Numerical Option pricing in Feller Lévy models

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Motivation

Theoretical background

Implementation

Numerical examples

Outlook
Markovian Projection of Semimartingales

(Bentata & Cont, C.R.A.S. 2009)

- \( Y = (Y_t)_{t \geq 0} \subseteq G \subseteq \mathbb{R}^q \) Ito Semimartingale in \( \mathbb{R}^q \),
- \( \exists! X = (X_t)_{t \geq 0} \subseteq G \subseteq \mathbb{R}^q \), Markovian Projection of \( Y \),

solution of martingale problem for generator

\[
(A_X(t, x; D)u)(t, x) := c^Y(t, x)u(x) + \gamma^Y(t, x)^\top \nabla_x u(t, x) + \frac{1}{2} \sigma^Y(t, x)\sigma^Y(t, x)^\top D^2 u(t, x) \\
+ \int_{y \in \mathbb{R}^d} \left( u(x + y) - u(x) - \frac{y \cdot \nabla_x u(x)}{1 + \|y\|^2} \right) N^Y(t, x; dy)
\]

- Numerical approximation of (weak!) solution \( u(x, t) \in \text{Domain}(A_X) \) in \( G \subseteq \mathbb{R}^q \) of fwd Kolmogoroff PIDE

\[
u_t + (A_X(t, x; D)u)(x) = f \in [0, T] \times G, \quad u|_{t=0} = u_0
\]
**Feller-processes I**

**Definition**
Assume $q = 1$ (e.g. $Y$ index of a $d$-dimensional disc. Semimart. Market). Then $X$ is a strong $\mathbb{R}$-valued Markov process and

$$(T_t g)(x) = \mathbb{E}[g(X_t) | X_0 = x].$$

$X$ is called Feller iff

1. $T_t : C_0(\mathbb{R}) \rightarrow C_0(\mathbb{R})$
2. $\lim_{t \rightarrow 0^+} \| u - T_t u \|_{L^\infty(\mathbb{R})} = 0$ for all $u \in C_0(\mathbb{R})$.

**Theorem**

*If* $u \in D(A_X)$ *and* $\sup_{x \in \mathbb{R}} u(x) = u(x_0) > 0$, *then* $(A_X u)(x_0) \leq 0$. 
Feller-processes II

Theorem

Let $A_X$ be the generator of a Feller-process with $C_0^\infty(\mathbb{R}) \subset D(A_X)$, then $A|_{C_0^\infty(\mathbb{R})}$ is a pseudodifferential operator (PDO):

$$(A_X u)(x) = -a(x, D)u(x) = -\left(2\pi\right)^{-\frac{1}{2}} \int_{\mathbb{R}} a(x, \xi) \hat{u}(\xi) e^{ix\xi} \, d\xi, \quad u \in C_0^\infty(\mathbb{R})$$

with symbol $a(x, \xi) : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ which is measurable and locally bounded in $(x, \xi)$ and which admits the Lévy-Khintchine representation.
Feller-processes III

\[ a(x, \xi) = c(x) - i\gamma(x)\xi + \frac{1}{2}(\sigma(x))^2\xi^2 \]
\[ + \int_{\mathbb{R}} \left( 1 - e^{iy\xi} + \frac{i y \xi}{1 + y^2} \right) N(x, dy), \]

where \( \sup_{x \in \mathbb{R}} \int_{\mathbb{R}} \min(1, y^2)N(x, dy) < \infty. \)

Examples:

1. Brownian motion (local vol) \( a(x, \xi) = \frac{1}{2}\sigma(x)^2\xi^2 \)
2. Lévy process \( a(x, \xi) = \\
c(x) - i\gamma(x)\xi + \frac{1}{2}(\sigma(x))^2\xi^2 + \int_{\mathbb{R}} \left( 1 - e^{iy\xi} + \frac{i y \xi}{1 + y^2} \right) \nu(dy) \)

Martingale problem: which \( a(x, \xi) \) define PDOs \( A(x, D) \) that generate Feller processes?
Feller-processes IV

Definition
(Symbolclass $S^{m(x)}_{\rho,\delta}(\mathbb{R})$)
Let $0 \leq \delta \leq \rho \leq 1$ and $m(x) \in C^\infty(\mathbb{R})$.
A symbol $a(x, \xi)$ belongs to $S^{m(x)}_{\rho,\delta}(\mathbb{R})$ iff

