j}(x_1, x_2)\right\},$$

where the coefficients $M_{k,j}(x_1, x_2)$ are given by

$$M_{k,j}(x_1, x_2) := \frac{2}{b-a}\int_{x_1}^{x_2} e^{ij\pi\frac{x-a}{b-a}} \cos\left(k\pi\frac{x-a}{b-a}\right) dx,$$

- With fundamental calculus, we can rewrite $M_{k,j}$ as

$$M_{k,j}(x_1, x_2) = -\frac{i}{\pi}\left(M^c_{k,j}(x_1, x_2) + M^s_{k,j}(x_1, x_2)\right),$$

CWI Centrum Wiskunde & Informatica

Hankel and Toeplitz

- Matrices $M_c = \{M^c_{k,j}(x_1, x_2)\}_{k,j=0}^{N-1}$ and $M_s = \{M^s_{k,j}(x_1, x_2)\}_{k,j=0}^{N-1}$ have special structure for which the FFT can be employed: M_c is a Hankel matrix,

$$M_c = \begin{bmatrix} m_0 & m_1 & m_2 & \cdots & m_{N-1} \\ m_1 & m_2 & \cdots & \cdots & m_N \\ \vdots & & & & \vdots \\ m_{N-2} & m_{N-1} & \cdots & & m_{2N-3} \\ m_{N-1} & \cdots & & m_{2N-3} & m_{2N-2} \end{bmatrix}_{N \times N}$$

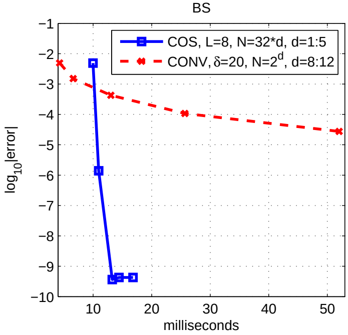
and M_s is a Toeplitz matrix,

$$M_s = \begin{bmatrix} m_0 & m_1 & \cdots & m_{N-2} & m_{N-1} \\ m_{-1} & m_0 & m_1 & \cdots & m_{N-2} \\ \vdots & & \ddots & & \vdots \\ m_{2-N} & \cdots & m_{-1} & m_0 & m_1 \\ m_{1-N} & m_{2-N} & \cdots & m_{-1} & m_0 \end{bmatrix}_{N \times N}$$

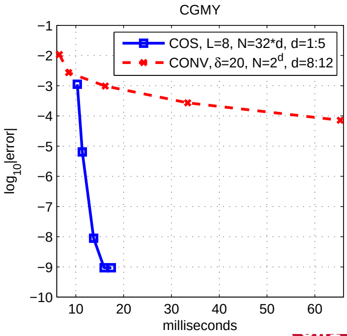
Bermudan puts with 10 early-exercise dates

Table: Test parameters for pricing Bermudan options

Test No.	Model	S_0	K	T	r	σ	Other Parameters
2	BS	100	110	1	0.1	0.2	–
3	CGMY	100	80	1	0.1	0	$C = 1, G = 5, M = 5, Y = 1.5$



(a) BS



(b) CGMY with $Y = 1.5$

American Options

- The option value must be greater than, or equal to the payoff, the Black-Scholes equation is replaced by an inequality, the option value must be a continuous function of S, the option delta (its slope) must be continuous.
- The problem for an American call option contract reads:

$$
\begin{aligned}
\frac{\partial V}{\partial t} &+ \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-\delta) S \frac{\partial V}{\partial S} - rV \leq 0 \\
V(S,t) &\geq \max(S-K,0) \\
V(S,T) &= \max(S-K,0) \\
\frac{\partial V}{\partial S} & \quad \text{is continuous}
\end{aligned}
$$

- Variational Inequality or Linear Complementarity Problem, with Projected Gauss-Seidel etc.

Pricing Discrete Barrier Options

- The price of an M-times monitored up-and-out option satisfies

$$\begin{cases} c(x, t_{m-1}) & = & e^{-r(t_m - t_{m-1})} \int_{\mathbb{R}} v(x, t_m) f(y|x) dy \\ v(x, t_{m-1}) & = & \begin{cases} e^{-r(T - t_{m-1})} Rb, & x \geq h \\ c(x, t_{m-1}), & x < h \end{cases} \end{cases}$$

  where $h = \ln(H/K)$, and $v(x, t_0) = e^{-r(t_m - t_{m-1})} \int_{\mathbb{R}} v(x, t_1) f(y|x) dy$.

