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New characterizations of the Owen and Banzhaf–Owen values using the intracoalitional balanced contributions property

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Abstract In this paper, several characterizations of the Owen and Banzhaf–Owen values are provided. All the characterizations make use of a property based on the principle of balanced contributions. This property is called the intracoalitional balanced contributions property and was defined by Calvo et al. [12].

Keywords Cooperative game · Shapley value · Banzhaf value · coalition structure · balanced contributions

1 Introduction

One of the main objectives in the Cooperative Game Theory field is the study of solutions (or values) for cooperative games with transferable utility (TU–games). These solutions stablish the payoff of each player in the corresponding TU–game. The Shapley value [28] and the Banzhaf value [11] are two of the best known concepts in this context.

A coalition structure is a partition of the set of players. There are many situations where the coalition structures make sense. It could be the case, for instance, of parliamentary coalitions formed by different political parties, alliances among countries in a negotiation process or unions of workers trying to improve a collective bargaining agreement.

Cooperative games with coalition structure (TU–games with coalition structure) were introduced by Aumann and Drèze [10]. They incorporate the given

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coalition structure to the classical notion of TU–game. As in the class of TU–games, it is interesting to find solutions (coalitional values) in order to obtain a suitable assignment for each player.

The coalition structure can be interpreted by the coalitional values in different ways. The coalitional value defined in Aumann and Drèze [10] assigns to each player his Shapley value in the subgame played by the union he belongs to. A different approach is taken into account by Owen [24] to define the so–called Owen value. In this case, the unions play a TU–game among themselves, called the quotient game, and after that the players in each union play an internal game. In the Owen value, the payoffs for the unions in the quotient game and the payoffs for the players inside the union are given by the Shapley value.

The Banzhaf–Owen value (also known as the modified Banzhaf value) was defined by Owen [25] following a similar procedure. This coalitional value only differs from the Owen value in the fact that the payoffs for the unions in the quotient game and the payoffs for the players within each union are computed by means of the Banzhaf value.

It is also possible to obtain other two coalitional values by alternating the Shapley and the Banzhaf values in both stages of the procedure. Thus, if we consider the Shapley value in the quotient game and the Banzhaf value applied to the TU–game played within the unions, we get the coalitional value defined in Amer et al. [9]. On the other hand, if the Banzhaf value is applied to the quotient game and the Shapley value to the internal game, the symmetric coalitional Banzhaf value [8] is obtained. Other coalitional values defined by means of this two–step procedure can be found in Casas–Méndez et al. [14], where the τ–value [29] is used in both stages; in Albizuri and Zarzuelo [3], where the coalitional semivalues are introduced by employing semivalues [15] in the two stages of the procedure; in Alonso–Meijide et al. [5], with the study of a subfamily of coalitional semivalues called the symmetric coalitional semivalues; or in Vidal–Puga [31], where the coalitional value is defined using the weighted Shapley value with weights given by the size of the unions in the quotient game and the Shapley value in the internal game.

This framework is focused on the study of two of these coalitional values: the Owen and the Banzhaf–Owen values. The first characterization of the Owen value can be found in [24], where it is also defined. However, there are many other characterizations of this value in the literature (see, for example, the papers by Hart and Kurz [19], Calvo et al. [12], Vázquez–Brage et al. [30], Hamiache [17] and [18], Albizuri [2], Khmelnitskaya and Yanovskaya [22], Casajus [13], Gómez–Rúa and Vidal–Puga [16], or Alonso–Meijide et al. [7]). As far as the Banzhaf–Owen value is concerned, although the first characterization of this value was given in Albizuri [1] for the class of simple games, this coalitional value was first characterized in the whole class of cooperative games with coalition structure in the paper by Amer et al. [9]. Later on, Alonso–Meijide et al. [4] and Alonso–Meijide et al. [7] proposed alternative characterizations of this coalitional value.

In this paper, both coalitional values are compared by means of appealing properties. Moreover, new characterizations of these two values are provided. All the characterizations make use of an interesting property, called intracoalitional balanced contributions and introduced in Calvo et al. [12]. According to this property, if we consider two players in the same coalition, the losses or gains for both agents when the other leaves the game are equal. It is based on a principle of balanced contributions, which is useful not only in the case of coalitional values but also in many other contexts.

The paper is organized as follows. In Section 2 we introduce notation and previous definitions required along the paper. The following two sections are devoted to study the relation between the property of intracoalitional balanced contributions and the Owen and Banzhaf–Owen values. So, in Section 3 we propose an expression in terms of the Shapley value for all the coalitional values satisfying intracoalitional balanced contributions. Finally, in Section 4 we provide the characterizations of the paper, where both values are characterized, trying to identify the similarities and differences of both coalitional values.

2 Notation and definitions

2.1 TU–games

A *cooperative game with transferable utility* (or *TU–game*) is a pair (N, v) defined by a finite set of players $N \subset \mathbb{N}$ (usually, $N = \{1, 2, \ldots, n\}$) and a function $v : 2^N \to \mathbb{R}$, that assigns to each coalition $S \subseteq N$ a real number $v(S)$, called the worth of S, and such that $v(\emptyset) = 0$. For any coalition $S \subseteq N$, we assume that $s = |S|$. In the sequel, $\mathscr{G}_N$ will denote the family of all TU–games on a given N and $\mathscr{G}$ will denote the family of all TU–games. Given $S \subseteq N$, we denote the restriction of a TU–game $(N, v) \in \mathscr{G}_N$ to S as (S, v).

A *simple game* is a TU–game (N, v) such that $v(S) \in \{0, 1\}$ for all $S \subseteq N$ and $v(T) \leq v(S)$ for all $T \subseteq S \subseteq N$. Given a simple game (N, v), a *winning coalition* is a coalition $S \subseteq N$ such that $v(S) = 1$. A winning coalition $S \subseteq N$ is a *minimal winning coalition* if $v(T) = 0$ for all $T \subsetneq S$.

A *weighted majority game* is a simple game (N, v) which depends on a quota $q > 0$ and a weight $w_i \geq 0$ for any player $i \in N$, since the worth of any coalition $S \subseteq N$ is given by

$$v(S) = \begin{cases} 1 & \text{if} \displaystyle\sum_{i \in S} w_i \geq q \\ 0 & \text{otherwise.} \end{cases}$$

Thus, it can be denoted by $[q; w_1, \ldots, w_n]$.

Let us fix a TU–game (N, v) and two players $i, j \in N$ with $i \neq j$. Let the set $\{i, j\}$ be considered as a new player $i^* \notin N$ (it means that the players i and j are amalgamated into one player i^*) and let us denote $N^{\{i,j\}} = (N \setminus \{i, j\}) \cup \{i^*\}$. The $\{i, j\}$*–amalgamated game* $\left(N^{\{i,j\}}, v^{\{i,j\}}\right) \in \mathscr{G}$ is defined for all $S \subseteq N^{\{i,j\}}$ by

$$v^{\{i,j\}}(S) = \begin{cases} v\left((S \setminus \{i^*\}) \cup \{i, j\}\right) & \text{if } i^* \in S \\ v(S) & \text{otherwise.} \end{cases}$$

A *value* is a map f that assigns to every TU–game $(N, v) \in \mathscr{G}$ a vector $f(N, v) = (f_i(N, v))_{i \in N}$, where each component $f_i(N, v)$ represents the payoff of i when he participates in the game.

The Shapley and the Banzhaf values are two widely–known tools in this context. In both cases, the payoff for each player can be computed as the weighted mean of the marginal contributions of the player. But, whereas in the Shapley value the weights are obtained assuming that all the orders of the players are equally likely, in the case of the Banzhaf value the weights are calculated by taking into account that the player is equally likely to join any coalition.

Definition 1 (Shapley [28]) The *Shapley value* is the value defined for all $(N,v) \in \mathscr{G}$ and all $i \in N$ by

$$Sh_i(N,v) = \sum_{S \subseteq N \setminus \{i\}} \frac{s!\,(n-s-1)!}{n!} \Big[v(S \cup \{i\}) - v(S) \Big].$$

Definition 2 (Banzhaf [11]) The *Banzhaf value* is the value defined for all $(N,v) \in \mathscr{G}$ and all $i \in N$ by

$$Bz_i(N,v) = \sum_{S \subseteq N \setminus \{i\}} \frac{1}{2^{n-1}} \Big[v(S \cup \{i\}) - v(S) \Big].$$

The Shapley value is one of the values that belongs to the family of weighted Shapley values. The values in this family are defined by means of a vector of weights $\omega \in \mathbb{R}^N_+$, where for each $i \in N$, ω_i is the weight of player i. If the weights of the players are equal, i.e., $\omega_i = \omega_j$ for all $i,j \in N$, $i \neq j$, then we get the Shapley value.