1. $a(x, \xi) \in C^\infty(\mathbb{R} \times \mathbb{R})$,
2. $m(x) = s + \tilde{m}(x)$ with $\tilde{m}(x) \in S(\mathbb{R})$, $s \in \mathbb{R}$,
3. for $\alpha, \beta \in \mathbb{N}_0$ there are constants $c_{\alpha,\beta}$ such that

$$\forall x, \xi \in \mathbb{R} : \left| D^\beta_x D^\alpha_\xi a(x, \xi) \right| \leq c_{\alpha,\beta} \langle \xi \rangle^{m(x)-\rho\alpha+\delta\beta},$$

where $\langle \xi \rangle := (1 + \xi^2)^{\frac{1}{2}}$, $\xi \in \mathbb{R}$.

The corresponding set of PDOs is denoted $\Psi^{m(x)}_{\rho,\delta}(\mathbb{R})$. 
Feller-processes V

Theorem
(Komatsu, Strook, Jacod 1976, Hoh 1998)
For every symbol $a(x, \xi) \in S^{m(x)}_{\rho, \delta}$ there exists a unique Feller process $X$ with generator $A_X$.

Definition
The PDO $\Lambda^{m(x)}$ with symbol $a(x, \xi) = \langle \xi \rangle^{m(x)} \in S^{m(x)}_{1, \delta}$, $\delta \in (0, 1)$ is called (variable order) Riesz potential.

Corollary
$(\Lambda^{m(x)})^\top \in \Psi^{m(x)}_{1, \delta}$ and $(\Lambda^{m(x)})^\top (\Lambda^{m(x)}) \in \Psi^{2m(x)}_{1, \delta}$. 
Alternative characterization of PDOs

A PDO in distributional sense can be written as:

\[ Au(x) = \int_{\mathbb{R}} K_A(x, y) u(y) \, dy, \]

\[ K_A(x, y) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i(x-y)\xi} a_A(x, \xi, y) \, d\xi, \]

where \( K_A(x, y) \) is an oscillatory integral i.e.

\[ K_A(x, y) = \lim_{\epsilon \to 0} \frac{1}{2\pi} \int_{\mathbb{R}} e^{i(x-y)\xi} a_A^{\epsilon}(x, \xi, y) \, d\xi, \]

\[ a_A^{\epsilon}(x, \xi, y) = a_A(x, \xi, y) \mu(\epsilon y, \epsilon \xi), \quad \mu \in C_0^\infty(\mathbb{R} \times \mathbb{R}), \mu(0, 0) = 1. \]

Kikuchi & Negoro 1997, Bass 2002:

\[ a((\Lambda m(x))^\top (\Lambda m(x)))(x, \xi, y) = \langle \xi \rangle^{m(x)+m(y)}. \]
Sobolev spaces of variable order I
In what follows we always assume \( m(x) \in (0, 1) \).

**Definition**
The Sobolev space of variable order is
\[
H^{m(x)}(\mathbb{R}) := \{ u \in L_2(\mathbb{R}) | \| u \|_{H^{m(x)}(\mathbb{R})} < \infty \}
\]
where
\[
\| u \|_{H^{m(x)}(\mathbb{R})}^2 := \| \Lambda^{m(x)} u \|_{L_2(\mathbb{R})}^2 + \| u \|_{L_2(\mathbb{R})}^2.
\]

We have the following continuous embedding result for \( \overline{m} = \sup_{x \in \mathbb{R}} m(x), \underline{m} = \sup_{x \in \mathbb{R}} m(x) \) due to Leopold ’91:

\[
H^{\overline{m}}(\mathbb{R}) \hookrightarrow H^{m(x)}(\mathbb{R}) \hookrightarrow H^{\underline{m}}(\mathbb{R}).
\]

On a bounded domain \( I \) we define the space

\[
\tilde{H}^{m(x)}(I) = \{ u \vert_I | u \in H^{m(x)}(\mathbb{R}), \ u \rvert_{\mathbb{R}\setminus I} = 0 \}.
\]
Sobolev spaces of variable order II

The norm on $\widetilde{H}^{m(x)}(I)$ is given as

$$\|u\|_{\widetilde{H}^{m(x)}(I)} = \|\tilde{u}\|_{H^{m(x)}(\mathbb{R})},$$

where $\tilde{u}$ denotes the zero extension of $u$ to $\mathbb{R}$. Intrinsically we could also write $\|\cdot\|_{\widetilde{H}^{m(x)}(I)}$ using the Sobolev-Slobodeckij norm i.e.

