ORIGINAL ARTICLE

Statistics

Supplementary Material: Bandwidth Selection for Nonparametric Kernel Prediction

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1.1 | Closed-expression for the criterion functions

Theorem 1. *If assumption (A1) from the paper holds, x is an interior point of the support of X, $(X_1, \ldots, X_n)$ an i.i.d. sample, and $f(x) \neq 0$, then the MSE_x of $\tilde{m}_h$ can be expressed as*

$$MSE_x(h) = \frac{n-1}{nf(x)^2}(K_h * q_x)^2(x) + \frac{1}{nf(x)^2}\left[(K_h)^2 * p_x\right](x), \quad (1)$$

where $p_x(z) = \left(\sigma^2(z) + (m(z) - m(x))^2\right) f(z)$, $q_x(z) = (m(z) - m(x)) f(z)$ and $\sigma^2(x) := Var(Y|X = x)$ stands for the volatility function.

Similarly, if $\hat{f}_{g_X}(x) \neq 0$, then the smoothed bootstrap version of the $MSE_x(h)$ is

$$MSE_x^*(h) = \frac{1}{n\hat{f}_{g_X}^2(x)}\left[\frac{g_Y^2\mu_2(K)}{n}\sum_{i=1}^{n}\left[(K_h)^2 * K_{g_X}\right](x - X_i) + \left[(K_h)^2 * \hat{p}_{x,g_X}\right](x)\right] + \frac{n-1}{n\hat{f}_{g_X}^2(x)}(K_h * \hat{q}_{x,g_X})^2(x), \tag{2}$$

where $\hat{p}_{x,g_X}(z) = \left(\hat{\sigma}_{g_X}^2(z) + (\tilde{m}_{g_X}(z) - \tilde{m}_{g_X}(x))^2\right)\hat{f}_{g_X}(z)$, $\hat{q}_{x,g_X}(z) = (\tilde{m}_{g_X}(z) - \tilde{m}_{g_X}(x))\hat{f}_{g_X}(z)$ *and* $\mu_r(K) = \int t^r K(t)\,dt$, $\hat{\sigma}_{g_X}^2(z) = \tilde{m}_{2,g_X}(z) - \tilde{m}_{g_X}^2(z)$, $\tilde{m}_{2,g_X}(z) = \frac{\sum_{i=1}^n K_{g_X}(z - X_i)\,Y_i^2}{\sum_{i=1}^n K_{g_X}(z - X_i)}$.

Proof Consider the following expression:

$$MSE_x(h) = \mathbb{E}\left[\left(\tilde{m}_h^{NW}(x) - m(x)\right)^2\right] = (\mathbb{E}[A_1])^2 + Var[A_1], \tag{3}$$

where $A_1 = \tilde{m}_h^{NW}(x) - m(x)$. Focusing first on the bias term:

$$\begin{aligned}
\mathbb{E}[A_1] &= \frac{1}{nf(x)}\sum_{i=1}^{n}\mathbb{E}[K_h(x - X_i)(Y_i - m(x))] \\
&= \frac{1}{f(x)}\mathbb{E}[K_h(x - X_1)(Y_1 - m(x))] \\
&= \frac{1}{f(x)}\mathbb{E}[\mathbb{E}[(K_h(x - X_1)(Y_1 - m(x)) \mid X_1]] \\
&= \frac{1}{f(x)}\mathbb{E}[K_h(x - X_1)(\mathbb{E}[Y_1 \mid X_1] - m(x))] \\
&= \frac{1}{f(x)}\mathbb{E}[K_h(x - X_1)(m(X_1) - m(x))] \\
&= \frac{1}{f(x)}\int K_h(x - y)(m(y) - m(x))f(y)\,dy \\
&= \frac{1}{f(x)}\int K_h(x - y)q_x(y)\,dy = \frac{1}{f(x)}[K_h * q_x](x),
\end{aligned} \tag{4}$$

where $q_x(z) = (m(z) - m(x))f(z)$. On the other hand,

$$\begin{aligned}
Var[A_1] &= \frac{1}{n^2f(x)^2}\sum_{i=1}^{n}Var[K_h(x - X_i)(Y_i - m(x))] \\
&= \frac{1}{nf(x)^2}Var[K_h(x - X_1)(Y_1 - m(x))] \\
&= \frac{1}{nf(x)^2}\mathbb{E}\left[K_h(x - X_1)^2(Y_1 - m(x))^2\right] \\
&\quad - \frac{1}{nf(x)^2}(\mathbb{E}[K_h(x - X_1)(Y_1 - m(x))])^2.
\end{aligned} \tag{5}$$

The first term of (5) turns out to be:

$$\begin{aligned}\mathbb{E}\left[K_h(x-X_1)^2(Y_1-m(x))^2\right] &= \mathbb{E}\left[\mathbb{E}\left[K_h(x-X_1)^2(Y_1-m(x))^2 \mid X_1\right]\right]\\ &= \mathbb{E}\left[K_h(x-X_1)^2\mathbb{E}\left[(Y_1-m(x))^2 \mid X_1\right]\right]\\ &= \mathbb{E}\left[K_h(x-X_1)^2\left[Var(Y_1-m(x) \mid X_1)\right.\right.\\ &\left.\left.+(\mathbb{E}\left[Y_1-m(x) \mid X_1\right])^2\right]\right]\\ &= \mathbb{E}\left[K_h(x-X_1)^2\right.\\ &\left.\cdot\left[Var(Y_1 \mid X_1)+(m(X_1)-m(x))^2\right]\right]\\ &= \int K_h(x-y)^2\left(\sigma^2(y)+(m(y)-m(x))^2\right)f(y)\,dy\\ &= \left[(K_h)^2 * p_x\right](x), \end{aligned} \tag{6}$$

where $p_x(z) = \left(\sigma^2(z)+(m(z)-m(x))^2\right)f(z)$. Collecting terms (4) and (6) and plugging them into (5), and then plugging (4) and (5) into (3), the first part of Theorem 1 is proven.

For the proving (2) for the bootstrap analogue, consider the following expression:

$$MSE_x^*(h) = \mathbb{E}^*\left[\left(\tilde{m}_h^{NW*}(x)-\tilde{m}_{g_X}^{NW}(x)\right)^2\right] = \left(\mathbb{E}^*\left[A_1^*\right]\right)^2 + Var^*[A_1^*], \tag{7}$$

where $A_1^* = \tilde{m}_h^{NW*}(x) - \hat{m}_{g_X}(x)$. Focusing on the bootstrap version of the bias term,