Definition 3 (Shapley [27] and Kalai and Samet [20]) The *weighted Shapley value* is the value defined for all $(N,v) \in \mathscr{G}$, all $\omega \in \mathbb{R}^N_+$ and all $i \in N$ by

$$Sh^{\omega}_i(N,v) = \sum_{S \subseteq N : i \in S} \frac{\omega_i}{\displaystyle\sum_{j \in S} \omega_j} \sum_{T \subseteq S} (-1)^{s-t} v(T).$$

2.2 TU–games with coalition structure

Let us consider a finite set of players N. A *coalition structure* over N is a partition of N, i.e., $C = \{C_1, \ldots, C_m\}$ is a coalition structure over N if it satisfies that $\bigcup_{h \in M} C_h = N$, where $M = \{1, \ldots, m\}$, and $C_h \cap C_r = \emptyset$ when $h \neq r$. There are two *trivial coalition structures*: $C^n = \{\{i\} : i \in N\}$, where each union is a singleton, and $C^N = \{N\}$, where the grand coalition forms.

Given $i \in N$, $\mathcal{C}(i)$ denotes the family of coalition structures over N where $\{i\}$ is a singleton union, that is, $C \in \mathcal{C}(i)$ if and only if $\{i\} \in C$.

Given $T \subseteq C_h$, $C_{|T}$ is the coalition structure where the union C_h is replaced by the subset T, i.e., $C_{|T} = (C \setminus \{C_h\}) \cup \{T\}$.

A *cooperative game with coalition structure* (or *TU–game with coalition structure*) is a triple (N,v,C) where (N,v) is a TU–game and C is a coalition structure over N. The set of all TU–games with coalition structure will be denoted by $\mathscr{CG}$, and by $\mathscr{CG}_N$ the subset where N is the player set.

Given $S \subseteq N$, such that $S = \bigcup_{h \in M} S_h$, with $\emptyset \neq S_h \subseteq C_h$ for all $h \in M$, we will denote the restriction of $(N,v,C) \in \mathscr{CG}_N$ to S as the TU–game with coalition structure (S,v,C_S), where $C_S = \{C_1, \ldots, C_m\}$.

If $(N,v,C) \in \mathscr{CG}$ and $C = \{C_1, \ldots, C_m\}$, the *quotient game* $\left(M, v^C\right)$ is the TU–game defined as $v^C(R) = v\left(\bigcup_{r \in R} C_r\right)$ for all $R \subseteq M$. It means that the quotient game is the game played by the unions, i.e., the TU–game induced by (N,v,C) by considering the elements of C as players. Notice that (M, v^C) coincides with (N, v) when $C = C^n$.

A *coalitional value* is a map g that assigns to every TU–game with coalition structure (N, v, C) a vector $g(N, v, C) = (g_i(N, v, C))_{i \in N} \in \mathbb{R}^N$, where $g_i(N, v, C)$ is the payoff for each player $i \in N$.

Given a value f on $\mathscr{G}$, a coalitional value g on $\mathscr{C}\mathscr{G}$ is a *coalitional f–value for singletons* when $g\left(N, v, C^n\right) = f(N, v)$. Thus, a coalitional value g is a *coalitional Shapley value for singletons* when $g\left(N, v, C^n\right) = Sh(N, v)$ and g is a *coalitional Banzhaf value for singletons* when $g\left(N, v, C^n\right) = Bz(N, v)$.

The Owen and Banzhaf–Owen values are two coalitional values which extend the Shapley and Banzhaf values to the context of cooperative games with coalition structure. Both coalitional values take into account that the players in the same union act together and, in this way, only contributions of each player to coalitions formed by full unions and agents in the union of the player are considered.

Definition 4 (Owen [24]) The *Owen value* is defined for all $(N, v, C) \in \mathscr{C}\mathscr{G}$ and all $i \in N$ by $Ow_i(N, v, C) =$

$$\sum_{R \subseteq M \setminus \{h\}} \sum_{T \subseteq C_h \setminus \{i\}} \frac{r!\,(m-r-1)!}{m!} \frac{t!\,(c_h-t-1)!}{c_h!} \Big[v\left(Q \cup T \cup \{i\}\right) - v(Q \cup T) \Big],$$

where $C_h \in C$ is the union such that $i \in C_h$, $m = |M|$, $c_h = |C_h|$, $t = |T|$, $r = |R|$ and $Q = \bigcup\limits_{r \in R} C_r$.

Another way to obtain the Owen value consists of computing the Shapley value twice, first applied to a quotient game and then applied to a TU–game inside the unions. According to this procedure, for all $(N, v, C) \in \mathscr{C}\mathscr{G}$, all $C_h \in C$ and all $i \in C_h$,

$$Ow_i(N, v, C) = Sh_i\left(C_h, \hat{v}^{Sh,C}\right) \tag{1}$$

and $\left(C_h, \hat{v}^{Sh,C}\right)$ is the TU–game such that $\hat{v}^{Sh,C}(T) = Sh_h\left(M, v^{C_{|T}}\right)$ for all $T \subseteq C_h$, $T \neq \emptyset$.

Since the Owen value satisfies that $Ow\left(N, v, C^n\right) = Sh(N, v)$, the Owen value is a *coalitional Shapley value for singletons.*

Definition 5 (Owen [25]) The *Banzhaf–Owen value* is defined for all $(N, v, C) \in \mathscr{C}\mathscr{G}$ and all $i \in N$ by

$$BzOw_i(N, v, C) = \sum_{R \subseteq M \setminus \{h\}} \sum_{T \subseteq C_h \setminus \{i\}} \frac{1}{2^{m-1}} \frac{1}{2^{c_h-1}} \Big[v\left(Q \cup T \cup \{i\}\right) - v(Q \cup T) \Big],$$

where $C_h \in C$ is the union such that $i \in C_h$, $m = |M|$, $c_h = |C_h|$ and $Q = \bigcup\limits_{r \in R} C_r$.

The Banzhaf–Owen value can be obtained according to a procedure in two stages. It is similar to the one used to compute the Owen value, just replacing the Shapley value with the Banzhaf value. Then, for all $(N, v, C) \in \mathscr{C}\mathscr{G}$, all $C_h \in C$ and all $i \in C_h$,

$$BzOw_i(N, v, C) = Bz_i\left(C_h, \hat{v}^{Bz,C}\right) \tag{2}$$

and $\left(C_h, \hat{v}^{Bz,C}\right)$ is the TU–game such that $\hat{v}^{Bz,C}(T) = Bz_h\left(M, v^{C_{|T}}\right)$ for all $T \subseteq C_h$, $T \neq \emptyset$.

The Banzhaf–Owen value is a *coalitional Banzhaf value for singletons* since $BzOw\left(N, v, C^n\right) = Bz(N, v)$.

3 The intracoalitional balanced contributions property

The property of intracoalitional balanced contributions was introduced by Calvo et al. [12] and states that, given two players in the same union, the amounts that both players gain or lose when the other leaves the coalitional game should be equal. This property is satisfied by the Owen and Banzhaf–Owen values, but also by many other coalitional values.

Intracoalitional balanced contributions **(IBC)**. For all $(N, v, C) \in \mathscr{CG}$ and all $i, j \in C_h \in C$, $i \neq j$,

$$g_i(N, v, C) - g_i\left(N \setminus \{j\}, v, C_{N\setminus\{j\}}\right) = g_j(N, v, C) - g_j\left(N \setminus \{i\}, v, C_{N\setminus\{i\}}\right).$$

In Calvo et al. [12], a characterization of the Owen value is provided by means of efficiency, coalitional balanced contributions and intracoalitional balanced contributions. The last two properties are based on the property of balanced contributions for TU–games. A value f satisfies this property if, for all $(N, v) \in \mathscr{G}$ and all $i, j \in N$, $f_i(N, v) - f_i\left(N \setminus \{j\}\right) = f_j(N, v) - f_j\left(N \setminus \{i\}\right)$. This property says that, given two players, the gains or losses obtained by both players when the other leaves the game coincide. Myerson [23] used it, together with efficiency, to characterize the Shapley value.

In Sánchez [26] it is proved that if a value on TU–games satisfies the balanced contributions axiom then this value can be expressed as the Shapley value of a particular TU–game. Following a similar reasoning, we can prove that the payoff of each player, according to any coalitional value satisfying the intracoalitional balanced contributions axiom, can be computed by means of the Shapley value applied to a TU–game restricted to the union the player belongs to.

