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SAM

Numerical Option pricing in Feller Lévy models

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Motivation

Theoretical background

Implementation

Numerical examples

Outlook

Markovian Projection of Semimartingales

(Bentata & Cont, C.R.A.S. 2009)

- $Y = (Y_t)_{t\geq 0} \subseteq G \subseteq \mathbb{R}^q$ Ito Semimartingale in $\mathbb{R}^q$,
- $\exists! X = (X_t)_{t\geq 0} \subset G \subseteq \mathbb{R}^q$, *Markovian Projection of Y*, solution of *martingale problem* for *generator*

$$(A_X(t,x;D)u)(t,x) :=$$
$$c^Y(t,x)u(x) + \gamma^Y(t,x)^\top \nabla_x u(t,x) + \frac{1}{2}\sigma^Y(t,x)\sigma^Y(t,x)^\top D^2 u(t,x)$$
$$+ \int_{y\in\mathbb{R}^d} \left(u(x+y) - u(x) - \frac{y\cdot\nabla_x u(x)}{1+||y||^2} \right) N^Y(t,x;dy)$$

- Numerical approximation of (weak!) solution $u(x,t) \in \mathrm{Domain}(A_X)$ in $G \subseteq \mathbb{R}^q$ of fwd Kolmogoroff PIDE

$$u_t + (A_X(t,x;D)u)(x) = f \in [0,T]\times G, \quad u|_{t=0} = u_0$$

Feller-processes I

Definition

Assume $q = 1$ (e.g. Y index of a d-dimensional disc. Semimart. Market). Then X is a strong $\mathbb{R}$-valued Markov process and

$$(T_t g)(x) = \mathbb{E}[g(X_t)|X_0 = x].$$

X is called Feller iff

1. $T_t : C_0(\mathbb{R}) \to C_0(\mathbb{R})$
2. $\lim_{t \to 0^+} \|u - T_t u\|_{L^\infty(\mathbb{R})} = 0$ for all $u \in C_0(\mathbb{R})$.

Theorem

if $u \in D(A_X)$ *and* $\sup_{x \in \mathbb{R}} u(x) = u(x_0) > 0$, *then* $(A_X u)(x_0) \leq 0$.

Feller-processes II

Theorem

Let A_X be the generator of a Feller-process with $C_0^\infty(\mathbb{R}) \subset D(A_X)$, then $A|_{C_0^\infty(\mathbb{R})}$ is a pseudodifferential operator (PDO):

$$\begin{aligned}(A_X u)(x) &= -a(x, D)u(x) \\ &= -(2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} a(x,\xi)\hat{u}(\xi)e^{ix\xi}\, d\xi, u \in C_0^\infty(\mathbb{R})\end{aligned}$$

with symbol $a(x,\xi) : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ which is measurable and locally bounded in (x, ξ) and which admits the Lévy-Khintchine representation.

Feller-processes III

$$a(x,\xi) = c(x) - i\gamma(x)\xi + \frac{1}{2}(\sigma(x))^2\xi^2 + \int_{\mathbb{R}}\left(1 - e^{iy\xi} + \frac{iy\xi}{1+y^2}\right) N(x,dy),$$

where $\sup_{x\in\mathbb{R}} \int_{\mathbb{R}} \min(1,y^2) N(x,dy) < \infty$.

Examples:

1. Brownian motion (local vol) $a(x,\xi) = \frac{1}{2}\sigma(x)^2\xi^2$
2. Lévy process $a(x,\xi) =$ $c(x) - i\gamma(x)\xi + \frac{1}{2}(\sigma(x))^2\xi^2 + \int_{\mathbb{R}}\left(1 - e^{iy\xi} + \frac{iy\xi}{1+y^2}\right)\nu(dy)$

Martingale problem: which $a(x,\xi)$ define PDOs $A(x,D)$ that generate Feller processes?

Feller-processes IV

Definition

(Symbolclass $S^{m(x)}_{\rho,\delta}(\mathbb{R})$)

Let $0 \le \delta \le \rho \le 1$ and $m(x) \in C^\infty(\mathbb{R})$.