$$\begin{aligned}\mathbb{E}^*\left[A_1^*\right] &= \frac{1}{n\hat{f}_{g_X}(x)}\sum_{i=1}^n \mathbb{E}^*\left[K_h(x-X_i^*)(Y_i^*-\hat{m}_{g_X}^{NW}(x))\right]\\ &= \frac{1}{\hat{f}_{g_X}(x)}\mathbb{E}^*\left[K_h(x-X_1^*)(Y_1^*-\hat{m}_{g_X}^{NW}(x))\right]\\ &= \frac{1}{\hat{f}_{g_X}(x)}\mathbb{E}^*\left[\mathbb{E}^*\left[(K_h(x-X_1^*)(Y_1^*-\hat{m}_{g_X}^{NW}(x)) \mid X_1^*\right]\right]\\ &= \frac{1}{\hat{f}_{g_X}(x)}\mathbb{E}^*\left[K_h(x-X_1^*)(\mathbb{E}^*[Y_1^* \mid X_1^*]-\hat{m}_{g_X}^{NW}(x))\right]\\ &= \frac{1}{\hat{f}_{g_X}(x)}\mathbb{E}^*\left[K_h(x-X_1^*)(\hat{m}_{g_X}^{NW}(X_1^*)-\hat{m}_{g_X}^{NW}(x))\right]\\ &= \frac{1}{\hat{f}_{g_X}(x)}\int K_h(x-y)(\hat{m}_{g_X}^{NW}(y)-\hat{m}_{g_X}^{NW}(x))\hat{f}_{g_X}(y)\,dy\\ &= \frac{1}{\hat{f}_{g_X}(x)}\int K_h(x-y)\hat{q}_{x,g}(y)\,dy = \frac{1}{\hat{f}_{g_X}(x)}\left[K_h * \hat{q}_{x,g}\right](x), \end{aligned} \tag{8}$$

where $\hat{q}_{x,g}(z) = (\hat{m}^{NW}_{g_X}(z) - \hat{m}^{NW}_{g_X}(x))\hat{f}_{g_X}(z)$. On the other hand,

$$\begin{aligned} Var^*[A_1^*] &= \frac{1}{n^2\hat{f}_{g_X}(x)^2}\sum_{i=1}^n Var^*\left[K_h(x-X_i^*)(Y_i^*-\hat{m}^{NW}_{g_X}(x))\right] \\ &= \frac{1}{n\hat{f}_{g_X}(x)^2} Var^*\left[K_h(x-X_1^*)(Y_1^*-\hat{m}^{NW}_{g_X}(x))\right] \\ &= \frac{1}{n\hat{f}_{g_X}(x)^2}\mathbb{E}^*\left[K_h(x-X_1^*)^2(Y_1^*-\hat{m}^{NW}_{g_X}(x))^2\right] \\ &\quad - \frac{1}{n\hat{f}_{g_X}(x)^2}\left(\mathbb{E}^*\left[K_h(x-X_1^*)(Y_1^*-\hat{m}^{NW}_{g_X}(x))\right]\right)^2. \end{aligned} \tag{9}$$

The first term of (9) turns out to be:

$$\begin{aligned} &\mathbb{E}^*\left[K_h(x-X_1^*)^2(Y_1^*-\hat{m}^{NW}_{g_X}(x))^2\right] \\ &= \mathbb{E}^*\left[\mathbb{E}^*\left[K_h(x-X_1^*)^2(Y_1^*-\hat{m}^{NW}_{g_X}(x))^2 \mid X_1^*\right]\right] \\ &= \mathbb{E}^*\left[K_h(x-X_1^*)^2\mathbb{E}^*\left[(Y_1^*-\hat{m}^{NW}_{g_X}(x))^2 \mid X_1^*\right]\right] \\ &= \mathbb{E}^*\left[K_h(x-X_1^*)^2\left[Var^*(Y_1^*-\hat{m}^{NW}_{g_X}(x) \mid X_1^*) + \left(\mathbb{E}^*\left[Y_1^*-\hat{m}^{NW}_{g_X}(x) \mid X_1^*\right]\right)^2\right]\right] \\ &= \mathbb{E}^*\left[K_h(x-X_1^*)^2\left[Var^*(Y_1^* \mid X_1^*) + \left(\hat{m}^{NW}_{g_X}(X_1^*) - \hat{m}^{NW}_{g_X}(x)\right)^2\right]\right] \\ &= \mathbb{E}^*\left[K_h(x-X_1^*)^2\left[\sigma^{*2}(X_1^*) + \left(\hat{m}^{NW}_{g_X}(X_1^*) - \hat{m}^{NW}_{g_X}(x)\right)^2\right]\right] \\ &= \mathbb{E}^*\left[K_h(x-X_1^*)^2\left[\hat{\sigma}^2_{g_X}(X_1^*) + g_Y^2\mu_2(K) + \left(\hat{m}^{NW}_{g_X}(X_1^*) - \hat{m}^{NW}_{g_X}(x)\right)^2\right]\right] \\ &= \int K_h(x-y)^2\left(\hat{\sigma}^2_{g_X}(y) + g_Y^2\mu_2(K) + (\hat{m}^{NW}_{g_X}(y) - \hat{m}^{NW}_{g_X}(x))^2\right)\hat{f}_{g_X}(y)\,dy \\ &= g_Y^2\mu_2(K)\int K_h(x-y)^2\hat{f}_{g_X}(y)dy \\ &+ \int K_h(x-y)^2\left(\hat{\sigma}^2_{g_X}(y) + (\hat{m}^{NW}_{g_X}(y) - \hat{m}^{NW}_{g_X}(x))^2\right)\hat{f}_{g_X}(y)\,dy \\ &= \frac{g_Y^2\mu_2(K)}{n}\sum_{i=1}^n\int K_h(x-y)^2K_{g_X}(y-X_i)dy + \int K_h(x-y)^2\hat{p}_{x,g}(y)\,dy \\ &= \frac{g_Y^2\mu_2(K)}{n}\sum_{i=1}^n\left[(K_h)^2 * K_{g_X}\right](x-X_i) + \int K_h(x-y)^2\hat{p}_{x,g}(y)\,dy \\ &= \frac{g_Y^2\mu_2(K)}{n}\sum_{i=1}^n\left[(K_h)^2 * K_{g_X}\right](x-X_i) + \left[(K_h)^2 * \hat{p}_{x,g}\right](x), \end{aligned} \tag{10}$$

where $\hat{p}_{x,g}(y) = \left(\hat{\sigma}^2_{g_X}(y) + (\hat{m}^{NW}_{g_X}(y) - \hat{m}^{NW}_{g_X}(x))^2\right)\hat{f}_{g_X}(y)$ and $\mu_2(K) = \int t^2K(t)\,dt$.

Collecting terms (8) and (10), and plugging them into (9), and then plugging (8) and (9) into (7), Theorem 1 is proven.