Definition 6 For all $(N, v, C) \in \mathscr{CG}$ and all $C_h \in C$, given a coalitional value g, we define the TU–game $\left(C_h, v^{g,C}\right)$ as

$$v^{g,C}(T) = \sum_{i\in T} g_i\left((N \setminus \{C_h\}) \cup T, v, C_{|T}\right) \text{ for all } T \subseteq C_h, T \neq \emptyset.$$

According to this TU–game, the worth of each subset of players in the union is given by the sum of their values in the coalitional game where the union is replaced with the subset.

Proposition 1
a) A coalitional value g satisfies IBC if and only if, for all $(N, v, C) \in \mathscr{CG}$ and all $i \in C_h$ with $C_h \in C$,

$$g_i(N, v, C) = \frac{v^{g,C}\left(C_h\right) - v^{g,C}\left(C_h \setminus \{i\}\right) + \sum_{j\in C_h\setminus\{i\}} g_i\left(N \setminus \{j\}, v, C_{N\setminus\{j\}}\right)}{|C_h|} \qquad (3)$$

b) A coalitional value g satisfies IBC if and only if, for all $(N,v,C) \in \mathscr{CG}$ and all $i \in C_h$ with $C_h \in C$, $g_i(N,v,C) = Sh_i\left(C_h, v^{g,C}\right)$.

In the first part of the proposition we propose a necessary and sufficient condition to be fulfilled by any coalitional value satisfying the property of IBC. This condition provides an interpretation of the coalitional value g for a player $i \in C_h$ as the average of $|C_h|$ quantities: the amounts player i gets if another player in the union leaves the game and the marginal contribution of i to his/her union in the TU–game $v^{g,C}$.

In the second part we express any coalitional value satisfying IBC as the Shapley value of a particular game. This condition is related to the first part of the proposition and it will we used in the characterizations of the next section.

Proof

Proof of part a)

First of all, it will be proved that if a coalitional value g satisfies IBC then (3) is true. Given $i \in C_h$, by IBC we know that for all $j \in C_h \setminus \{i\}$,

$$g_i(N,v,C) - g_i\left(N \setminus \{j\}, v, C_{N\setminus\{j\}}\right) = g_j(N,v,C) - g_j\left(N \setminus \{i\}, v, C_{N\setminus\{i\}}\right).$$

If we add these equations over $j \in C_h \setminus \{i\}$, we obtain that

$$\begin{aligned}&(|C_h|-1)\, g_i(N,v,C) - \sum_{j\in C_h\setminus\{i\}} g_i\left(N \setminus \{j\}, v, C_{N\setminus\{j\}}\right)\\ &= \sum_{j\in C_h\setminus\{i\}} g_j(N,v,C) - \sum_{j\in C_h\setminus\{i\}} g_j\left(N \setminus \{i\}, v, C_{N\setminus\{i\}}\right).\end{aligned}$$

Taking into account the definition of the game $\left(C_h, v^{g,C}\right)$, the previous equation can be replaced by

$$\begin{aligned}&(|C_h|-1)\, g_i(N,v,C) - \sum_{j\in C_h\setminus\{i\}} g_i\left(N \setminus \{j\}, v, C_{N\setminus\{j\}}\right)\\ &= v^{g,C}\left(C_h\right) - g_i(N,v,C) - v^{g,C}\left(C_h \setminus \{i\}\right).\end{aligned}$$

Thus

$$g_i(N,v,C) = \frac{v^{g,C}(C_h) - v^{g,C}\left(C_h \setminus \{i\}\right) + \sum_{j\in C_h\setminus\{i\}} g_i\left(N \setminus \{j\}, v, C_{N\setminus\{j\}}\right)}{|C_h|}.$$

Suppose now that g is a coalitional value that satisfies (3). We will prove that g satisfies IBC. The proof will be done by induction on $|C_h|$. We assume that $|C_h| > 1$ since this is the situation where IBC gives some information about the coalitional value.

Suppose that $C_h = \{i, j\}$. Then equation (3) can be written as

$$\begin{aligned}
&g_i(N, v, C)\\
&= \frac{v^{g,C}(C_h) - v^{g,C}\left(C_h \setminus \{i\}\right) + g_i\left(N \setminus \{j\}, v, C_{N\setminus\{j\}}\right)}{2}\\
&= \frac{g_i(N, v, C) + g_j(N, v, C) - g_j\left(N \setminus \{i\}, v, C_{N\setminus\{i\}}\right) + g_i\left(N \setminus \{j\}, v, C_{N\setminus\{j\}}\right)}{2},
\end{aligned}$$

what implies that IBC is true.

Suppose now that $|C_h| > 2$. Let us choose $i, j \in C_h$. Then

$$\begin{aligned}
&|C_h| \left[g_i\left(N \setminus \{j\}, v, C_{N\setminus\{j\}}\right) - g_j\left(N \setminus \{i\}, v, C_{N\setminus\{i\}}\right)\right]\\
&= (|C_h| - 1)\left[g_i\left(N \setminus \{j\}, v, C_{N\setminus\{j\}}\right) - g_j\left(N \setminus \{i\}, v, C_{N\setminus\{i\}}\right)\right]\\
&+ g_i\left(N \setminus \{j\}, v, C_{N\setminus\{j\}}\right) - g_j\left(N \setminus \{i\}, v, C_{N\setminus\{i\}}\right)\\
&= v^{g,C_{N\setminus\{j\}}}\left(C_h \setminus \{j\}\right) - v^{g,C_{N\setminus\{j\}}}\left(C_h \setminus \{i, j\}\right)\\
&+ \sum_{k\in C_h\setminus\{i,j\}} g_i\left(N \setminus \{j, k\}, v, C_{N\setminus\{j,k\}}\right) - v^{g,C_{N\setminus\{i\}}}\left(C_h \setminus \{i\}\right)\\
&+ v^{g,C_{N\setminus\{i\}}}\left(C_h \setminus \{i, j\}\right) - \sum_{k\in C_h\setminus\{i,j\}} g_j\left(N \setminus \{i, k\}, v, C_{N\setminus\{i,k\}}\right)\\
&+ g_i\left(N \setminus \{j\}, v, C_{N\setminus\{j\}}\right) - g_j\left(N \setminus \{i\}, v, C_{N\setminus\{i\}}\right)\\
&= v^{g,C_{N\setminus\{j\}}}(C_h \setminus \{j\}) + \sum_{k\in C_h\setminus\{i,j\}} g_i\left(N \setminus \{j, k\}, v, C_{N\setminus\{j,k\}}\right)\\
&+ g_i\left(N \setminus \{j\}, v, C_{N\setminus\{j\}}\right) - v^{g,C_{N\setminus\{i\}}}(C_h \setminus \{i\})\\
&- \sum_{k\in C_h\setminus\{i,j\}} g_j\left(N \setminus \{i, k\}, v, C_{N\setminus\{i,k\}}\right) - g_j\left(N \setminus \{i\}, v, C_{N\setminus\{i\}}\right).
\end{aligned}$$

On the other hand, by the induction hypothesis we know that for all $k \in C_h \setminus \{i, j\}$,

$$\begin{aligned}
&g_i\left(N \setminus \{j, k\}, v, C_{N\setminus\{j,k\}}\right) - g_j\left(N \setminus \{i, k\}, v, C_{N\setminus\{i,k\}}\right)\\
&= g_i\left(N \setminus \{k\}, v, C_{N\setminus\{k\}}\right) - g_j\left(N \setminus \{k\}, v, C_{N\setminus\{k\}}\right).
\end{aligned}$$

Thus, by induction hypothesis and (3),

$$\begin{aligned}
&|C_h| \left[g_i\left(N \setminus \{j\}, v, C_{N\setminus\{j\}}\right) - g_j\left(N \setminus \{i\}, v, C_{N\setminus\{i\}}\right)\right]\\
&= \sum_{k\in C_h\setminus\{i\}} g_i\left(N \setminus \{k\}, v, C_{N\setminus\{k\}}\right) - v^{g,C_{N\setminus\{i\}}}\left(C_h \setminus \{i\}\right)\\
&- \left[\sum_{k\in C_h\setminus\{j\}} g_j\left(N \setminus \{k\}, v, C_{N\setminus\{k\}}\right) - v^{g,C_{N\setminus\{j\}}}\left(C_h \setminus \{j\}\right)\right]\\
&= \sum_{k\in C_h\setminus\{i\}} g_i\left(N \setminus \{k\}, v, C_{N\setminus\{k\}}\right) + v^{g,C}(C_h) - v^{g,C_{N\setminus\{i\}}}\left(C_h \setminus \{i\}\right)\\
&- \left[\sum_{k\in C_h\setminus\{j\}} g_j\left(N \setminus \{k\}, v, C_{N\setminus\{k\}}\right) + v^{g,C}(C_h) - v^{g,C_{N\setminus\{j\}}}\left(C_h \setminus \{j\}\right)\right]\\
&= |C_h| \left[g_i(N, v, C) - g_j(N, v, C)\right]. \quad \square
\end{aligned}$$