- The technique:
  - Recover $V_n(t_1)$ recursively, from $V_n(t_M)$, $V_n(t_{M-1}), \cdots, V_n(t_2)$ in $O((M-1)N \log_2(N))$ operations.
  - Split the integration range at the barrier level (no Newton required)
  - Insert $V_n(t_1)$ in the COS formula to get $v(x, t_0)$, in $O(N)$ operations.

CWI Centrum Wiskunde & Informatica

Monthly-monitored Barrier Options

Table: Test parameters for pricing barrier options

Test No.	Model	S_0	K	T	r	q	Other Parameters
1	NIG	100	100	1	0.05	0.02	$\alpha = 15, \beta = -5, \delta = 0.5$

Option Type	Ref. Val.	N N	time (milli-sec.)	error
DOP	2.139931117	2^7	3.7	1.28e-3
		2^8	5.4	4.65e-5
		2^9	8.4	1.39e-7
		2^{10}	14.7	1.38e-12
DOC	8.983106036	2^7	3.7	1.09e-3
		2^8	5.3	3.99e-5
		2^9	8.3	9.47e-8
		2^{10}	14.8	5.61e-13

Credit Default Swaps (with W. Schoutens, H. Jönsson)

- Credit default swaps (CDSs), the basic building block of the credit risk market, offer investors the opportunity to either buy or sell default protection on a reference entity.
- The protection buyer pays a premium periodically for the possibility to get compensation if there is a credit event on the reference entity until maturity or the default time, which ever is first.
- If there is a credit event the protection seller covers the losses by returning the par value. The premium payments are based on the CDS spread.

Centrum Wiskunde & Informatica

CDS and COS

- CDS spreads are based on a series of default/survival probabilities, that can be efficiently recovered using the COS method. It is also very flexible w.r.t. the underlying process as long as it is Lévy.
- The flexibility and the efficiency of the method are demonstrated via a calibration study of the iTraxx Series 7 and Series 8 quotes.

Lévy Default Model

- Definition of default: For a given recovery rate, R, default occurs the first time the firm's value is below the "reference value" RV_0.
- As a result, the survival probability in the time period $(0, t]$ is nothing but the price of a digital down-and-out barrier option without discounting.

$$\begin{aligned} \mathrm{P}_{surv}(t) &= \mathrm{P}_{\mathbb{Q}}\left(X_s > \ln R, \text{for all } 0 \leq s \leq t\right) \\ &= \mathrm{P}_{\mathbb{Q}}\left(\min_{0\leq s\leq t} X_s > \ln R\right) \\ &= \mathbb{E}_{\mathbb{Q}}\left[\mathbf{1}\left(\min_{0\leq s\leq t} X_s > \ln R\right)\right] \end{aligned}$$

CWI Centrum Wiskunde & Informatica

Survival Probability

- Assume there are only a finite number of observing dates.

$$\mathrm{P}_{surv}(\tau) = \mathbb{E}_{\mathbb{Q}}\left[\mathbf{1}\Big(X_{\tau_1} \in [\ln R, \infty)\Big) \cdot \mathbf{1}\Big(X_{\tau_2} \in [\ln R, \infty)\Big) \cdots \mathbf{1}\Big(X_{\tau_M} \in [\ln R, \infty)\Big)\right.$$

where $\tau_k = k\Delta\tau$ and $\Delta\tau := \tau/M$.

- The survival probability then has the following recursive expression:

$$\begin{cases} \mathrm{P}_{surv}(\tau) & := & p(x=0,\tau_0) \\ p(x,\tau_m) & := & \int_{\ln R}^{\infty} f_{X_{\tau_{m+1}}|X_{\tau_m}}(y|x)p(y,\tau_{m+1})\, dy, \quad m = M-1,\cdots,2,1,0, \\ p(x,\tau_M) & := & 1, x > ln(R);\; p(x,\tau_M) := 0, x \leq ln(R) \end{cases}$$

$f_{X_{\tau_{m+1}}|X_{\tau_m}}(\cdot|\cdot)$ denotes the conditional probability density of $X_{\tau_{m+1}}$ given X_{τ_m}.

CWI Centrum Wiskunde & Informatica

The Fair Spread of a Credit Default Swap

- The *fair spread*, C, of a CDS at the initialization date is the spread that equalizes the present value of the premium leg and the present value of the protection leg, i.e.