Proposition 1. *If F_1 is the distribution function of the target population and $\hat{m}_h$ the estimated regression function. Then,* *an upper bound for expression (6) in the paper is given by*

$$\mathbb{E}\left[\left(\int (\hat{m}_h(x) - m(x))\ dF_1(x)\right)^2\right] \quad \leq \quad \mathbb{E}\left[\int (\hat{m}_h(x) - m(x))^2\ dF_1(x)\right]. \tag{11}$$

On the other hand, the average prediction error is given by

$$\mathbb{E}\left[\frac{1}{n_1}\sum_{i=1}^{n_1}\left(Y_i^1 - \hat{m}_h\left(X_i^1\right)\right)^2\right] \quad = \quad \int \sigma^2(x)\, dF_1(x) + \mathbb{E}\left[\int (\hat{m}_h(x) - m(x))^2\ dF_1(x)\right]. \tag{12}$$

Proof On the one hand, consider Jensen's inequality and the convex function $\psi(z) = z^2$. Then,

$$\begin{aligned}
\int (\hat{m}_h(x) - m(x))^2\ dF_1(x) &= \mathbb{E}\left[\psi\left(\hat{m}_h\left(X^1\right) - m\left(X^1\right)\right)\Big|_{\left(X_1^0,Y_1^0\right),\ldots,\left(X_{n_0}^0,Y_{n_0}^0\right)}\right] \\
&\geq \psi\left(\mathbb{E}\left[\hat{m}_h\left(X^1\right) - m\left(X^1\right)\Big|_{\left(X_1^0,Y_1^0\right),\ldots,\left(X_{n_0}^0,Y_{n_0}^0\right)}\right]\right) \\
&= \left(\int (\hat{m}_h(x) - m(x))\ dF_1(x)\right)^2.
\end{aligned}$$

Then, expression (11) holds.

On the other hand, the average prediction error introduced in Definition 1 in the paper can be expressed as:

$$\begin{aligned}
&\mathbb{E}\left[\frac{1}{n_1}\sum_{i=1}^{n_1}\left(Y_i^1 - \hat{m}_h\left(X_i^1\right)\right)^2\right] = \frac{1}{n_1}\sum_{i=1}^{n_1}\mathbb{E}\left[\left(Y_i^1 - \hat{m}_h\left(X_i^1\right)\right)^2\right] \\
= \ & \mathbb{E}\left[\left(Y_1^1 - \hat{m}_h\left(X_1^1\right)\right)^2\right] = \mathbb{E}\left[\left(Y_1^1 - m\left(X_1^1\right) + m\left(X_1^1\right) - \hat{m}_h\left(X_1^1\right)\right)^2\right] \\
= \ & A_1 + A_2 + 2A_3,
\end{aligned}$$

where

$$\begin{aligned}
A_1 &:= \mathbb{E}\left[\left(Y_1^1 - m\left(X_1^1\right)\right)^2\right], \\
A_2 &:= \mathbb{E}\left[\left(m\left(X_1^1\right) - \hat{m}_h\left(X_1^1\right)\right)^2\right], \\
A_3 &:= \mathbb{E}\left[\left(Y_1^1 - m\left(X_1^1\right)\right)\cdot\left(m\left(X_1^1\right) - \hat{m}_h\left(X_1^1\right)\right)\right].
\end{aligned}$$

Term A_1 does not depend on h. In fact,

$$\begin{aligned}
A_1 &= \mathbb{E}\left[\mathbb{E}\left[\left(Y_1^1 - m\left(X_1^1\right)\right)^2\Big|_{X_1^1}\right]\right] = \mathbb{E}\left[Var\left(Y_1^1\Big|_{X_1^1}\right)\right] \\
&= \mathbb{E}\left[\sigma^2\left(X_1^1\right)\right] = \int \sigma^2(x)\, dF_1(x).
\end{aligned}$$

Moreover, carrying on with further computations in term A_2:

$$\begin{aligned} A_2 &= \mathbb{E}\left[\left(m\left(X_1^1\right) - \hat{m}_h\left(X_1^1\right)\right)^2\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[\left(m\left(X_1^1\right) - \hat{m}_h\left(X_1^1\right)\right)^2\Big|_{\left(X_1^0,Y_1^0\right),\ldots,\left(X_{n_0}^0,Y_{n_0}^0\right)}\right]\right] \\ &= \mathbb{E}\left[\int \left(\hat{m}_h(x) - m(x)\right)^2 dF_1(x)\right]. \end{aligned}$$

Finally, term A_3 results in:

$$\begin{aligned} A_3 &= \mathbb{E}\left[\mathbb{E}\left[\left(m\left(X_1^1\right) - \hat{m}_h\left(X_1^1\right)\right)\cdot\left(Y_1^1 - m\left(X_1^1\right)\right)\Big|_{\left(X_1^0,Y_1^0\right),\ldots,\left(X_{n_0}^0,Y_{n_0}^0\right),X_1^1}\right]\right] \\ &= \mathbb{E}\left[\left(m\left(X_1^1\right) - \hat{m}_h\left(X_1^1\right)\right)\cdot\left(\mathbb{E}\left[Y_1^1\Big|_{X_1^1}\right] - m\left(X_1^1\right)\right)\right] = 0, \end{aligned}$$

as a consequence of $\mathbb{E}\left[Y_1^1\Big|_{X_1^1}\right] = m\left(X_1^1\right)$. Then, expression (12) holds.

Theorem 2. *Assume (A1) from the paper, let $\{(X_1^0, Y_1^0), \ldots, (X_{n_0}^0, Y_{n_0}^0)\}$ be a simple random sample coming from the source population, and $\{X_1^1, \ldots, X_{n_1}^1\}$ a simple random sample coming from the target population. Then the MASE prediction error is*

$$MASE_{\tilde{m}_h,X^1}(h) = \frac{1}{n_1}\sum_{j=1}^{n_1}\frac{1}{f^0(X_j^1)^2}\left[\left(1-\frac{1}{n_0}\right)\cdot\left[K_h * q^0_{X_j^1}\right]^2(X_j^1) + \frac{1}{n_0}\left[(K_h)^2 * p^0_{X_j^1}\right](X_j^1)\right], \tag{13}$$

where $q_x^0(z) = (m(z) - m(x))f^0(z)$ and $p_x^0(y) = \left(\sigma_0^2(y) + (m(y) - m(x))^2\right)f^0(y)$. Similarly

$$\begin{aligned} MASE^*_{\tilde{m}_h,X^1}(h) = \frac{1}{n_1}\sum_{j=1}^{n_1}\frac{1}{\hat{f}^0_{g_X}(X_j^1)^2}&\left[\left(1-\frac{1}{n_0}\right)\cdot\left[K_h * \hat{q}^0_{X_j^1,g_X}\right]^2(X_j^1)\right. \\ &\left.+\frac{1}{n_0}\left[(K_h)^2 * \hat{p}^0_{X_j^1,g_X}\right](X_j^1) + \frac{g_Y^2\mu_2(K)}{n_0}\left[(K_h)^2 * \hat{f}^0_{g_X}\right](X_j^1)\right], \end{aligned} \tag{14}$$

where $\hat{p}^0_{x,g_X}(y) = \left(\hat{\sigma}^2_{0,g_X}(y) + (\hat{m}_{g_X}(y) - \hat{m}_{g_X}(x))^2\right)\hat{f}^0_{g_X}(y)$ and $\hat{q}^0_{x,g_X}(z) = (\hat{m}_{g_X}(z) - \hat{m}_{g_X}(x))\hat{f}^0_{g_X}(z)$, and $\hat{\sigma}^2_{0,g_X}(z) = \tilde{m}_{2,g_X}(z) - \tilde{m}^2_{g_X}(z)$ with $\tilde{m}_{2,g_X}(z) = \dfrac{\sum_{i=1}^n K_{g_X}\left(z - X_i^0\right)\left(Y_i^0\right)^2}{\sum_{i=1}^n K_{g_X}\left(z - X_i^0\right)}$.