Proof of part b)

It is straightforward to prove that every coalitional value g, which satisfies that $g_i(N,v,C) = Sh_i\left(C_h, v^{g,C}\right)$ for all $(N,v,C) \in \mathscr{CG}$ and all $i \in C_h$ with $C_h \in C$, also satisfies the property IBC. In fact, since the Shapley value satisfies the property of balanced contributions (for more information, see Myerson [23]), we have that for all $i,j \in C_h$,

$$\begin{aligned}
&g_i(N,v,C) - g_i\left(N \setminus \{j\}, v, C_{N\setminus\{j\}}\right)\\
&= Sh_i\left(C_h, v^{g,C}\right) - Sh_i\left(C_h \setminus \{j\}, v^{g,C_{N\setminus\{j\}}}\right)\\
&= Sh_i\left(C_h, v^{g,C}\right) - Sh_i\left(C_h \setminus \{j\}, v^{g,C}\right)\\
&= Sh_j\left(C_h, v^{g,C}\right) - Sh_j\left(C_h \setminus \{i\}, v^{g,C}\right)\\
&= Sh_j\left(C_h, v^{g,C}\right) - Sh_j\left(C_h \setminus \{i\}, v^{g,C_{N\setminus\{i\}}}\right)\\
&= g_j(N,v,C) - g_j\left(N \setminus \{i\}, v, C_{N\setminus\{i\}}\right).
\end{aligned}$$

Next, we will prove that a coalitional value g satisfying property IBC can be expressed as $g_i(N,v,C) = Sh_i\left(C_h, v^{g,C}\right)$ for all $(N,v,C) \in \mathscr{CG}$ and all $i \in C_h$ with $C_h \in C$. The proof will be done by induction on $|C_h|$.

Suppose that $C_h = \{i\}$. Then, in this case, $Sh_i\left(C_h, v^{g,C}\right) = Sh_i\left(\{i\}, v^{g,C}\right) = v^{g,C}(\{i\}) = g_i(N,v,C)$.

Suppose now that $|C_h| > 1$. By part a), we know that g satisfies equation (3). It means that, applying the induction hypothesis,

$$\begin{aligned}
&|C_h| g_i(N,v,C)\\
&= v^{g,C}(C_h) - v^{g,C}\left(C_h \setminus \{i\}\right) + \sum_{j \in C_h \setminus \{i\}} g_i\left(N \setminus \{j\}, v, C_{N\setminus\{j\}}\right)\\
&= v^{g,C}(C_h) - v^{g,C}\left(C_h \setminus \{i\}\right) + \sum_{j \in C_h \setminus \{i\}} Sh_i\left(C_h \setminus \{j\}, v^{g,C_{N\setminus\{j\}}}\right)\\
&= v^{g,C}(C_h) - v^{g,C}\left(C_h \setminus \{i\}\right)\\
&+ \sum_{j \in C_h \setminus \{i\}} \sum_{T \subseteq C_h \setminus \{i,j\}} \frac{t!\,(c_h - t - 2)!}{(c_h - 1)!}\left[v^{g,C_{N\setminus\{j\}}}\left(T \cup \{i\}\right) - v^{g,C_{N\setminus\{j\}}}(T)\right]\\
&= v^{g,C}(C_h) - v^{g,C}\left(C_h \setminus \{i\}\right)\\
&+ \sum_{j \in C_h \setminus \{i\}} \sum_{T \subseteq C_h \setminus \{i,j\}} \frac{t!\,(c_h - t - 2)!}{(c_h - 1)!}\left[v^{g,C}(T \cup \{i\}) - v^{g,C}(T)\right]
\end{aligned}$$

$$= v^{g,C}(C_h) - v^{g,C}\left(C_h \setminus \{i\}\right)$$
$$+ \sum_{T \subsetneq C_h \setminus \{i\}} \; \sum_{j \in C_h \setminus (T \cup \{i\})} \frac{t!\,(c_h - t - 2)!}{(c_h - 1)!} \Big[v^{g,C}\left(T \cup \{i\}\right) - v^{g,C}(T) \Big]$$

$$= v^{g,C}(C_h) - v^{g,C}\left(C_h \setminus \{i\}\right) + \sum_{T \subsetneq C_h \setminus \{i\}} \frac{t!\,(c_h - t - 1)!}{(c_h - 1)!} \Big[v^{g,C}\left(T \cup \{i\}\right) - v^{g,C}(T) \Big].$$

Therefore,

$$g_i(N, v, C) = \sum_{T \subseteq C_h \setminus \{i\}} \frac{t!\,(c_h - t - 1)!}{c_h!} \Big[v^{g,C}(T \cup \{i\}) - v^{g,C}(T) \Big] = Sh_i\left(C_h, v^{g,C}\right). \square$$

Remark 1

- Since the Owen value Ow satisfies IBC, by Proposition 1 we obtain an alternative expression for this coalitional value. Then, for all $(N, v, C) \in \mathscr{CG}$ and all $i \in C_h$ with $C_h \in C$,
$$Ow_i(N, v, C) = Sh_i\left(C_h, v^{Ow,C}\right),$$
where $v^{Ow,C}(T) = \sum_{i \in T} Ow_i\left((N \setminus \{C_h\}) \cup T, v, C_{|T}\right)$ for all $T \subseteq C_h$.

  Moreover, we know that for all $T \subseteq C_h$,
$$v^{Ow,C}(T) = \sum_{i \in T} Ow_i\left((N \setminus \{C_h\}) \cup T, v, C_{|T}\right) = Sh_h\left(M, v^{C_{|T}}\right) = \hat{v}^{Sh,C}(T).$$
It implies that $Ow_i(N, v, C) = Sh_i\left(C_h, \hat{v}^{Sh,C}\right)$, which coincides with the expression given in (1).
- It is straightforward to prove that the Banzhaf–Owen value satisfies IBC. Then, by Proposition 1, we can also obtain an alternative expression to compute the Banzhaf–Owen value $BzOw$. Thus, for all $(N, v, C) \in \mathscr{CG}$ and all $i \in C_h$ with $C_h \in C$,
$$BzOw_i(N, v, C) = Sh_i\left(C_h, v^{BzOw,C}\right), \qquad (4)$$
where $v^{BzOw,C}(T) = \sum_{i \in T} BzOw_i\left((N \setminus \{C_h\}) \cup T, v, C_{|T}\right)$ for all $T \subseteq C_h$.

Example 1 In this example the Owen and Banzhaf–Owen values are computed according to different expressions provided in this paper.

To this aim, let us consider the set of players $N = \{1, 2, 3, 4, 5\}$ and the weighted majority game $[7; 5, 2, 2, 2, 1]$.

It is easy to prove that the set of minimal winning coalitions associated with this weighted majority game is $\{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3, 4, 5\}\}$.

Let us fix the coalition structure $C = \{C_1, C_2\}$ with $C_1 = \{2, 3, 4\}$ and $C_2 = \{1, 5\}$. Thus, with this weighted majority game (N, v) and the coalition structure, it is possible to define a TU–game with coalition structure (N, v, C).

Both coalitional values can be easily computed with the formulas given in Definitions 4 and 5, obtaining that

$$Ow(N, v, C) = \left(\frac{1}{4}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{4}\right) \text{ and } BzOw(N, v, C) = \left(\frac{1}{4}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{4}\right).$$

Table 1 TU–games $\hat{v}^{Sh,C}$, $\hat{v}^{Bz,C}$ and $v^{BzOw,C}$ evaluated in the coalitions of union C_1

T	$\{2\}$	$\{3\}$	$\{4\}$	$\{2,3\}$	$\{2,4\}$	$\{3,4\}$	$\{2,3,4\}$
$\hat{v}^{Sh,C}(T)$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$\hat{v}^{Bz,C}(T)$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$v^{BzOw,C}(T)$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{8}$

Table 2 TU–games $\hat{v}^{Sh,C}$, $\hat{v}^{Bz,C}$ and $v^{BzOw,C}$ evaluated in the coalitions of union C_2

T	$\{1\}$	$\{5\}$	$\{1,5\}$
$\hat{v}^{Sh,C}(T)$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$\hat{v}^{Bz,C}(T)$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$v^{BzOw,C}(T)$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

In the case of the Owen value, we could also apply the expression given in (1) to obtain the payoffs for the players in each coalition. Then, we know that for each $i \in C_h$, with $h \in \{1,2\}$, $Ow_i(N,v,C) = Sh_i\left(C_h, \hat{v}^{Sh,C}\right)$. According to this formula and the TU–game $\hat{v}^{Sh,C}$ computed in Tables 1 and 2, we know that

$$Ow_i(N,v,C) = \frac{1}{6} \text{ for } i \in \{2,3,4\} \text{ and } Ow_i(N,v,C) = \frac{1}{4} \text{ for } i \in \{1,5\}.$$

However, the Banzhaf–Owen value can be obtained in two different ways: either using the TU–game $\hat{v}^{Bz,C}(T)$ or $v^{BzOw,C}(T)$.