$$\|u\|_{\widetilde{H}^{m(x)}(I)}^2 := \|u\|^2_{L^2(I)} + |u|^2_{H^{m(x)}(I)},$$

$$|u|_{\widetilde{H}^{m(x)}(I)}^2 := \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|\tilde{u}(x) - \tilde{u}(y)|^2}{|x - y|^{1+m(x)+m(y)}} \, dx \, dy.$$
Wavelets I

Aim: Prove a norm equivalence on $\tilde{H}^m(x)(I)$ and obtain a preconditioner for the wavelet matrix of $N(x, dz)$. We require the following properties of the wavelets:

1. Biorthogonality i.e. $\psi_{l,k}, \tilde{\psi}_{l',k'}$ satisfy

$$\langle \psi_{l,k}, \tilde{\psi}_{l',k'} \rangle = \delta_{l,l'}\delta_{k,k'}.$$  

2. Local support:

$$\text{diam supp}\psi_{l,k} \leq C2^{-l}, \quad \text{diam supp}\tilde{\psi}_{l,k} \leq C2^{-l}.$$  

3. Conformity:

$$\mathcal{W}^l \subset \tilde{H}^1(I), \quad \tilde{\mathcal{W}}^l \subset \tilde{H}^\delta(I) \quad \text{for some } \delta > 0, \ l \geq -1.$$  

4. Density: $\bigoplus_{l=-1}^\infty \mathcal{W}^l, \bigoplus_{l=-1}^\infty \tilde{\mathcal{W}}^l$ dense in $L_2(I)$.  


Wavelets II

Example: piecewise linear, biorthogonal wavelets

\[ V_3 \]

(\text{Nodal basis})

\[ W_0 \]

\[ W_1 \]

\[ W_2 \]

\[ W_3 \]
Vanishing moments:

For inner wavelets:
\[ \langle \psi_{l,k}, x^\alpha \rangle = 0 \quad \alpha = 0, \ldots, p^* + 1, l \geq 0. \]

For inner dual wavelets:
\[ \langle \tilde{\psi}_{l,k}, x^\alpha \rangle = 0 \quad \alpha = 0, \ldots, p + 1, l \geq 0. \]

For the boundary wavelets:
\[ \langle \tilde{\psi}_{l,k}, x^\alpha \rangle = 0 \quad \alpha = 1, \ldots, p + 1, l \geq 0. \]
Estimates for extended symbols

Theorem
For any $\delta \in (0, 1)$ the Schwartz kernel $K_{\Lambda m(x)}^\top(\Lambda m(x))$ satisfies the Calderón-Zygmund type estimate

$$\left| D_x^\alpha D_y^\beta K_{\Lambda m(x)}^\top(\Lambda m(x))(x, y) \right| \leq C_{\alpha, \beta, \delta} |x - y|^{-(1+m(x)+m(y)+(1-\delta)(\alpha+\beta))}$$

where $x \neq y$ and $|x - y|$ is small. For large values of $|x - y|$ the kernel decays faster than $|x - y|^{-N}$, for any $N \in \mathbb{N}$.

Proof:
- Littlewood Paley decomposition of unity
- Decomposition of the symbol
Norm Equivalences I

We consider the infinite matrix \((\lambda = (l, k), \lambda' = (l, k')):\)

\[
\mathbf{M} := \left( \langle \Lambda^m(x) \psi_{\lambda'}, \Lambda^m(x) \psi_{\lambda} \rangle \right)_{\lambda, \lambda' \in \mathcal{I}} = \left( \langle (\Lambda^m(x))^T \Lambda^m(x) \psi_{\lambda'}, \psi_{\lambda} \rangle \right)_{\lambda, \lambda' \in \mathcal{I}}.
\]

The following variables will be useful:

\[
\overline{m}_\lambda = \sup \{m(x) : x \in \Omega_\lambda\}, \quad m_\lambda = \inf \{m(x) : x \in \Omega_\lambda\},
\]

\[
\Omega_\lambda = \bigcup_{l' > l} \{ \text{supp}\psi_{\lambda'} : \text{supp}\psi_{\lambda} \cap \text{supp}\psi_{\lambda'} \neq \emptyset \}.
\]
Norm Equivalences II

Several cases have to be considered and the entries of $M$ have to be estimated in each case.