Proof The target is to work out an explicit expression for the prediction error MASE, given by:

$$
\begin{aligned}
MASE_{\tilde{m}_h^{NW},X^1}(h) &= \frac{1}{n_1}\sum_{j=1}^{n_1}\left[\mathbb{E}_0\left[\left(\tilde{m}_h^{NW}(X_j^1)-m(X_j^1)\right)^2\right]\right] = \\
&= \frac{1}{n_1}\sum_{j=1}^{n_1}\left[Var_0\left[\frac{1}{n_0 f^0(X_j^1)}\sum_{i=1}^{n_0}K_h(X_j^1-X_i^0)(Y_i^0-m(X_j^1))\right]\right. \\
&+\left.\left(\mathbb{E}_0\left[\frac{1}{n_0 f^0(X_j^1)}\sum_{i=1}^{n_0}K_h(X_j^1-X_i^0)(Y_i^0-m(X_j^1))\right]\right)^2\right] = \\
&= \frac{1}{n_1}\sum_{j=1}^{n_1}\left[Var_0[A_1^0]+\left(\mathbb{E}_0\left[A_1^0\right]\right)^2\right] \\
&= \frac{1}{n_1}\sum_{j=1}^{n_1}Var_0[A_1^0]+\frac{1}{n_1}\sum_{j=1}^{n_1}\left(\mathbb{E}_0\left[A_1^0\right]\right)^2. \qquad (15)
\end{aligned}
$$

Focusing now on the second term of (15):

$$
\begin{aligned}
&\frac{1}{n_1}\sum_{j=1}^{n_1}\left(\mathbb{E}_0\left[A_1^0\right]\right)^2 \\
&= \frac{1}{n_1}\sum_{j=1}^{n_1}\left(\frac{1}{n_0 f^0(X_j^1)}\sum_{i=1}^{n_0}\mathbb{E}_0\left[K_h(X_j^1-X_i^0)(Y_i^0-m(X_j^1))\right]\right)^2 \\
&= \frac{1}{n_1}\sum_{j=1}^{n_1}\left(\frac{1}{f^0(X_j^1)}\mathbb{E}_0\left[K_h(X_j^1-X_1^0)(Y_1^0-m(X_j^1))\right]\right)^2 \\
&= \frac{1}{n_1}\sum_{j=1}^{n_1}\left(\frac{1}{f^0(X_j^1)}\mathbb{E}_0\left[\mathbb{E}_0\left[K_h(X_j^1-X_1^0)(Y_1^0-m(X_j^1))\mid X_1^0\right]\right]\right)^2 \\
&= \frac{1}{n_1}\sum_{j=1}^{n_1}\left(\frac{1}{f^0(X_j^1)}\mathbb{E}_0\left[K_h(X_j^1-X_1^0)\left(\mathbb{E}_0\left[Y_1^0\mid X_1^0\right]-m(X_j^1)\right)\right]\right)^2 \\
&= \frac{1}{n_1}\sum_{j=1}^{n_1}\left(\frac{1}{f^0(X_j^1)}\mathbb{E}_0\left[K_h(X_j^1-X_1^0)\left(m(X_1^0)-m(X_j^1)\right)\right]\right)^2 \\
&= \frac{1}{n_1}\sum_{j=1}^{n_1}\left(\frac{1}{f^0(X_j^1)}\int K_h(X_j^1-y)\left(m(y)-m(X_j^1)\right)f^0(y)\,dy\right)^2 \\
&= \frac{1}{n_1}\sum_{j=1}^{n_1}\left(\frac{1}{f^0(X_j^1)}\int K_h(X_j^1-y)q_{X_j^1}^0(y)\,dy\right)^2 \\
&= \frac{1}{n_1}\sum_{j=1}^{n_1}\left(\frac{1}{f^0(X_j^1)}\left[K_h * q_{X_j^1}^0\right](X_j^1)\right)^2 \qquad (16)
\end{aligned}
$$

where $q_x^0(z) = (m(z) - m(x))f^0(z)$. On the other hand,

$$\frac{1}{n_1}\sum_{j=1}^{n_1} Var_0[A_1^0] = \frac{1}{n_1}\sum_{j=1}^{n_1} Var_0\left[\frac{1}{n_0 f^0(X_j^1)}\sum_{i=1}^{n_0} K_h(X_j^1 - X_i^0)(Y_i^0 - m(X_j^1))\right]$$
$$= \frac{1}{n_1}\sum_{j=1}^{n_1}\left[\frac{1}{n_0 f^0(X_j^1)^2} Var_0\left[K_h(X_j^1 - X_1^0)(Y_1^0 - m(X_j^1))\right]\right]$$
$$= \frac{1}{n_0 n_1}\sum_{j=1}^{n_1}\frac{1}{f^0(X_j^1)^2}\left[\mathbb{E}_0\left[K_h(X_j^1 - X_1^0)^2(Y_1^0 - m(X_j^1))^2\right] - \left(\mathbb{E}_0\left[K_h(X_j^1 - X_1^0)(Y_1^0 - m(X_j^1))\right]\right)^2\right]. \quad (17)$$

The first term of (17) turns out to be:

$$\mathbb{E}_0\left[K_h(X_j^1 - X_1^0)^2(Y_1^0 - m(X_j^1))^2\right]$$
$$= \mathbb{E}_0\left[\mathbb{E}_0\left[K_h(X_j^1 - X_1^0)^2(Y_1^0 - m(X_j^1))^2 \mid X_1^0\right]\right]$$
$$= \mathbb{E}_0\left[K_h(X_j^1 - X_1^0)^2\mathbb{E}_0\left[(Y_1^0 - m(X_j^1))^2 \mid X_1^0\right]\right]$$
$$= \mathbb{E}_0\left[K_h(X_j^1 - X_1^0)^2\left[Var_0(Y_1^0 - m(X_j^1) \mid X_1^0) + \left(\mathbb{E}_0\left[Y_1^0 - m(X_j^1) \mid X_1^0\right]\right)^2\right]\right]$$
$$= \mathbb{E}_0\left[K_h(X_j^1 - X_1^0)^2\left[Var_0(Y_1^0 \mid X_1^0) + \left(m(X_1^0) - m(X_j^1)\right)^2\right]\right]$$
$$= \int K_h(X_j^1 - y)^2\left(\sigma_0^2(y) + (m(y) - m(X_j^1))^2\right) f^0(y)\,dy$$
$$= \left[(K_h)^2 * p^0_{X_j^1}\right](X_j^1), \quad (18)$$

where $p_x^0(y) = \left(\sigma_0^2(y) + (m(y) - m(x))^2\right) f^0(y)$.

Collecting terms (16) and (18) and plugging them into (17), and then plugging (16) and (17) into (15), the first part of Theorem 2 holds.