On one hand, we can consider the expression given in (2). So, the Banzhaf–Owen value can be computed for each $i \in C_h$, with $h \in \{1,2\}$, as $BzOw_i(N,v,C) = Bz_i\left(C_h, \hat{v}^{Bz,C}\right)$. Taking into account the TU–game $\hat{v}^{Bz,C}$ computed in Tables 1 and 2, it is easy to deduce that

$$BzOw_i(N,v,C) = \frac{1}{8} \text{ for } i \in \{2,3,4\} \text{ and } BzOw_i(N,v,C) = \frac{1}{4} \text{ for } i \in \{1,5\}.$$

On the other hand, if we choose the formula from expression (4), we have that $BzOw_i(N,v,C) = Sh_i\left(C_h, v^{BzOw,C}\right)$ for all $i \in C_h$, with $h \in \{1,2\}$. Then, if we compute the Shapley value of the TU–game $v^{BzOw,C}$ for each union, we also obtain that

$$BzOw_i(N,v,C) = \frac{1}{8} \text{ for } i \in \{2,3,4\} \text{ and } BzOw_i(N,v,C) = \frac{1}{4} \text{ for } i \in \{1,5\}.$$

Note that $\left(C_h, \hat{v}^{Sh,C}\right) = \left(C_h, v^{Ow,C}\right)$ for all $(N,v,C) \in \mathcal{CG}$ and all $C_h \in C$. However, this is not the case of the Banzhaf–Owen value. In fact, for the TU–game with coalition structure considered in the example,

$$Bz_1\left(M, v^C\right) = \hat{v}^{Bz,C}\left(C_1\right) = \frac{1}{2} \neq \frac{3}{8} = v^{BzOw,C}\left(C_1\right) = \sum_{i \in C_1} BzOw_i(N,v,C).$$

4 The characterizations

Below, we introduce the properties that, together with the property of intracoalitional balanced contributions, will be used to study the behavior of the Owen and Banzhaf–Owen values.

Efficiency (**E**). For all $(N,v,C) \in \mathscr{CG}$, $\sum_{i\in N} g_i(N,v,C) = v(N)$.
Efficiency says that the worth of the grand coalition should be distributed among the players.

2–Efficiency within unions (**2–EWU**). For all $(N,v,C) \in \mathscr{CG}$, any $C_h \in C$, and $i,j \in C_h$, $i \neq j$,

$$g_i(N,v,C) + g_j(N,v,C) = g_{i^*}\left(N^{\{i,j\}}, v^{\{i,j\}}, C^{\{i,j\}}\right),$$

where $\left(N^{\{i,j\}}, v^{\{i,j\}}\right)$ is the $\{i,j\}$–amalgamated game of (N,v) and $C^{\{i,j\}} = \left\{C_1^{\{i,j\}}, \ldots, C_m^{\{i,j\}}\right\}$ is the coalition structure over $N^{\{i,j\}}$ such that $C_h^{\{i,j\}} = (C_h \setminus \{\{i,j\}\}) \cup \{i^*\}$ and $C_r^{\{i,j\}} = C_r$ for $r \neq h$.
According to this property, the sum of the payoffs of two players in the same union coincides with the payoff of their representative i^* in the amalgamated game.

Coherence (**C**). For all $(N,v) \in \mathscr{G}$, $g\left(N,v,C^N\right) = g\left(N,v,C^n\right)$.
Coherence means that the situations where all the players belong to the same union and when all of them act as singletons are indistinguishable.

Neutrality for the amalgamated game (**NAG**). If $(N,v,C) \in \mathscr{CG}$ and $i,j \in C_h \in C$, with $i \neq j$, then

$$g_k(N,v,C) = g_k\left(N^{\{i,j\}}, v^{\{i,j\}}, C^{\{i,j\}}\right) \text{ for all } k \in N \backslash C_h.$$

Neutrality for the amalgamated game says that if two players in the same union join their forces then the players outside the union are not affected.

1–Quotient game (**1–QG**). If $(N,v,C) \in \mathscr{CG}$ and $C \in \mathcal{C}(i)$ for some $i \in N$, then $g_i(N,v,C) = g_h\left(M, v^C, C^m\right)$, where $C_h = \{i\}$.
This property means that the amount an isolated player gets should not depend on the structure of the other unions.

Quotient game (**QG**). For all $(N,v,C) \in \mathscr{CG}$ and all $C_h \in C$, $\sum_{i\in C_h} g_i(N,v,C) = g_h\left(M, v^C, C^m\right)$.
In the case of the quotient game property, the sum of the payoffs of the players in a union coincides with the payoff of this union in the quotient game.

Note that both 2–EWU and NAG are refered to the amalgamated game. However, whereas 2–EWU deals with the players in the same union who merge, NAG is focused on the study of the payoffs of the players outside the union. On the other hand, property 1–QG is weaker than properties NAG and QG. In fact, 1–QG is a particular case of property QG for the coalitions formed by one player. An interesting comparison of 1–QG and QG can be found in Alonso–Meijide et al. [4].

In the next proposition, different relations between the properties and the corresponding payoff for each coalition are studied. These results, together with Proposition 1, will be used later to give characterizations for the Owen and the Banzhaf–Owen value.

Proposition 2

a) If g is a coalitional value that satisfies E and NAG then g satisfies QG.

b) A coalitional value g is a coalitional Shapley value for singletons and satisfies QG if and only if $\sum_{i \in C_h} g_i(N,v,C) = Sh_h\left(M, v^C\right)$ for all $C_h \in C$.

c) If g is a coalitional Shapley value for singletons and satisfies E and NAG then $\sum_{i \in C_h} g_i(N,v,C) = Sh_h\left(M, v^C\right)$ for all $C_h \in C$.

d) If g is a coalitional Banzhaf value for singletons and satisfies 2–EWU and 1–QG then $\sum_{i \in C_h} g_i(N,v,C) = \sum_{i \in C_h} BzOw_i(N,v,C)$ for all $C_h \in C$.

Proof

Proof of part a)

Let us consider a coalitional value g that satisfies properties E and NAG. We will prove that g satisfies QG.

Then, let us fix a TU–game with coalition structure $(N,v,C) \in \mathscr{CG}$ and $C_h \in C$. We distinguish two cases:

- *First case:* $C = C^n$. In this particular case, $C_h = \{i\}$ and $(N,v) = \left(M, v^C\right)$. It implies that $g_i\left(N,v,C^n\right) = g_h\left(M, v^C, C^m\right)$.
- *Second case:* $|C_r| = 1$ for all $r \neq h$ and $|C_h| > 1$. In this case we apply a recursive procedure, which consists of amalgamating two players in the union C_h and then a third player in the same union is amalgamated to the new one, and so on. The procedure is applied $c_h - 1$ times to build the TU–game with coalition structure $\left(N^{(c_h-1)}, v^{(c_h-1)}, C^{(c_h-1)}\right)$, where $\left|C_h^{(c_h-1)}\right| = 1$, $C_r^{(c_h-1)} = C_r$ for all $r \neq h$ and

$$v^{(c_h-1)}(S) = \begin{cases} v\left(\left(S \setminus C_h^{(c_h-1)}\right) \cup C_h\right) & \text{if } C_h^{(c_h-1)} \subseteq S \\ v(S) & \text{otherwise.} \end{cases}$$