1. **Case**: $\psi_\lambda, \psi_{\lambda'}$ have $p^* + 1$ vanishing moments and disjoint support.

Taylor expansion of the Schwartz kernel in $(x, y)$ and the Caldéron-Zygmund estimate give:

$$\left| \left\langle \Lambda^m(x) \psi_\lambda', \Lambda^m(x) \psi_\lambda \right\rangle \right| \lesssim C_\delta 2^{-(l+l')(\frac{1}{2}+p+1)} \times \text{dist}(\text{supp}\psi_\lambda', \text{supp}\psi_\lambda)^{-(1+2m+2(1-\delta)(p+1))}.$$
2. Case: Diagonal entries. Using the continuous embedding theorem for Sobolev spaces of variable order it can be shown:

\[ 2^{2lm} \lesssim \left| \Lambda^m(x) \psi_{l,k}, \Lambda^m(x) \psi_{l,k} \right| \lesssim 2^{lm}. \]

3. Case: \( \text{supp} \psi_\lambda \cap \text{supp} \psi'_\lambda \neq \emptyset \).

\[ \left| \langle \Lambda^m(x) \psi_{l,k}, \Lambda^m(x) \psi_{l',k'} \rangle \right| \lesssim 2^{lm + l'm'} 2^{-|l-l'|s}, \]

where \( s \) satisfies: \( 0 \leq m - s \) and \( m + s \leq 1 \).
Norm Equivalences IV

Theorem

Let \( D^{-m(x)} := (2^{-\frac{1}{2m}} \delta_{\lambda, \lambda'}) \lambda, \lambda' \) and

\[
A := D^{-m(x)} M D^{-m(x)}.
\]

Then \( A \) is compressible i.e. there exists \( s > 0 \) s.t.

\[
|A_{\lambda, \lambda'}| \lesssim 2^{-|l-l'| \left( s + \frac{1}{2} \right)} \left( 1 + \text{dist}(\text{supp} \psi'_{\lambda'}, \text{supp} \psi_{\lambda}) \right)^{-1 - 2(d-m)(1-\delta)}.
\]
Norm Equivalences V

As compressible matrices have a bounded spectral norm and $D^{-m(x)} D m(x)$ also has a bounded spectral norm, we obtain the norm equivalence:

$$\|u\|_{\tilde{H} m(x)(I)} \sim u^\top D^{2m(x)} u.$$ 

**Open Pbm:** Estimate matrix entries without PDO theory?

Problems:

- Global definition of $m(x)$
- Calderon-Zygmund - type Estimates for kernel.
Implementation of the PIDE

- Implementation of FFT methods for the PDO not feasible, due to nonstationarity of $X$
- Alternative: solve PIDE in “$x$-space” $\mathbb{R}^q$
- Weak solutions: FEM
- Compression of Jump Measure: Wavelets
Assumptions

Let $N(x, dz) = k(x, z)dz$. Assume that the jump density $k(x, z)$ satisfies: there exist constants $\beta^- > 0$ and $\beta^+ > 1$, $0 \leq \delta \leq \rho \leq 1$ independent of $x$ s.t.

1. 

\[
    k(x, z) \leq C \begin{cases} 
    e^{-\beta^- |z|}, & z < -1 \\
    e^{-\beta^+ z}, & z > 1.
    \end{cases}
\]

2. 

\[
    \frac{1}{2} (k(x, z) + k(x, -z)) \geq C \frac{1}{|z|^{2m(x)}}, \quad 0 < |z| \leq 1.
\]

3. 

\[
    \left| D_x^\beta D_z^\alpha k(x, z) \right| \leq c\alpha!\beta! |z|^{-1-2m(x)-\alpha\rho-\beta\delta} \quad \forall \alpha, \beta \in \mathbb{N}_0, z \neq 0.
\]
Martingale condition

- Assume the risk-neutral dynamics of the underlying asset to be given by

\[ S_t = S_0 e^{rt + X}, \]

where \( X \) is a Feller process with characteristic triple \((\gamma(x), \sigma(x), k(x, z) \, dz)\) under a risk neutral measure \( Q \) such that \( e^X \) is a martingale with respect to the canonical filtration of \( X \).