For proving the second part, the aim is to compute a closed-form expression for the bootstrap version of the

MASE:

$$
\begin{aligned}
MASE^*_{\tilde{m}_h^{NW},X^1}(h) &= \frac{1}{n_1}\sum_{j=1}^{n_1}\left[\mathbb{E}_0^*\left[\left(\tilde{m}_h^{NW*}(X_j^1)-\hat{m}_{gX}^{NW}(X_j^1)\right)^2\right]\right] = \\
&= \frac{1}{n_1}\sum_{j=1}^{n_1}\left[Var_0^*\left[\frac{1}{n_0\hat{f}_{gX}^0(X_j^1)}\sum_{i=1}^{n_0}K_h(X_j^1-X_i^{0*})\right.\right. \\
&\left.\cdot(Y_i^{0*}-\hat{m}_{gX}^{NW}(X_j^1))\right]+\left(\mathbb{E}_0^*\left[\frac{1}{n_0\hat{f}_{gX}^0(X_j^1)}\sum_{i=1}^{n_0}K_h(X_j^1-X_i^{0*})\right.\right. \\
&\left.\left.\left.\cdot(Y_i^{0*}-\hat{m}_{gX}^{NW}(X_j^1))\right]\right)^2\right] \\
&= \frac{1}{n_1}\sum_{j=1}^{n_1}\left[Var_0^*[A_1^{0*}]+\left(\mathbb{E}_0^*\left[A_1^{0*}\right]\right)^2\right] \\
&= \frac{1}{n_1}\sum_{j=1}^{n_1}Var_0^*[A_1^{0*}]+\frac{1}{n_1}\sum_{j=1}^{n_1}\left(\mathbb{E}_0^*\left[A_1^{0*}\right]\right)^2. \qquad (19)
\end{aligned}
$$

Focusing now on the second term of (19):

$$
\begin{aligned}
&\frac{1}{n_1}\sum_{j=1}^{n_1}\left(\mathbb{E}_0^*\left[A_1^{0*}\right]\right)^2 \\
&= \frac{1}{n_1}\sum_{j=1}^{n_1}\left(\frac{1}{n_0\hat{f}_{gX}^0(X_j^1)}\sum_{i=1}^{n_0}\mathbb{E}_0^*\left[K_h(X_j^1-X_i^{0*})(Y_i^{0*}-\hat{m}_{gX}^{NW}(X_j^1))\right]\right)^2 \\
&= \frac{1}{n_1}\sum_{j=1}^{n_1}\left(\frac{1}{f^0(X_j^1)}\mathbb{E}_0^*\left[K_h(X_j^1-X_1^{0*})(Y_1^{0*}-\hat{m}_{gX}^{NW}(X_j^1))\right]\right)^2 \\
&= \frac{1}{n_1}\sum_{j=1}^{n_1}\left(\frac{1}{\hat{f}_{gX}^0(X_j^1)}\mathbb{E}_0^*\left[\mathbb{E}_0^*\left[K_h(X_j^1-X_1^{0*})(Y_1^{0*}-\hat{m}_{gX}^{NW}(X_j^1))\mid X_1^{0*}\right]\right]\right)^2 \\
&= \frac{1}{n_1}\sum_{j=1}^{n_1}\left(\frac{1}{f^{0*}(X_j^1)}\mathbb{E}_0^*\left[K_h(X_j^1-X_1^{0*})\left(\mathbb{E}_0^*\left[Y_1^{0*}\mid X_1^{0*}\right]-\hat{m}_{gX}^{NW}(X_j^1)\right)\right]\right)^2 \\
&= \frac{1}{n_1}\sum_{j=1}^{n_1}\left(\frac{1}{\hat{f}_{gX}^0(X_j^1)}\mathbb{E}_0^*\left[K_h(X_j^1-X_1^{0*})\left(\hat{m}_{gX}^{NW}(X_1^{0*})-\hat{m}_{gX}^{NW}(X_j^1)\right)\right]\right)^2 \\
&= \frac{1}{n_1}\sum_{j=1}^{n_1}\left(\frac{1}{\hat{f}_{gX}^0(X_j^1)}\int K_h(X_j^1-y)\left(\hat{m}_{gX}^{NW}(y)-\hat{m}_{gX}^{NW}(X_j^1)\right)\hat{f}_{gX}^0(y)\,dy\right)^2 \\
&= \frac{1}{n_1}\sum_{j=1}^{n_1}\left(\frac{1}{\hat{f}_{gX}^0(X_j^1)}\int K_h(X_j^1-y)\hat{q}^0_{X_j^1,g}(y)\,dy\right)^2 \\
&= \frac{1}{n_1}\sum_{j=1}^{n_1}\left(\frac{1}{\hat{f}_{gX}^0(X_j^1)}\left[K_h*\hat{q}^0_{X_j^1,g}\right](X_j^1)\right)^2, \qquad (20)
\end{aligned}
$$

where $\hat{q}_x^0(z) = (\hat{m}_{g_X}^{NW}(z) - \hat{m}_{g_X}^{NW}(x))\hat{f}_{g_X}^0(z)$. On the other hand,

$$\begin{aligned}
&\frac{1}{n_1}\sum_{j=1}^{n_1} Var_0^*\left[A_1^{0*}\right] \\
&= \frac{1}{n_1}\sum_{j=1}^{n_1} Var_0^*\left[\frac{1}{n_0\hat{f}_{g_X}^0(X_j^1)}\sum_{i=1}^{n_0} K_h(X_j^1 - X_i^{0*})(Y_i^{0*} - \hat{m}_{g_X}^{NW}(X_j^1))\right] \\
&= \frac{1}{n_1}\sum_{j=1}^{n_1}\left[\frac{1}{n_0\hat{f}_{g_X}^0(X_j^1)^2} Var_0^*\left[K_h(X_j^1 - X_1^{0*})(Y_1^{0*} - \hat{m}_{g_X}^{NW}(X_j^1))\right]\right] \\
&= \frac{1}{n_0 n_1}\sum_{j=1}^{n_1}\frac{1}{\hat{f}_{g_X}^0(X_j^1)^2}\left[\mathbb{E}_0^*\left[K_h(X_j^1 - X_1^{0*})^2(Y_1^{0*} - \hat{m}_{g_X}^{NW}(X_j^1))^2\right] - \right. \\
&\left.\left(\mathbb{E}_0^*\left[K_h(X_j^1 - X_1^{0*})(Y_1^{0*} - \hat{m}_{g_X}^{NW}(X_j^1))\right]\right)^2\right]. \qquad (21)
\end{aligned}$$