Moreover, if we apply NAG at each step, at the end of the procedure we obtain that for all $i \in N \setminus C_h$,

$$g_i(N,v,C) = g_i\left(N^{(c_h-1)}, v^{(c_h-1)}, C^{(c_h-1)}\right).$$

Note that the game $\left(N^{(c_h-1)}, v^{(c_h-1)}, C^{(c_h-1)}\right)$ is now in the conditions of the first case. Then, for all $C_r \in C$, with $r \neq h$, and $i \in C_r$,

$$g_i(N, v, C) = g_i\left(N^{(c_h-1)}, v^{(c_h-1)}, C^{(c_h-1)}\right) = g_r\left(M, v^C, C^m\right).$$

On the other hand, by E we know that

$$\sum_{i \in C_h} g_i(N, v, C) = v(N) - \sum_{i \in N \setminus C_h} g_i(N, v, C).$$

So, it is easy to deduce that

$$\begin{aligned}\sum_{i \in C_h} g_i(N, v, C) &= v(N) - \sum_{i \in N \setminus C_h} g_i(N, v, C) \\ &= v^C(M) - \sum_{r \in M \setminus \{h\}} g_r\left(M, v^C, C^m\right) \\ &= g_h\left(M, v^C, C^m\right).\end{aligned}$$

– *Third case:* There exists at least one union $C_r \in C$, $r \neq h$, such that $|C_r| > 1$. Let us consider the set $M_h = \{r \in M \setminus \{h\} : |C_r| > 1\}$, with $m_h = |M_h|$. In this case we apply a procedure, which consists of m_h stages. The procedure at each stage is similar to the recursive procedure used in the second case until we get, after $\sum_{r \in M_h} c_r - m_h$ steps, the TU–game with coalition structure denoted by $\left(N^{\left(\sum_{r \in M_h} c_r - m_h\right)}, v^{\left(\sum_{r \in M_h} c_r - m_h\right)}, C^{\left(\sum_{r \in M_h} c_r - m_h\right)}\right)$.

  This TU–game with coalition structure satisfies that $\left|C_r^{\left(\sum_{r \in M_h} c_r - m_h\right)}\right| = 1$ for all $r \in M_h$ and $C_r^{\left(\sum_{r \in M_h} c_r - m_h\right)} = C_r$ for all $r \in M \setminus M_h$. Moreover, if we apply NAG at each step, we finally have that, for all $i \in C_h$,

$$g_i\left(N^{\left(\sum_{r \in M_h} c_r - m_h\right)}, v^{\left(\sum_{r \in M_h} c_r - m_h\right)}, C^{\left(\sum_{r \in M_h} c_r - m_h\right)}\right) = g_i(N, v, C).$$

  Since the game $\left(N^{\left(\sum_{r \in M_h} c_r - m_h\right)}, v^{\left(\sum_{r \in M_h} c_r - m_h\right)}, C^{\left(\sum_{r \in M_h} c_r - m_h\right)}\right)$ is in the conditions of the first or second case, it is easy to prove that

$$\begin{aligned}&\sum_{i \in C_h} g_i(N, v, C) \\ &= \sum_{i \in C_h} g_i\left(N^{\left(\sum_{r \in M_h} c_r - m_h\right)}, v^{\left(\sum_{r \in M_h} c_r - m_h\right)}, C^{\left(\sum_{r \in M_h} c_r - m_h\right)}\right) \\ &= v^C(M) - \sum_{r \in M \setminus \{h\}} g_r\left(M, v^C, C^m\right) \\ &= g_h\left(M, v^C, C^m\right).\end{aligned}$$

Proof of part b)

Let us consider the TU–game with coalition structure $(N, v, C) \in \mathscr{CG}$ and $C_h \in C$. It is straightforward to prove that any coalitional value g satisfying $\sum_{i \in C_h} g_i(N, v, C) = Sh_h\left(M, v^C\right)$ is a coalitional value for singletons and satisfies QG.

On the other hand, if g is a coalitional value for singletons and satisfies QG, then

$$\sum_{i \in C_h} g_i(N, v, C) = g_h\left(M, v^C, C^m\right) = Sh_h\left(M, v^C\right). \quad \square$$

Proof of part c)

According to part a), if a coalitional value g satisfies E and NAG then it satisfies QG. Thus, if we apply part b) we obtain that $\sum_{i \in C_h} g_i(N, v, C) = Sh_h\left(M, v^C\right)$ for all $(N, v, C) \in \mathscr{CG}$ and $C_h \in C$. $\square$

Proof of part d)

Let us consider the TU–game with coalition structure $(N, v, C) \in \mathscr{CG}$ and $C_h \in C$. We distinguish three cases:

- *First case:* $C = C^n$. Since g is a coalitional Banzhaf value for singletons we have that $g(N, v, C) = Bz(N, v) = BzOw(N, v, C)$.
- *Second case:* $|C_h| = 1$ and there exists at least one union C_r, $r \neq h$, such that $|C_r| > 1$. Suppose that $C_h = \{i\}$. By 1–QG we obtain that $g_i(N, v, C) = g_h\left(M, v^C, C^m\right)$.
  According to the first case and taking into account that $BzOw$ also satisfies 1–QG, $g_i(N, v, C) = g_h\left(M, v^C, C^m\right) = BzOw_h\left(M, v^C, C^m\right) = BzOw_i(N, v, C)$.
- *Third case:* $|C_h| = c_h > 1$. In this particular case we follow a recursive procedure, which consists of $c_h - 1$ steps:
  Initially, we consider the TU–game with coalition structure $\left(N^{(0)}, v^{(0)}, C^{(0)}\right) = (N, v, C)$. We also assume that $C_h^{(0)} = \left\{i_1^{(0)}, \ldots, i_{c_h}^{(0)}\right\}$. After $c_h - 1$ steps we obtain $\left(N^{(c_h-1)}, v^{(c_h-1)}, C^{(c_h-1)}\right)$, that satisfies that $\left|C_h^{(c_h-1)}\right| = 1$. The procedure at each step p, with $1 \leq p \leq c_h - 1$, is as follows:
  Suppose that, after step $p-1$, we obtained the game $\left(N^{(p-1)}, v^{(p-1)}, C^{(p-1)}\right)$ with $\left|C_h^{(p-1)}\right| = c_h - p + 1$. Let us assume that $C_h^{(p-1)} = \left\{i_1^{(p-1)}, \ldots, i_{c_h-p+1}^{(p-1)}\right\}$. At step p we take players $i_1^{(p-1)}$, $i_2^{(p-1)} \in C_h^{(p-1)}$ and define $\left(N^{(p)}, v^{(p)}, C^{(p)}\right)$ as the $\left\{i_1^{(p-1)}, i_2^{(p-1)}\right\}$–amalgamated game of $\left(N^{(p-1)}, v^{(p-1)}, C^{(p-1)}\right)$. By definition of amalgamated game, we know that $\left|C_h^{(p)}\right| = c_h - p$ and $C_r^{(p)} = C_r$ for all $r \neq h$.

Moreover, by 2–EWU we know that

$$\begin{aligned} & g_{i_1^{(p-1)}}\left(N^{(p-1)}, v^{(p-1)}, C^{(p-1)}\right) + g_{i_2^{(p-1)}}\left(N^{(p-1)}, v^{(p-1)}, C^{(p-1)}\right) \\ & = g_{i_1^{(p-1)} i_2^{(p-1)}}\left(N^{(p)}, v^{(p)}, C^{(p)}\right). \end{aligned}$$

We consider that the set $\left\{i_1^{(p-1)}, i_2^{(p-1)}\right\}$ is represented by the player $i_1^{(p)}$ and that $i_k^{(p)} = i_{k+1}^{(p-1)}$ for $2 \leq k \leq c_h - p$. If $p < c_h - 1$, we go to next step.
After $c_h - 1$ steps, we obtain the game $\left(N^{(c_h-1)}, v^{(c_h-1)}, C^{(c_h-1)}\right)$, where $\left|C_h^{(c_h-1)}\right| = 1$ and $C_r^{(c_h-1)} = C_r$ for all $r \neq h$. In addition,

$$\begin{aligned} & g_{i_1^{(c_h-1)}}\left(N^{(c_h-1)}, v^{(c_h-1)}, C^{(c_h-1)}\right) \\ & = g_{i_1^{(c_h-2)}}\left(N^{(c_h-2)}, v^{(c_h-2)}, C^{(c_h-2)}\right) + g_{i_2^{(c_h-2)}}\left(N^{(c_h-2)}, v^{(c_h-2)}, C^{(c_h-2)}\right) \\ & \vdots \\ & = g_{i_1^{(0)}}\left(N^{(0)}, v^{(0)}, C^{(0)}\right) + \ldots + g_{i_{c_h}^{(0)}}\left(N^{(0)}, v^{(0)}, C^{(0)}\right) \\ & = \sum_{i \in C_h} g_i(N, v, C). \end{aligned}$$