- Under the stated assumptions the martingale condition for \( X \) is equivalent to:

\[ \frac{\sigma(x)^2}{2} + \gamma(x) + \int_{\mathbb{R}} (e^z - 1 - z) k(x, z) \, dz = 0 \quad \forall x \in \mathbb{R}. \]
Derivation of the PIDE I

Let $X$ be a pure jump process without drift. Then

$$a(x, \xi) = \int_\mathbb{R} (1 - e^{iz\xi} + iz\xi) k(x, z) dz.$$ 

We can derive for all $u(x) \in S(\mathbb{R})$:

$$\left( A_X u \right)(x) = -\frac{1}{2\pi} \int_\mathbb{R} e^{ix\xi} a(x, \xi) \hat{u}(\xi) \, d\xi$$

$$= \int_\mathbb{R} (u(x + z) - u(x) - z\partial_x u(x)) k(x, z) \, dz.$$ 

For sufficiently smooth $u$ this can be written as:

$$\left( A_X u \right)(x) = \int_\mathbb{R} u''(x + z) k^{(-2)}(x, z) \, dz,$$

where $k^{(-i)}$ is the $i$-th antiderivative w.r.t. $z$. 
Derivation of the PIDE II

The bilinear form for a test function $v \in C_0^\infty(\mathbb{R})$ reads:

$$b(u, v) = \int_{\mathbb{R}} (A_X u)(x)v(x) \, dx$$

$$= - \int_{\mathbb{R}} \int_{\mathbb{R}} u'(x + z)v'(x)k^{(-2)}(x, z) \, dz \, dx$$

$$- \int_{\mathbb{R}} \int_{\mathbb{R}} u'(x + z)v(x)k_x^{(-2)}(x, z) \, dz \, dx.$$
Numerical quadrature

Density $k(x, z)$ of $N(x, dz)$ satisfies conditions of Chernov, von Peterdorff & Schwab 2009

- Composite Gauss quadrature used to deal with singularity at $x = y$.
- Idea: Geometric Quadrature Node refinement towards singularity of $k(x, z)$.
- Exponential convergence of the tensorized (composite) Gauss quadrature
- Regularized, “singularity-free” expression of Dirichlet Form of $X$ (Reich, Schwab & Winter ’09).
Model problem I

We consider CGMY-type processes:

\[ k(x, z) = C \begin{cases} 
  e^{-\beta - z} z^{-1 - \alpha(x)}, & z > 0 \\
  e^{-\beta + |z|} |z|^{-1 - \alpha(x)}, & z < 0,
\end{cases} \]

\[ \alpha(x) = ke^{-x^2} + 0.5. \]

\[(a) \quad \alpha(x) = 1.75 \]

\[(b) \quad \alpha(x) = 1.25e^{-x^2} + 0.5 \]
Model problem II

\[ k(x, z) = C \begin{cases} 
  e^{-\beta^+ |z| |z|^{-1-\alpha(x)}}, & z < 0, \\
  e^{-\beta^- z^{z-1-\alpha(x)}}, & z > 0 
\end{cases} \]

\[ \alpha(x) = 0.5 + k \begin{cases} 
  0.4x, & 0.25 > x > 0 \\
  0.8x - 0.1, & 0.5 > x \geq 0.25 \\
  -0.4x + 0.5, & 0.75 > x \geq 0.5 \\
  -0.8x + 0.8, & 1 > x \geq 0.75 \\
  0.5, & \text{else} 
\end{cases} \]
Stiffness matrices

Stiffness matrices for Example I with $Y(x) = 1.25e^{-x^2} + 0.5$:

(a) $k^{(-2)}(x, z)$

(b) $k_x^{(-2)}(x, z)$

Figure: Stiffness matrices
Preconditioning

Figure: Condition numbers for different levels and choices of $k$. 

(a) Model problem I

(b) Model problem II
Compression scheme

Figure: Number of non-zero entries of the compressed/uncompressed stiffness matrix versus number of degrees of freedom corresponding to the Lévy kernel in model problem I and $k = 1.25$. 
**Option prices**

**Figure:** Option prices for several models for a European put option with $T = 1$ and $K = 100$. 
Outlook

- Early exercise contracts (American, Russian)
- Optimal Control Problems
- Quadratic Hedging
- Analysis of model risk via hierarchical models, i.e. Local Vol, Additive Lévy, Local Lévy $\subseteq$ Feller-Lévy,
- Multidimensional models
- Preconditioning methods for $\alpha(x) \approx 2$
- Computable model sensitivity indicators
- (piecewise) smooth time dependent coefficients
- Fast Calibration (P. Carr 2009)