The first term of (21) turns out to be:

$$\begin{aligned}
&\mathbb{E}_0^*\left[K_h(X_j^1 - X_1^{0*})^2(Y_1^{0*} - \hat{m}_{g_X}^{NW}(X_j^1))^2\right] \\
&= \mathbb{E}_0^*\left[\mathbb{E}_0^*\left[K_h(X_j^1 - X_1^{0*})^2(Y_1^{0*} - \hat{m}_{g_X}^{NW}(X_j^1))^2 \mid X_1^{0*}\right]\right] \\
&= \mathbb{E}_0^*\left[K_h(X_j^1 - X_1^{0*})^2\mathbb{E}_0^*\left[(Y_1^{0*} - \hat{m}_{g_X}^{NW}(X_j^1))^2 \mid X_1^{0*}\right]\right] \\
&= \mathbb{E}_0^*\left[K_h(X_j^1 - X_1^{0*})^2\left[Var_0^*(Y_1^{0*} - \hat{m}_{g_X}^{NW}(X_j^1) \mid X_1^{0*})\right.\right. \\
&\left.\left. + \left(\mathbb{E}_0^*\left[Y_1^{0*} - \hat{m}_{g_X}^{NW}(X_j^1) \mid X_1^{0*}\right]\right)^2\right]\right] \\
&= \mathbb{E}_0^*\left[K_h(X_j^1 - X_1^{0*})^2\left[Var_0^*(Y_1^{0*} \mid X_1^{0*}) + \left(\hat{m}_{g_X}^{NW}(X_1^{0*}) - \hat{m}_{g_X}^{NW}(X_j^1)\right)^2\right]\right] \\
&= \mathbb{E}_0^*\left[K_h(X_j^1 - X_1^{0*})^2\left[\sigma_0^{*2}(X_1^{0*}) + \left(\hat{m}_{g_X}^{NW}(X_1^{0*}) - \hat{m}_{g_X}^{NW}(X_j^1)\right)^2\right]\right] \\
&= \mathbb{E}_0^*\left[K_h(X_j^1 - X_1^{0*})^2\left[\hat{\sigma}_{0,g}^2(X_1^{0*}) + g_Y^2\mu_2(K) + \left(\hat{m}_{g_X}^{NW}(X_1^{0*}) - \hat{m}_{g_X}^{NW}(X_j^1)\right)^2\right]\right] \\
&= \int K_h(X_j^1 - y)^2\left(\hat{\sigma}_{0,g}^2(y) + g_Y^2\mu_2(K) + (\hat{m}_{g_X}^{NW}(y) - \hat{m}_{g_X}^{NW}(X_j^1))^2\right)\hat{f}_{g_X}^0(y)\,dy \\
&= g_Y^2\mu_2(K)\int K_h(X_j^1 - y)^2\hat{f}_{g_X}^0(y)\,dy \\
&+ \int K_h(X_j^1 - y)^2\left(\hat{\sigma}_{0,g}^2(y) + (\hat{m}_{g_X}^{NW}(y) - \hat{m}_{g_X}^{NW}(X_j^1))^2\right)\hat{f}_{g_X}^0(y)\,dy \\
&= \frac{g_Y^2\mu_2(K)}{n_0}\sum_{i=1}^{n_0}\int K_h(X_j^1 - y)^2 K_{g_X}(y - X_i^0)\,dy \\
&+ \int K_h(X_j^1 - y)^2\hat{p}_{X_j^1,g}^0(y)\,dy \\
&= \frac{g_Y^2\mu_2(K)}{n_0}\sum_{i=1}^{n_0}\left[(Kh)^2 * K_{g_X}\right](X_j^1 - X_i^0) + \left[K_h * \hat{p}_{X_j^1,g}^0\right](X_j^1), \qquad (22)
\end{aligned}$$

where $\hat{p}_{x,g}^0(y) = \left(\hat{\sigma}_{0,g}^2(y) + (\hat{m}_{g_X}^{NW}(y) - \hat{m}_{g_X}^{NW}(x))^2\right)\hat{f}_{g_X}^0(y)$.

Collecting terms (20) and (22) and plugging them into (21), and then plugging (20) and (21) into (19), Theorem 2 holds.

Corollary 1. *If K is a Gaussian kernel, then expression (14) can be rewritten as follows:*

$$MASE^{*}_{\tilde{m}_h,X^1}(h) = \frac{1}{n_1}\sum_{j=1}^{n_1}\frac{1}{\hat{f}^0_g(X^1_j)^2}\left[\frac{n_0-1}{n_0^3}\cdot\left[\sum_{i=1}^{n_0}K_h * K_{gx}\left(X^1_j - X^0_i\right)\cdot\left(Y^0_i - \hat{m}_{gx}\left(X^1_j\right)\right)\right]^2 + \frac{1}{n_0^2}\sum_{i=1}^{n_0}\left[(K_h)^2 * K_{gx}\right](X^1_j - X^0_i)\cdot\left[Y^0_i - \hat{m}_{gx}\left(X^1_j\right)\right]^2 + \frac{g^2\mu_2(K)}{n_0^2}\sum_{i=1}^{n_0}\left[(K_h)^2 * K_g\right](X^1_j - X^0_i)\right]. \tag{23}$$

Proof Carrying on with computations in expression (14), we have:

$$\begin{aligned}\hat{q}^0_{x,gx}(z) &= \left(\hat{m}_{gx}(z) - \hat{m}_{gx}(x)\right)\hat{f}^0_{gx}(z) = \frac{1}{n_0}\sum_{i=1}^{n_0}\left(K_{gx}(z - X^0_i)Y^0_i - \hat{m}_{gx}(x)\sum_{i=1}^{n_0}K_{gx}\left(z - X^0_i\right)\right)\\ &= \frac{1}{n_0}\sum_{i=1}^{n_0}K_{gx}\left(z - X^0_i\right)\left(Y^0_i - \hat{m}_{gx}(x)\right).\end{aligned}$$

Then,

$$\begin{aligned}\left[K_h * \hat{q}^0_{X^1_j,gx}\right]^2(X^1_j) &= K_h * \hat{q}^0_{X^1_j,gx}(X^1_j)^2\\ &= \left[K_h * \frac{1}{n_0}\sum_{i=1}^{n_0}K_{gx}\left(\cdot - X^0_i\right)\cdot\left(Y^0_i - \hat{m}_{gx}(X^1_j)\right)\right](X^1_j)^2\\ &= \left[\frac{1}{n_0}\sum_{i=1}^{n_0}\left[K_h * K_{gx}\left(\cdot - X^0_i\right)\right](X^1_j)\cdot\left(Y^0_i - \hat{m}_{gx}(X^1_j)\right)\right]^2\\ &= \left[\frac{1}{n_0}\sum_{i=1}^{n_0}K_h * K_{gx}\left(X^1_j - X^0_i\right)\cdot\left(Y^0_i - \hat{m}_{gx}(X^1_j)\right)\right]^2.\end{aligned} \tag{24}$$

On the other hand, using that $\hat{\sigma}^2_{0,gx}(z) := \hat{m}_{2,gx}(z) - \hat{m}_{gx}(z)^2$, where $\hat{m}_{k,gx}(z)$ is defined as $\hat{m}_{k,gx}(z) = \frac{\sum_{i=1}^{n_0}K_{gx}(z - X^0_i)(Y^0_i)^k}{\sum_{i=1}^{n_0}K_{gx}(z - X^0_i)}$, then:

$$\begin{aligned}\hat{p}^0_{x,gx}(z) &= \left[\hat{\sigma}^2_{0,gx}(z) + \left(\hat{m}_{gx}(z) - \hat{m}_{gx}(x)\right)^2\right]\hat{f}^0_{gx}(z)\\ &= \left[\hat{m}_{2,gx}(z) - 2\hat{m}_{gx}(z)\hat{m}_{gx}(x) + \hat{m}_{gx}(x)^2\right]\hat{f}_{gx}(z) = \frac{1}{n_0}\sum_{i=1}^{n_0}K_{gx}(z - X^0_i)(Y^0_i)^2\\ &- 2\hat{m}_g(x)\frac{1}{n_0}\sum_{i=1}^{n_0}K_{gx}(z - X^0_i)Y^0_i + \hat{m}_{gx}(x)^2\frac{1}{n_0}\sum_{i=1}^{n_0}K_{gx}(y - X^0_i)\\ &= \frac{1}{n_0}\sum_{i=1}^{n_0}K_{gx}(z - X^0_i)\left[\left(Y^0_i\right)^2 - 2\hat{m}_{gx}(x)Y^0_i + \hat{m}_{gx}(x)^2\right].\end{aligned}$$