Then, the TU–game with coalition structure $\left(N^{(c_h-1)}, v^{(c_h-1)}, C^{(c_h-1)}\right)$ is in the conditions of the first or the second case (it depends on the size of the coalitions). In both cases, we know that $g_{i_1^{(c_h-1)}}\left(N^{(c_h-1)}, v^{(c_h-1)}, C^{(c_h-1)}\right)$ $= BzOw_{i_1^{(c_h-1)}}\left(N^{(c_h-1)}, v^{(c_h-1)}, C^{(c_h-1)}\right)$.
Therefore, since $BzOw$ also satisfies 2–EWU,

$$\begin{aligned} \sum_{i \in C_h} g_i(N, v, C) & = g_{i_1^{(c_h-1)}}\left(N^{(c_h-1)}, v^{(c_h-1)}, C^{(c_h-1)}\right) \\ & = BzOw_{i_1^{(c_h-1)}}\left(N^{(c_h-1)}, v^{(c_h-1)}, C^{(c_h-1)}\right) \\ & = \sum_{i \in C_h} BzOw_i(N, v, C). \quad \square \end{aligned}$$

Theorem 1
a) The Owen value is the only coalitional Shapley value for singletons that satisfies QG and IBC.
b) The Owen value is the only coalitional Shapley value for singletons that satisfies E, NAG and IBC.
c) The Owen value is the only coalitional value that satisfies E, C, NAG and IBC.

Proof

By the results contained in other papers where the Owen value is studied, we know that the Owen value satisfies E, C, NAG, QG and IBC. Then, it only remains to prove the uniqueness.

Proof of uniqueness in part a)

Since g is in the conditions of Proposition 1 we can assume that, for all $(N,v,C)\in\mathscr{CG}$ and all $i\in C_h$ with $C_h\in C$, $g_i(N,v,C)=Sh_i\left(C_h,v^{g,C}\right)$.

On the other hand, since g is in the conditions of part b) from Proposition 2, we know that for all $T\subseteq C_h$,

$$v^{g,C}(T)=\sum_{i\in T}g_i\left((N\setminus C_h)\cup T,v,C_{|T}\right)=Sh_h\left(M,v^{C_{|T}}\right)=\hat{v}^{Sh,C}(T).$$

Then, we know that for all $(N,v,C)\in\mathscr{CG}$ and all $i\in C_h$ with $C_h\in C$,

$$g_i(N,v,C)=Sh_i\left(C_h,\hat{v}^{Sh,C}\right)=Ow_i(N,v,C).\quad\square$$

Proof of uniqueness in part b)

Since g is in the conditions of Proposition 1 we can assume that, for all $(N,v,C)\in\mathscr{CG}$ and all $i\in C_h$ with $C_h\in C$, $g_i(N,v,C)=Sh_i\left(C_h,v^{g,C}\right)$.

Moreover, g is also in the conditions of Proposition 2 part c). If we apply this result to the definition of the game $v^{g,C}$, we obtain that for all $T\subseteq C_h$,

$$v^{g,C}(T)=\sum_{i\in T}g_i\left((N\setminus C_h)\cup T,v,C_{|T}\right)=Sh_h\left(M,v^{C_{|T}}\right)=\hat{v}^{Sh,C}(T).$$

Then, we know that for all $(N,v,C)\in\mathscr{CG}$ and all $i\in C_h$ with $C_h\in C$,

$$g_i(N,v,C)=Sh_i\left(C_h,\hat{v}^{Sh,C}\right)=Ow_i(N,v,C).\quad\square$$

Proof of uniqueness in part c)

Let us consider $(N,v)\in\mathscr{G}$. By IBC applied to (N,v,C^N), we know that for all $i,j\in N$,

$$g_i(N,v,C^N)-g_i\left(N\setminus\{j\},v,C^{N\setminus\{j\}}\right)=g_j(N,v,C^N)-g_j\left(N\setminus\{i\},v,C^{N\setminus\{i\}}\right).$$

And, taking into account the characterization for the Shapley value given by Myerson [23], this result jointly with E implies that $g(N,v,C^N)=Sh(N,v)$ for all $(N,v)\in\mathscr{G}$. By C, we also know that $g(N,v,C^n)=g(N,v,C^N)=Sh(N,v)$ for all $(N,v)\in\mathscr{G}$.

Then, since g is in the conditions of the part a) of this theorem, we can assert that for all $(N,v,C)\in\mathscr{CG}$ and all $i\in C_h$ with $C_h\in C$,

$$g_i(N,v,C)=Sh_i\left(C_h,\hat{v}^{Sh,C}\right)=Ow_i(N,v,C).\quad\square$$

Remark 2

- Note that part b) of Theorem 1 can be shown as a consecuence of part a). The reason that motivates this argument is the fact that axioms E and NAG imply axiom QG.

However, axioms E and NAG are not equivalent to axiom QG. In fact, axiom QG does not imply E or NAG:

- The symmetric coalitional Banzhaf value introduced by Alonso–Meijide and Fiestras–Janeiro [8] and defined for all $(N,v,C) \in \mathscr{CG}$ and all $i \in C_h$, with $C_h \in C$, by

$$\sum_{R \subseteq M \setminus \{h\}} \sum_{T \subseteq C_h \setminus \{i\}} \frac{1}{2^{m-1}} \frac{t!(c_h - t - 1)!}{c_h!} \left[v(Q \cup T \cup \{i\}) - v(Q \cup T) \right]$$

satisfies axioms NAG and QG but, however, does not satisfy axiom E.
- The coalitional value defined for all $(N,v,C) \in \mathscr{CG}$ and all $i \in C_h$, with $C_h \in C$, by

$$\gamma_i^1(N,v,C) = \begin{cases} \dfrac{Sh_h\left(M, v^C\right)}{|C_h|} & \text{if } |C_r| = 1 \text{ for all } r \in M \setminus \{h\} \\ Ow_i(N,v,C) & \text{otherwise} \end{cases}$$

satisfies axioms E and QG but it does not satisfy axiom NAG.

- While parts b) and c) only differ in one property, it is not possible to obtain a similar characterization to a) for the Owen value just replacing the condition of coalitional Shapley value for singletons by axiom C.

  In fact, the coalitional value given by $\gamma_i^2(N,v,C) = 0$ for all $(N,v,C) \in \mathscr{CG}$ and all $i \in N$ also satisfies the properties C, QG and IBC.

Remark 3 *(Independence of the properties in Theorem 1)*
- The axioms given in part a) are independent:
  - The symmetric coalitional Banzhaf value satisfies both QG and IBC but it is not a coalitional Shapley value for singletons.
  - The coalitional value given for all $(N,v,C) \in \mathscr{CG}$ by $\gamma^3(N,v,C) = Sh(N,v)$ is a coalitional Shapley value for singletons that satisfies IBC and fails QG.
  - The two–step Shapley value defined by Kamijo [21] and given for all $(N,v,C) \in \mathscr{CG}$ and all $i \in C_h$, with $C_h \in C$, by the formula

$$Sh_i\left(C_h, v\right) + \frac{1}{c_h} \left[Sh_h\left(M, v^C\right) - v(C_h) \right]$$

is a coalitional Shapley value for singletons that satisfies QG and fails IBC.

- The axioms given in part b) are independent:
  - The coalitional value given for all $(N,v,C) \in \mathscr{CG}$ and all $i \in C_h$, with $C_h \in C$, by $\gamma_i^4(N,v,C) = Sh_i\left(C_h, v_1\right)$, where (C_h, v_1) is the TU–game such that $v_1(T) = \dfrac{v\left(T \cup N \setminus C_h\right)}{m}$ for all $T \subseteq C_h$, $T \neq \emptyset$, satisfies E, NAG and IBC but it is not a coalitional Shapley value for singletons.
  - The coalitional value studied by Alonso–Meijide et al. [6] and defined for all $(N,v,C) \in \mathscr{CG}$ and all $i \in C_h$, with $C_h \in C$, by

$$\sum_{R \subseteq M \setminus \{h\}} \sum_{T \subseteq C_h \setminus \{i\}} \frac{(r+t)!(m + c_h - r - t - 2)!}{(m + c_h - 1)!} \left[v(Q \cup T \cup \{i\}) - v(Q \cup T) \right]$$

is a coalitional Shapley value for singletons that satisfies both NAG and IBC but fails E.

 – The coalitional value γ^3 is a coalitional Shapley value for singletons that satisfies E and IBC but fails NAG.
 – The two–step Shapley value is a coalitional Shapley value for singletons that satisfies E and NAG but fails IBC.