A symbol $a(x,\xi)$ belongs to $S^{m(x)}_{\rho,\delta}(\mathbb{R})$ iff

1. $a(x,\xi) \in C^\infty(\mathbb{R} \times \mathbb{R})$,
2. $m(x) = s + \widetilde{m}(x)$ with $\widetilde{m}(x) \in S(\mathbb{R})$, $s \in \mathbb{R}$,
3. for $\alpha, \beta \in \mathbb{N}_0$ there are constants $c_{\alpha,\beta}$ such that

$$\forall x, \xi \in \mathbb{R}: \quad \left| D_x^\beta D_\xi^\alpha a(x,\xi) \right| \le c_{\alpha,\beta} \langle \xi \rangle^{m(x)-\rho\alpha+\delta\beta},$$

where $\langle \xi \rangle := (1+\xi^2)^{\frac{1}{2}}$, $\xi \in \mathbb{R}$.

The corresponding set of PDOs is denoted $\Psi^{m(x)}_{\rho,\delta}(\mathbb{R})$.

Feller-processes V

Theorem

(Komatsu, Strook, Jacod 1976, Hoh 1998)

For every symbol $a(x,\xi) \in S^{m(x)}_{\rho,\delta}$ *there exists a unique Feller process* X *with generator* A_X*.*

Domain of A_X? Answer: Sobolev spaces of variable order.

Definition

The PDO $\Lambda^{m(x)}$ with symbol $a(x,\xi) = \langle\xi\rangle^{m(x)} \in S^{m(x)}_{1,\delta}$, $\delta \in (0,1)$ is called *(variable order) Riesz potential*.

Corollary

$(\Lambda^{m(x)})^\top \in \Psi^{m(x)}_{1,\delta}$ *and* $(\Lambda^{m(x)})^\top(\Lambda^{m(x)}) \in \Psi^{2m(x)}_{1,\delta}$.

Alternative characterization of PDOs

A PDO in distributional sense can be written as:

$$\begin{aligned} Au(x) &= \int_{\mathbb{R}} K_A(x,y)u(y)\,dy, \\ K_A(x,y) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{i(x-y)\xi} a_A(x,\xi,y)\,d\xi, \end{aligned}$$

where $K_A(x,y)$ is an oscillatory integral i.e.

$$\begin{aligned} K_A(x,y) &= \lim_{\epsilon\to 0}\frac{1}{2\pi}\int_{\mathbb{R}} e^{i(x-y)\xi} a_A^{\epsilon}(x,\xi,y)\,d\xi, \\ a_A^{\epsilon}(x,\xi,y) &= a_A(x,\xi,y)\mu(\epsilon y,\epsilon\xi), \quad \mu \in C_0^{\infty}(\mathbb{R}\times\mathbb{R}),\ \mu(0,0)=1. \end{aligned}$$

Kikuchi & Negoro 1997, Bass 2002:

$$a_{(\Lambda^{m(x)})^{\top}(\Lambda^{m(x)})}(x,\xi,y) = \langle\xi\rangle^{m(x)+m(y)}.$$

Sobolev spaces of variable order I

In what follows we always assume $m(x) \in (0,1)$.

Definition

The Sobolev space of variable order is
$H^{m(x)}(\mathbb{R}) := \{u \in L_2(\mathbb{R}) |\, \|u\|_{H^{m(x)}(\mathbb{R})} < \infty\}$ where

$$\|u\|^2_{H^{m(x)}(\mathbb{R})} := \left\|\Lambda^{m(x)} u\right\|^2_{L_2(\mathbb{R})} + \|u\|^2_{L_2(\mathbb{R})} .$$

We have the following continuous embedding result for $\overline{m} = \sup_{x\in\mathbb{R}} m(x)$, $\underline{m} = \sup_{x\in\mathbb{R}} m(x)$ due to Leopold '91:

$$H^{\overline{m}}(\mathbb{R}) \hookrightarrow H^{m(x)}(\mathbb{R}) \hookrightarrow H^{\underline{m}}(\mathbb{R}).$$

On a bounded domain I we define the space

$$\widetilde{H}^{m(x)}(I) \quad = \quad \{u|_I \,\big|\, u \in H^{m(x)}(\mathbb{R}), \quad u|_{\mathbb{R}\setminus\overline{I}} = 0\}.$$