Therefore, straightforward calculations lead to:

$$\begin{aligned}(K_h)^2 * \hat{p}^0_{X^1_j,gx}(X^1_j) &= \frac{1}{n_0}\sum_{i=1}^{n_0}(K_h)^2 * K_{gx}(X^1_j - X^0_i)\cdot\left[\left(Y^0_i\right)^2 - 2\hat{m}_{gx}(X^1_j)Y^0_i + \hat{m}_{gx}(X^1_j)^2\right]\\ &= \frac{1}{n_0}\sum_{i=1}^{n_0}(K_h)^2 * K_{gx}(X^1_j - X^0_i)\cdot\left(Y^0_i - \hat{m}_{gx}(X^1_j)\right)^2.\end{aligned} \tag{25}$$

Finally, collecting expressions (24) and (25) and plugging them in expression (14), Corollary 1 is concluded.

1.2 | Asymptotic theory

1.2.1 | Asymptotic expression for the criterion functions

Lemma 1. *Under regularity conditions (B1)-(B4) from the paper, the function $MISE^a$ can be expressed as*

$$MISE^a(h) = \frac{R(K)}{n_0\, h}\int \gamma(x)dx + \frac{h^4\mu_2(K)^2}{4}\int \beta(x)\,dx + O(h^6) + O\left(\frac{h}{n_0}\right). \tag{26}$$

where $\beta(x) = \left[m''(x)^2 + \frac{4m'(x)m''(x)\left(f^0\right)'(x)}{f^0(x)} + \frac{4m'(x)^2\left(f^0\right)'(x)^2}{f^0(x)^2}\right] f^1(x)$ *and* $\gamma(x) = \frac{\sigma^2(x) f^1(x)}{f^0(x)}$. *The asymptotic version of expression (26) is given by:*

$$AMISE^a(h) \quad = \quad \frac{R(K)}{n_0\, h}\int \gamma(x)dx + \frac{h^4}{4}\mu_2(K)^2\int \beta(x)\,dx. \tag{27}$$

Proof Using a Taylor expansion and a change of variable, we obtain:

$$\begin{aligned}
&\mathbb{E}\left[\int\left(\tilde{m}_h^{NW}(x) - m(x)\right)^2 dF_1(x)\right] \\
&= \mathbb{E}\left[\int \frac{1}{n_0 f^0(x)}\sum_{i=1}^{n_0} K_h\left(x - X_i^0\right)\left(Y_i^0 - m(x)\right)\right]^2 dF_1(x) \\
&= \frac{1}{n_0^2}\int \frac{1}{f^0(x)^2}\mathbb{E}\left[\sum_{i=1}^{n_0} K_h\left(x - X_i^0\right)\left(Y_i^0 - m(x)\right)\right]^2 dF_1(x) \\
&= \frac{1}{n_0^2}\int \frac{1}{f^0(x)^2}\left[Var\left[\sum_{i=1}^{n_0} K_h\left(x - X_i^0\right)\left(Y_i^0 - m(x)\right)\right]\right. \\
&\left.+\left(\mathbb{E}\left[\sum_{i=1}^{n_0} K_h\left(x - X_i^0\right)\left(Y_i^0 - m(x)\right)\right]\right)^2\right] dF_1(x) \\
&= \frac{1}{n_0^2}\int \frac{1}{f^0(x)^2}\left[\sum_{i=1}^{n_0} Var\left[K_h\left(x - X_i^0\right)\left(Y_i^0 - m(x)\right)\right]\right. \\
&\left.+\left(\sum_{i=1}^{n_0}\mathbb{E}\left[K_h\left(x - X_i^0\right)\left(Y_i^0 - m(x)\right)\right]\right)^2\right] dF_1(x) \\
&= \frac{1}{n_0^2}\int \frac{1}{f^0(x)^2}\left[n_0 Var\left[K_h\left(x - X_1^0\right)\left(Y_1^0 - m(x)\right)\right]\right. \\
&\left.+n_0^2\left(\mathbb{E}\left[K_h\left(x - X_1^0\right)\left(Y_1^0 - m(x)\right)\right]\right)^2\right] dF_1(x)
\end{aligned}$$