- The axioms given in part c) are independent:
 – The Banzhaf–Owen value satisfies C, NAG and IBC but fails E.
 – The coalitional value γ^4 does not satisfy C but satisfies E, NAG and IBC.
 – The coalitional value γ^3 satisfies E, C and IBC and fails NAG.
 – The two–step Shapley value is a coalitional value that satisfies E, C and NAG but fails IBC.

Theorem 2
a) The Banzhaf–Owen value is the only coalitional Banzhaf value for singletons that satisfies 2–EWU, 1–QG and IBC.
b) The Banzhaf–Owen value is the only coalitional Banzhaf value for singletons that satisfies 2–EWU, NAG and IBC.

Proof

By the results contained in other papers where the Banzhaf–Owen value is studied, we know that the Banzhaf–Owen value satisfies 2–EWU, NAG, 1–QG and IBC. Then, it only remains to prove the uniqueness.

Proof of uniqueness in part a)

Since g is in the conditions of Proposition 1 we can assume that, for all $(N, v, C) \in \mathscr{CG}$ and all $i \in C_h$ with $C_h \in C$, $g_i(N, v, C) = Sh_i\left(C_h, v^{g,C}\right)$.

Moreover, g is in the conditions of Proposition 2 part d). If we apply this result to the definition of the game $v^{g,C}$, we obtain that for all $T \subseteq C_h$,

$$v^{g,C}(T) = \sum_{i\in T} g_i\left((N \setminus C_h) \cup T, v, C_{|T}\right) = \sum_{i\in T} BzOw_i\left((N \setminus C_h) \cup T, v, C_{|T}\right).$$

Then, according to Remark 1, we know that for all $(N, v, C) \in \mathscr{CG}$ and all $i \in C_h$ with $C_h \in C$,

$$g_i(N, v, C) = BzOw_i(N, v, C). \quad \square$$

Proof of uniqueness in part b)

It is straightforward to prove that NAG implies 1–QG. Then, if a coalitional value g is a coalitional Banzhaf value for singletons that satisfies 2–EWU, NAG and IBC, this coalitional value is in the conditions of the part a) of this theorem. It means that for all $(N, v, C) \in \mathscr{CG}$ and all $i \in C_h$ with $C_h \in C$,

$$g_i(N, v, C) = BzOw_i(N, v, C). \quad \square$$

Remark 4 *(Independence of the properties in Theorem 2)*
Since axiom NAG implies axiom 1–QG, the independence of the properties in both parts can be shown with the same coalitional values:

– The coalitional value defined by Amer et al. [9] and defined for all $(N,v,C) \in \mathscr{C}\mathscr{G}$ and all $i \in C_h$, with $C_h \in C$, by

$$\sum_{R \subseteq M \setminus \{h\}} \sum_{T \subseteq C_h \setminus \{i\}} \frac{r!(m-r-1)!}{m!} \frac{1}{2^{c_h-1}} \left[v(Q \cup T \cup \{i\}) - v(Q \cup T) \right]$$

satisfies 2–EWU, NAG and IBC but, however, it is not a coalitional Banzhaf value for singletons.

– The symmetric coalitional Banzhaf value is a coalitional Banzhaf value for singletons, satisfies NAG and IBC but fails 2–EWU.

– The coalitional value given for all $(N,v,C) \in \mathscr{C}\mathscr{G}$ by $\gamma^5(N,v,C) = Bz(N,v)$ is a coalitional Banzhaf value for singletons that satisfies 2–EWU and IBC but fails 1–QG.

– The coalitional value given, for all $(N,v,C) \in \mathscr{C}\mathscr{G}$ and all $i \in C_h$, with $C_h \in C$, by

$$\gamma_i^6(N,v,C) = \begin{cases} Bz_i(C_h,v) & \text{if } C \neq C^n \\ \\ Bz_i(N,v) & \text{if } C = C^n \end{cases}$$

is a coalitional Banzhaf value for singletons that satisfies 2–EWU and NAG. However, it does not satisfy axiom IBC.

5 Concluding remarks

The main purpose of this framework is to study the behaviour of two well–known coalitional values: the Owen and Banzhaf–Owen values. In order to achieve this objective, we make use of several properties and compare both coalitional values, trying to deduce their main differences and similarities. As a consequence of this study, some characterizations of these two values are obtained. All the results are summarized in Table 3.

Table 3 Properties satisfied by the Owen and the Banzhaf–Owen values

Properties/Axioms	Owen	Banzhaf–Owen
Coalitional Shapley value for singletons	$\checkmark^{(1a,1b)}$	$\times$
Coalitional Banzhaf value for singletons	$\times$	$\checkmark^{(2a,2b)}$
Efficiency (E)	$\checkmark^{(1b,1c)}$	$\times$
2–Efficiency within unions (2–EWU)	$\times$	$\checkmark^{(2a,2b)}$
Coherence (C)	$\checkmark^{(1c)}$	$\checkmark$
Neutrality for the amalgamated game (NAG)	$\checkmark^{(1b,1c)}$	$\checkmark^{(2b)}$
1–Quotient game (1–QG)	$\checkmark$	$\checkmark^{(2a)}$
Quotient game (QG)	$\checkmark^{(1a)}$	$\times$
Intracoalitional balanced contributions (IBC)	$\checkmark^{(1a,1b,1c)}$	$\checkmark^{(2a,2b)}$

In all the characterizations, the property of intracoalitional balanced contributions (IBC) is a key axiom. In fact, we have provided an expression for all the coalitional values satisfying this property. However, we have taken into account many other interesting properties in this framework, most of them crucial in the context of coalitional values. All of them appear in Table 3.

The characterizations of the framework are collected in Section 4. In this section, we investigate the relation between the Owen and Banzhaf–Owen values. While in Theorem 1 part b) the Owen value is characterized as the only coalitional Shapley value for singletons satisfying efficiency (E), neutrality of the amalgamated game (NAG) and intracoalitional balanced contributions (IBC), in Theorem 2 part b) a parallel characterization is obtained for the Banzhaf–Owen value, just replacing coalitional Shapley value for singletons and efficiency with coalitional Banzhaf value for singletons and 2–efficiency within unions (2–EWU).

In Theorem 1 part c), a new characterization of the Owen value is obtained. This characterization only differs in one property from the characterization given in Theorem 1 part b). However, it is not possible to find a similar result for the Banzhaf–Owen value, since it is not possible to replace coalitional Bazhaf value for singletons with coherence (C) in Theorem 2 part b). Although the Banzhaf–Owen value satisfies 2–EWU, C, NAG and IBC, other coalitional values also satisfy these properties:

– Given $\alpha \in \mathbb{R}$, the coalitional value defined for all $(N, v, C) \in \mathscr{CG}$ and all $i \in N$ by
$$\sum_{R \subseteq M\setminus\{h\}} \sum_{T \subseteq C_h\setminus\{i\}} \frac{\alpha}{2^{m+c_h-2}} \Big[v\left(Q \cup T \cup \{i\}\right) - v(Q \cup T) \Big],$$
satisfies 2–EWU, C, NAG and IBC. Since the case $\alpha = 1$ corresponds to the Banzhaf–Owen value, it suffices to consider any $\alpha \neq 1$ to obtain another coalitional value that satisfies these properties.

On the other hand, Theorem 2 part b) can be seen as a result of Theorem 2 part a), since property 1–QG is weaker than NAG. But, although the Owen value also satisfies 1–QG, we cannot characterize the Owen value as the only Shapley coalitional value which satisfies E, 1–QG and IBC. Then, it is not possible to find for the Owen value a parallel characterization to the one given for the Banzhaf–Owen value in Theorem 2 part a). In fact:

– The coalitional value given for all $(N, v, C) \in \mathscr{CG}$ and all $i \in C_h$, with $C_h \in C$, by $\gamma_i^7(N, v, C) = Sh_i\left(C_h, v_5\right)$, where $v_5(T)$
$$v_5(T) = \begin{cases} v_4(T) = Sh_h^{\omega^{C_{|T}}}\left(M, v^{C_{|T}}\right) & \text{if } |T| > 1 \text{ and} \\ & |C_r| > 1 \text{ for all } r \in M \setminus \{h\} \\ \hat{v}^{Sh,C}(T) = Sh_h\left(M, v^{C_{|T}}\right) & \text{otherwise} \end{cases}$$
for all $T \subseteq C_h$, $T \neq \emptyset$, is a Shapley coalitional value that satisfies E, 1–QG and IBC. As it was to be expected, this coalitional value does not satisfy NAG.

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