Sobolev spaces of variable order II

The norm on $\widetilde{H}^{m(x)}(I)$ is given as

$$\|u\|_{\widetilde{H}^{m(x)}(I)} = \|\widetilde{u}\|_{H^{m(x)}(\mathbb{R})},$$

where $\widetilde{u}$ denotes the zero extension of u to $\mathbb{R}$.
Intrinsically we could also write $\|\cdot\|_{\widetilde{H}^{m(x)}(I)}$ using the Sobolev-Slobodeckij norm i.e.

$$\begin{aligned} \|u\|^2_{\widetilde{H}^{m(x)}(I)} &:= \|u\|^2_{L_2(I)} + |u|^2_{\widetilde{H}^{m(x)}(I)}, \\ |u|^2_{\widetilde{H}^{m(x)}(I)} &:= \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|\widetilde{u}(x)-\widetilde{u}(y)|^2}{|x-y|^{1+m(x)+m(y)}}\,dx\,dy. \end{aligned}$$

Wavelets I

Aim: Prove a norm equivalence on $\widetilde{H}^{m(x)}(I)$ and obtain a preconditioner for the wavelet matrix of $N(x,dz)$. We require the following properties of the wavelets:

1. Biorthogonality i.e. $\psi_{l,k}, \widetilde{\psi}_{l',k'}$ satisfy

$$\langle \psi_{l,k}, \widetilde{\psi}_{l',k'} \rangle = \delta_{l,l'}\delta_{k,k'}.$$

2. Local support:

$$\text{diam supp}\psi_{l,k} \leq C2^{-l}, \quad \text{diam supp}\widetilde{\psi}_{l,k} \leq C2^{-l}.$$

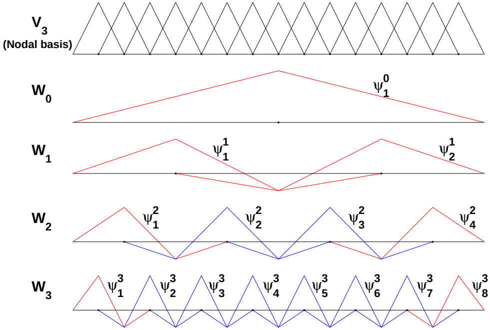
3. Conformity:

$$\mathcal{W}^l \subset \widetilde{H}^1(I), \widetilde{\mathcal{W}}^l \subset \widetilde{H}^\delta(I) \quad \text{for some } \delta > 0,\ l \geq -1.$$

4. Density: $\bigoplus_{l=-1}^{\infty} \mathcal{W}^l$, $\bigoplus_{l=-1}^{\infty} \widetilde{\mathcal{W}}^l$ dense in $L_2(I)$.

Wavelets II

Example: piecewise linear, biorthogonal wavelets



Wavelets III

Vanishing moments:

For inner wavelets:

$$\langle \psi_{l,k}, x^\alpha \rangle = 0 \quad \alpha = 0, \dots, p^* + 1, l \geq 0.$$

For inner dual wavelets:

$$\langle \widetilde{\psi}_{l,k}, x^\alpha \rangle = 0 \quad \alpha = 0, \dots, p + 1, l \geq 0.$$

For the boundary wavelets:

$$\langle \widetilde{\psi}_{l,k}, x^\alpha \rangle = 0 \quad \alpha = 1, \dots, p + 1, l \geq 0.$$

Estimates for extended symbols

Theorem

For any $\delta \in (0,1)$ *the Schwartz kernel* $K_{(\Lambda^{m(x)})^\top(\Lambda^{m(x)})}$ *satisfies the Calderón-Zygmund type estimate*

$$\left| D_x^\alpha D_y^\beta K_{(\Lambda^{m(x)})^\top(\Lambda^{m(x)})}(x,y) \right| \leq C_{\alpha,\beta,\delta} \, |x-y|^{-(1+m(x)+m(y)+(1-\delta)(\alpha+\beta))}$$

where $x \neq y$ *and* $|x-y|$ *is small. For large values of* $|x-y|$ *the kernel decays faster than* $|x-y|^{-N}$*, for any* $N \in \mathbb{N}$*.*

Proof:

- Littlewood Paley decomposition of unity
- Decomposition of the symbol

Norm Equivalences I

We consider the infinite matrix ($\lambda = (l,k)$, $\lambda' = (l,k')$):

$$\mathbf{M} := \Big(\langle \Lambda^{m(x)} \psi_{\lambda'}, \Lambda^{m(x)} \psi_\lambda \rangle \Big)_{\lambda,\lambda' \in \mathcal{I}} = \Big(\langle (\Lambda^{m(x)})^\top \Lambda^{m(x)} \psi_{\lambda'}, \psi_\lambda \rangle \Big)_{\lambda,\lambda' \in \mathcal{I}} .$$

The following variables will be useful:

$$\begin{aligned} \overline{m}_\lambda &= \sup\{m(x) : x \in \Omega_\lambda\}, \quad \underline{m}_\lambda = \inf\{m(x) : x \in \Omega_\lambda\}, \\ \Omega_\lambda &= \bigcup_{l' > l} \{\text{supp}\psi_{\lambda'} : \text{supp}\psi_\lambda \cap \text{supp}\psi_{\lambda'} \neq \emptyset\} . \end{aligned}$$

Norm Equivalences II

Several cases have to be considered and the entries of $\mathbf{M}$ have to be estimated in each case.

1. Case: $\psi_\lambda, \psi_{\lambda'}$ have $p^* + 1$ vanishing moments and disjoint support.

Taylor expansion of the Schwartz kernel in (x, y) and the Caldéron-Zygmund estimate give:

$$\left|\langle \Lambda^{m(x)}\psi_{\lambda'}, \Lambda^{m(x)}\psi_\lambda\rangle\right| \lesssim C_\delta 2^{-(l+l')(\frac{1}{2}+p+1)} \times \text{dist}(\text{supp}\psi'_\lambda, \text{supp}\psi_\lambda)^{-(1+2\overline{m}+2(1-\delta)(p+1))}.$$

Norm Equivalences III

2. Case: Diagonal entries. Using the continuous embedding theorem for Sobolev spaces of variable order it can be shown:

$$2^{2l\underline{m}_\lambda} \lesssim \left|\langle \Lambda^{m(x)}\psi_{l,k}, \Lambda^{m(x)}\psi_{l,k}\rangle\right| \lesssim 2^{2l\overline{m}_\lambda}.$$

3. Case: supp$\psi_\lambda \cap$ supp$\psi'_\lambda \neq \emptyset$.

$$\left|\langle \Lambda^{m(x)}\psi_{l,k}, \Lambda^{m(x)}\psi_{l',k'}\rangle\right| \lesssim 2^{l\overline{m}_\lambda + l'\overline{m}'_\lambda} 2^{-|l-l'|s},$$

where s satisfies: $0 \leq \underline{m} - s$ and $\overline{m} + s \leq 1$.

Norm Equivalences IV

Theorem

Let $\mathbf{D}^{-m(x)} := (2^{-l\overline{m}_\lambda}\delta_{\lambda,\lambda'})_{\lambda,\lambda'}$ *and*

$$\mathbf{A} := \mathbf{D}^{-m(x)}\mathbf{M}\mathbf{D}^{-m(x)}.$$

Then $\mathbf{A}$ *is compressible i.e. there exists* $s > 0$ *s.t.*

$$\left|A_{\lambda,\lambda'}\right| \lesssim 2^{-|l-l'|(s+\frac{1}{2})}(1 + \mathit{dist}(\mathit{supp}\psi'_\lambda, \mathit{supp}\psi_\lambda)^{-1-2(d-\overline{m})(1-\delta)}.$$

Norm Equivalences V

As compressible matrices have a bounded spectral norm and $\mathbf{D}^{-m(x)}\mathbf{D}^{m(x)}$ also has a bounded spectral norm, we obtain the norm equivalence:

$$\|u\|_{\widetilde{H}^{m(x)}(I)} \sim u^\top \mathbf{D}^{2m(x)} u.$$

Open Pbm: Estimate matrix entries without PDO theory?
Problems:

- Global definition of $m(x)$
- Calderon-Zygmund - type Estimates for kernel.

Implementation of the PIDE

- Implementation of FFT methods for the PDO not feasible, due to nonstationarity of X
- Alternative: solve PIDE in "x-space" $\mathbb{R}^q$
- Weak solutions: FEM
- Compression of Jump Measure: Wavelets

Assumptions

Let $N(x, dz) = k(x, z)dz$. Assume that the jump density $k(x, z)$ satisfies: there exist constants $\beta^- > 0$ and $\beta^+ > 1$, $0 \le \delta \le \rho \le 1$ independent of x s.t.