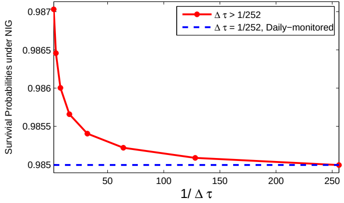
$$C = \frac{(1-R)\left(\int_0^T \exp(-r(s)s)\mathrm{d}\mathrm{P}_{def}(s)\right)}{\int_0^T \exp(-r(s)s)\mathrm{P}_{surv}(s)\mathrm{d}s},$$

- It is actually based on a series of survival probabilities on different time intervals:

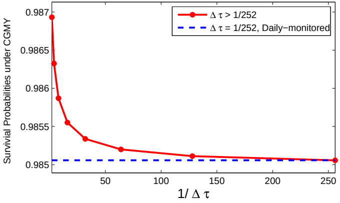
$$C = \frac{(1-R)\sum_{j=0}^{J}\frac{1}{2}\left[\exp(-r_j t_j)+\exp(-r_{j+1}t_{j+1})\right]\left[\mathrm{P}_{surv}(t_j)-\mathrm{P}_{surv}(t_{j+1})\right]}{\sum_{j=0}^{J}\frac{1}{2}\left[\exp(-r_j t_j)\mathrm{P}_{surv}(t_j)+\exp(-r_{j+1}t_{j+1})\mathrm{P}_{surv}(t_{j+1})\right]\Delta t}+\epsilon,$$

Convergence of Survival Probabilities

- Ideally, the survival probabilities should be monitored daily, i.e. $\Delta\tau = 1/252$. That is, $M = 252T$, which is a bit too much for $T = 5, 7, 10$ years.
- For Black-Scholes' model, there exist rigorous proof of the convergence of discrete barrier options to otherwise identical continuous options [Kou,2003].
- We observe similar convergence under NIG, CGMY:



(c)



(d)

- Convergence of the 1-year survival probability w.r.t. $\Delta\tau$.

Error Convergence

- The error convergence of the COS method is usually exponential in N

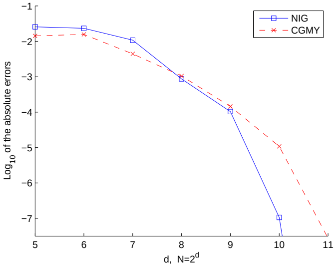


Figure: Convergence of $\mathrm{P}_{surv}(\Delta\tau = 1/48)$ w.r.t. N for NIG and CGMY

Calibration Setting

- The data sets: weekly quotes from iTraxx Series 7 (S7) and 8 (S8). After cleaning the data we were left with 119 firms from Series 7 and 123 firms from Series 8. Out of these firms 106 are common to both Series.
- The interest rates: EURIBOR swap rates.
- We have chosen to calibrate the models to CDSs spreads with maturities 1, 3, 5, 7, and 10 years.

The Objective Function

- To avoid the ill-posedness of the inverse problem we defined here, the objective function is set to

$$F_{obj} = \text{rmse} + \gamma \cdot ||\mathbf{X}_2 - \mathbf{X}_1||_2,$$

where

$$\text{rmse} = \sqrt{\sum_{\text{CDS}} \frac{(\text{market CDS spread} - \text{model CDS spread})^2}{\text{number of CDSs on each day}}},$$

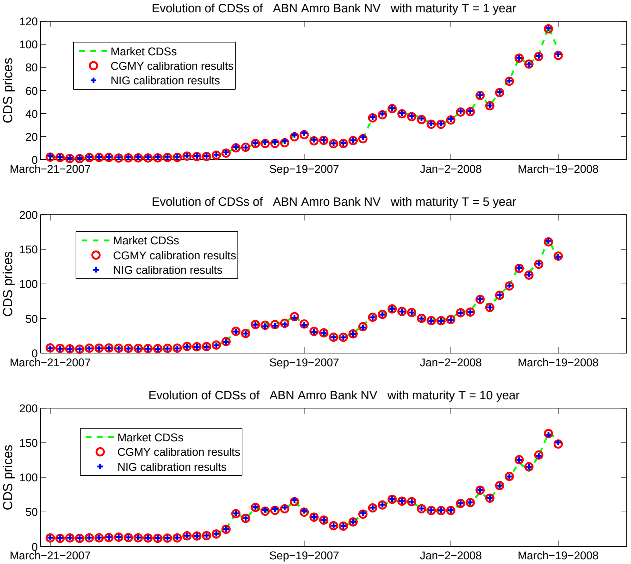
$|| \cdot ||_2$ denotes the L_2–norm operator, and $\mathbf{X}_2$ and $\mathbf{X}_1$ denote the parameter vectors of two neighbor data sets.