1.
$$k(x, z) \le C \begin{cases} e^{-\beta^-|z|}, & z < -1 \\ e^{-\beta^+ z}, & z > 1. \end{cases}$$

2.
$$\frac{1}{2}(k(x, z) + k(x, -z)) \ge C \frac{1}{|z|^{2m(x)}}, \quad 0 < |z| \le 1.$$

3.
$$\left| D_x^\beta D_z^\alpha k(x, z) \right| \le c\alpha!\beta! \, |z|^{-1-2m(x)-\alpha\rho-\beta\delta} \quad \forall \alpha, \beta \in \mathbb{N}_0, z \ne 0.$$

Martingale condition

- Assume the risk-neutral dynamics of the underlying asset to be given by

$$S_t = S_0 e^{rt+X},$$

where X is a Feller process with characteristic triple $(\gamma(x), \sigma(x), k(x,z)\mathrm{d}z)$ under a risk neutral measure $\mathbb{Q}$ such that e^X is a martingale with respect to the canonical filtration of X.

- Under the stated assumptions the martingale condition for X is equivalent to:

$$\frac{\sigma(x)^2}{2} + \gamma(x) + \int_{\mathbb{R}} (e^z - 1 - z) k(x,z)\,\mathrm{d}z = 0 \quad \forall x \in \mathbb{R}.$$

Derivation of the PIDE I

Let X be a pure jump process without drift. Then

$$a(x,\xi) = \int_{\mathbb{R}} (1 - e^{iz\xi} + iz\xi) k(x,z) dz.$$

We can derive for all $u(x) \in S(\mathbb{R})$:

$$\begin{aligned}(A_X u)(x) &= -\frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} a(x,\xi) \hat{u}(\xi)\, d\xi \\ &= \int_{\mathbb{R}} (u(x+z) - u(x) - z\partial_x u(x)) k(x,z)\, dz.\end{aligned}$$

For sufficiently smooth u this can be written as:

$$(A_X u)(x) = \int_{\mathbb{R}} u''(x+z) k^{(-2)}(x,z)\, dz,$$

where $k^{(-i)}$ is the i-th antiderivative w.r.t. z.

Derivation of the PIDE II

The bilinear form for a test function $v \in C_0^\infty(\mathbb{R})$ reads:

$$\begin{aligned} b(u,v) &= \int_{\mathbb{R}} (A_X u)(x) v(x)\, dx \\ &= -\int_{\mathbb{R}} \int_{\mathbb{R}} u'(x+z) v'(x) k^{(-2)}(x,z)\, dz\, dx \\ &\quad -\int_{\mathbb{R}} \int_{\mathbb{R}} u'(x+z) v(x) k_x^{(-2)}(x,z)\, dz\, dx. \end{aligned}$$

Numerical quadrature

Density $k(x,z)$ of $N(x,dz)$ satisfies conditions of Chernov, von Peterdorff & Schwab 2009

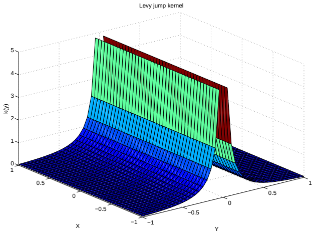
- Composite Gauss quadrature used to deal with singularity at $x = y$.
- Idea: Geometric Quadrature Node refinement towards singularity of $k(x,z)$.
- Exponential convergence of the tensorized (composite) Gauss quadrature
- Regularized, “singularity-free” expression of Dirichlet Form of X (Reich, Schwab & Winter ’09).

Model problem I

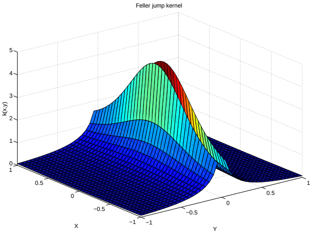
We consider CGMY-type processes:

$$k(x,z) = C \begin{cases} e^{-\beta^- z} z^{-1-\alpha(x)}, & z > 0 \\ e^{-\beta^+ |z|} \, |z|^{-1-\alpha(x)}, & z < 0, \end{cases}$$

$$\alpha(x) = k e^{-x^2} + 0.5.$$



(a) $\alpha(x) = 1.75$



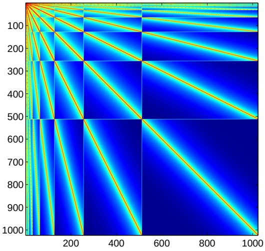
(b) $\alpha(x) = 1.25e^{-x^2} + 0.5$

Model problem II

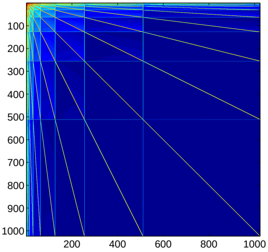
$$
\begin{aligned}
k(x,z) &= C \begin{cases} e^{-\beta^- z} z^{-1-\alpha(x)}, & z > 0 \\ e^{-\beta^+ |z|} \, |z|^{-1-\alpha(x)}, & z < 0, \end{cases} \\
\alpha(x) &= 0.5 + k \begin{cases} 0.4x, & 0.25 > x > 0 \\ 0.8x - 0.1, & 0.5 > x \geq 0.25 \\ -0.4x + 0.5, & 0.75 > x \geq 0.5 \\ -0.8x + 0.8, & 1 > x \geq 0.75 \\ 0.5, & \text{else} \end{cases} .
\end{aligned}
$$

Stiffness matrices

Stiffness matrices for Example I with $Y(x) = 1.25e^{-x^2} + 0.5$:



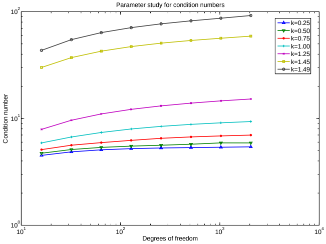
(a) $k^{(-2)}(x,z)$



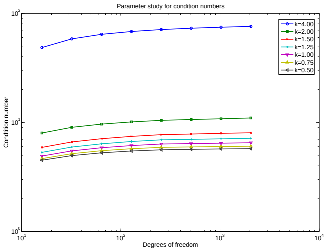
(b) $k_x^{(-2)}(x,z)$

Figure: Stiffness matrices

Preconditioning



(a) Model problem I



(b) Model problem II

Figure: Condition numbers for different levels and choices of k.

Compression scheme

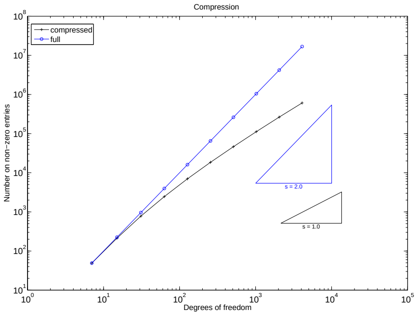


Figure: Number of non-zero entries of the compressed/uncompressed stiffness matrix versus number of degrees of freedom corresponding to the Lévy kernel in model problem I and $k = 1.25$.

Option prices

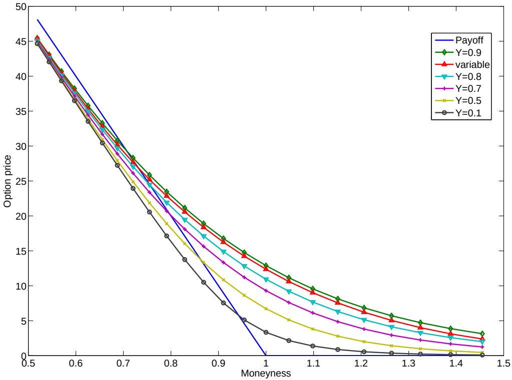


Figure: Option prices for several models for a European put option with $T = 1$ and $K = 100$.

Outlook

- Early exercise contracts (American, Russian)
- Optimal Control Problems
- Quadratic Hedging
- Analysis of model risk via hierarchical models, i.e. Local Vol, Additive Lévy , Local Lévy $\subseteq$ Feller-Lévy,
- Multidimensional models
- Preconditioning methods for $\alpha(x) \approx 2$
- Computable model sensitivity indicators
- (piecewise) smooth time dependent coefficients
- Fast Calibration (P. Carr 2009)