Good Fit to Market Data

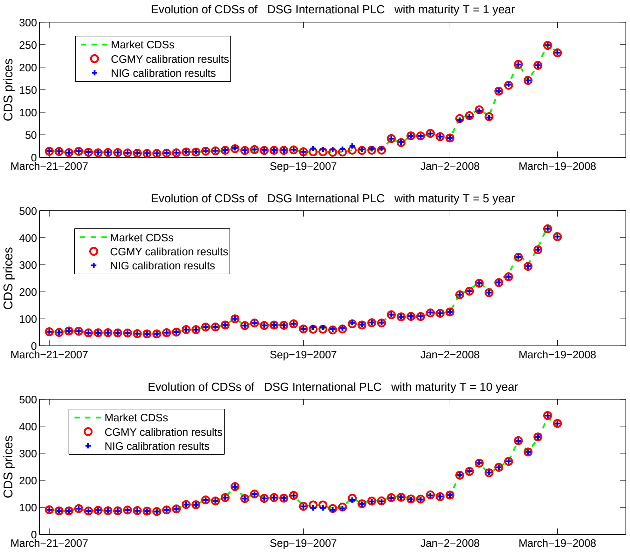
Table: Summary of calibration results of all 106 firms in both S7 and S8 of iTraxx quotes

RMSEs	NIG in S7	CGMY in S7	NIG in S8	CGMY in S8
Average (bp.)	0.89	0.79	1.65	1.54
Min. (bp.)	0.22	0.29	0.27	0.46
Max. (bp.)	2.29	1.97	4.27	3.52

A Typical Example



An Extreme Case



NIG Parameters for “ABN AMRO Bank”

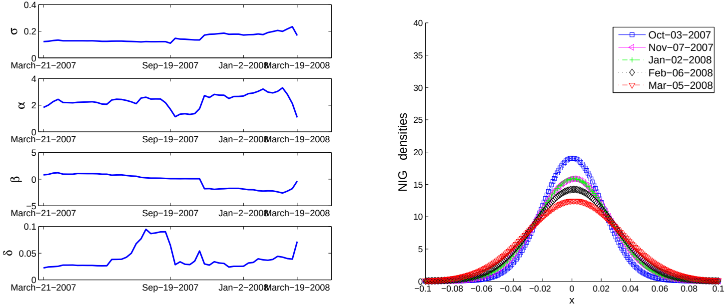
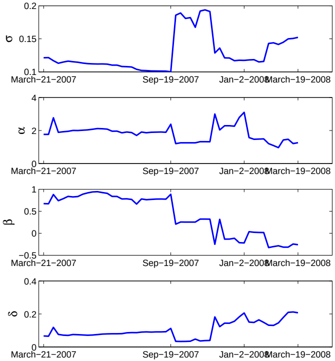


Figure: Evolution of the NIG parameters and densities of “ABN AMRO Bank”

Centrum Wiskunde & Informatica

NIG Parameters for “DSG International PLC”



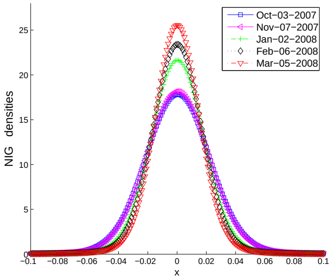


Figure: Evolution of the NIG parameters and densities of “DSG International PLC”

NIG vs. CGMY

Both Lévy processes gave good fits, but

- The NIG model returns more consistent measures from time to time and from one company to another.
- From a numerical point of view, the NIG model is also more preferable.
  - Small N (e.g. $N = 2^{10}$) can be applied.
  - The NIG model is much less sensitive to the initial guess of the optimum-searching procedure.
  - Fast convergence to the optimal parameters are observed (usually within 200 function evaluations). However, averagely 500 to 600 evaluations for the CGMY model are needed.

Conclusions

- The COS method is efficient for density recovery, for pricing European, Bermudan and discretely -monitored barrier options
- Convergence is exponential, usually with small N
- We relate the credit default spreads to a series survival/default probabilities with different maturities, and generalize the COS method to value these survival probabilities efficiently.
- Calibration results are also discussed. Both the NIG and the CGMY models give very good fits to the market CDSs, but the NIG model turns out to be more advantageous.

Black, F., and J. Cox (1976) Valuing corporate securities: some effects on bond indenture provisions, *J. Finance*, **31**, pp. 351–367.

Centrum Wiskunde & Informatica