$$= \frac{1}{n_0} \int \frac{1}{f^0(x)^2} Var\left[K_h\left(x - X_1^0\right)\left(Y_1^0 - m(x)\right)\right] dF_1(x)$$
$$+ \int \frac{1}{f^0(x)^2}\left(\mathbb{E}\left[K_h\left(x - X_1^0\right)\left(Y_1^0 - m(x)\right)\right]\right)^2 dF_1(x)$$
$$= \frac{1}{n_0} \int \frac{1}{f^0(x)^2}\left[\mathbb{E}\left[K_h\left(x - X_1^0\right)^2\left(Y_1^0 - m(x)\right)^2\right]\right.$$
$$\left. - \left(\mathbb{E}\left[K_h\left(x - X_1^0\right)\left(Y_1^0 - m(x)\right)\right]\right)^2\right] dF_1(x)$$
$$+ \int \frac{1}{f^0(x)^2}\left(\mathbb{E}\left[K_h\left(x - X_1^0\right)\left(Y_1^0 - m(x)\right)\right]\right)^2 dF_1(x)$$
$$= \frac{1}{n_0} \int \frac{1}{f^0(x)^2}\mathbb{E}\left[K_h\left(x - X_1^0\right)^2\left(Y_1^0 - m(x)\right)^2\right] dF_1(x)$$
$$+ \frac{n_0 - 1}{n_0} \int \frac{1}{f^0(x)^2}\left(\mathbb{E}\left[K_h\left(x - X_1^0\right)\left(Y_1^0 - m(x)\right)\right]\right)^2 dF_1(x)$$
$$= \frac{1}{n_0} \int \frac{1}{f^0(x)^2}\mathbb{E}\left[\mathbb{E}\left[K_h\left(x - X_1^0\right)^2\left(Y_1^0 - m(x)\right)^2 \mid X_1^0\right]\right] dF_1(x)$$
$$+ \frac{n_0 - 1}{n_0} \int \frac{1}{f^0(x)^2}\left(\mathbb{E}\left[\mathbb{E}\left[K_h\left(x - X_1^0\right)\left(Y_1^0 - m(x)\right) \mid X_1^0\right]\right]\right)^2 dF_1(x)$$
$$= \frac{1}{n_0} \int \frac{1}{f^0(x)^2}\mathbb{E}\left[K_h\left(x - X_1^0\right)^2 \mathbb{E}\left[\left(Y_1^0 - m(x)\right)^2 \mid X_1^0\right]\right] dF_1(x)$$
$$+ \frac{n_0 - 1}{n_0} \int \frac{1}{f^0(x)^2}\left(\mathbb{E}\left[K_h\left(x - X_1^0\right)\left(\mathbb{E}\left[Y_1^0 \mid X_1^0\right] - m(x)\right)\right]\right)^2 dF_1(x)$$
$$= \frac{1}{n_0} \int \frac{1}{f^0(x)^2}\mathbb{E}\left[K_h\left(x - X_1^0\right)^2\left(Var\left(Y_1^0 - m(x) \mid X_1^0\right)\right.\right.$$
$$\left.\left. + \left(\mathbb{E}\left[Y_1^0 - m(x) \mid X_1^0\right]\right)^2\right)\right] dF_1(x)$$
$$+ \frac{n_0 - 1}{n_0} \int \frac{1}{f^0(x)^2}\left(\mathbb{E}\left[K_h\left(x - X_1^0\right)\left(m(X_1^0) - m(x)\right)\right]\right)^2 dF_1(x)$$
$$= \frac{1}{n_0} \int \frac{1}{f^0(x)^2}\mathbb{E}\left[K_h\left(x - X_1^0\right)^2\left(Var\left(Y_1^0 \mid X_1^0\right) + \left(\mathbb{E}\left[Y_1^0 \mid X_1^0\right] - m(x)\right)^2\right)\right] dF_1(x)$$
$$+ \frac{n_0 - 1}{n_0} \int \frac{1}{f^0(x)^2}\left(\mathbb{E}\left[K_h\left(x - X_1^0\right)\left(m(X_1^0) - m(x)\right)\right]\right)^2 dF_1(x)$$
$$= \frac{1}{n_0} \int \frac{1}{f^0(x)^2}\mathbb{E}\left[K_h\left(x - X_1^0\right)^2\left(\sigma^2(X_1^0) + \left(m(X_1^0) - m(x)\right)^2\right)\right] dF_1(x)$$
$$+ \frac{n_0 - 1}{n_0} \int \frac{1}{f^0(x)^2}\left(\mathbb{E}\left[K_h\left(x - X_1^0\right)\left(m(X_1^0) - m(x)\right)\right]\right)^2 dF_1(x)$$
$$= \frac{1}{n_0 h^2} \int \frac{1}{f^0(x)^2}\mathbb{E}\left[K\left(\frac{x - X_1^0}{h}\right)^2\left(\sigma^2(X_1^0) + \left(m(X_1^0) - m(x)\right)^2\right)\right] dF_1(x)$$
$$+ \frac{n_0 - 1}{n_0 h^2} \int \frac{1}{f^0(x)^2}\left(\mathbb{E}\left[K\left(\frac{x - X_1^0}{h}\right)\left(m(X_1^0) - m(x)\right)\right]\right)^2 dF_1(x)$$
$$= \frac{1}{n_0 h^2} \int \frac{1}{f^0(x)^2} \int K\left(\frac{y - x}{h}\right)^2\left(\sigma^2(y) + (m(y) - m(x))^2\right) f^0(y)\, dy\, dF_1(x)$$
$$+ \frac{n_0 - 1}{n_0 h^2} \int \frac{1}{f^0(x)^2}\left(\int K\left(\frac{y - x}{h}\right)(m(y) - m(x))\, f^0(y)\, dy\right)^2 dF_1(x)$$
$$= \frac{1}{n_0 h} \int \frac{1}{f^0(x)^2} \int K(u)^2\left(\sigma^2(x + hu) + (m(x + hu) - m(x))^2\right)$$
$$f^0(x + hu)\, du\, dF_1(x) + \frac{n_0 - 1}{n_0} \int \frac{1}{f^0(x)^2}$$
$$\left(\int K(u)\,(m(x + hu) - m(x))\, f^0(x + hu)\, du\right)^2 dF_1(x). \tag{28}$$

Let us denote $\psi(y) = (m(y) - m(x))\, f^0(y)$. Then, $\psi(x) = 0; \psi'(x) = m'(y) f(y) + (m(y) - m(x)) \left(f^0\right)'(y); \psi''(y) = m''(y) f^0(y) + 2m'(y) \left(f^0\right)'(y) + (m(y) - m(x)) \left(f^0\right)''(y)$. Using Taylor expansion:

$$\begin{aligned}\int K(u)\psi(x+hu)\, du &= \int K(u)\psi(x)\, du + h \int u\, K(u)\, \psi'(x)\, du \\ &+ \frac{h^2}{2} \int u^2 K(u)\psi''(x)\, du + \frac{h^3}{3!} \int u^3 K(u)\psi'''(x)\, du + O(h^4) \\ &= \frac{h^2}{2}\psi''(x)\mu_2(K) + O(h^4),\end{aligned}$$

which leads to:

$$\begin{aligned}&\frac{n_0 - 1}{n_0} \int \frac{1}{f^0(x)^2} \left(\int K(u)\, \psi(x+hu)\, du \right)^2 dF_1(x) \\ &= \frac{n_0 - 1}{n_0} \int \frac{1}{f^0(x)^2} \frac{h^4}{4} \psi''(x)^2 \mu_2(K)^2 dF_1(x) + O(h^6) \\ &= \frac{(n_0 - 1)h^4}{n_0} \frac{1}{4} \mu_2(K)^2 \int \left[m''(x)^2 + \frac{4m'(x)m''(x)\left(f^0\right)'(x)}{f^0(x)} \right. \\ &\left. + \frac{4m'(x)^2 \left(f^0\right)'(x)^2}{f^0(x)^2} \right] f^1(x) dx + O(h^6).\end{aligned} \tag{29}$$

On the other hand, denote $\phi(y) = \left[\sigma^2(y) + (m(y) - m(x))^2\right] f^0(y)$. Then, $\phi(x) = \sigma^2(x) f^0(x)$ and using Taylor expansion:

$$\begin{aligned}&\frac{1}{n_0\, h} \int \frac{1}{f^0(x)^2} \int K(u)^2 \phi(x+hu)\, du\, dF_1(x) \\ &= \frac{R(K)}{n_0\, h} \int \frac{1}{f^0(x)^2} \sigma^2(x) f^0(x) f^1(x)\, dx + O\left(\frac{h}{n_0}\right) \\ &= \frac{R(K)}{n_0\, h} \int \frac{\sigma^2(x)\, f^1(x)}{f^0(x)} dx + O\left(\frac{h}{n_0}\right).\end{aligned} \tag{30}$$

Assembling terms (29) and (30), and inserting them in expression (28), we have that Lemma 1 holds.

5[Technical result] Consider a sequence of bandwidths $c\, n_0^{-1/5}, c > 0$ and the function $MISE^a(h)$, given in (26) it turns out:

$$\begin{aligned}MISE^a\left(c\, n_0^{-1/5}\right) &= R(K) n_0^{-4/5} c^{-1} \int \frac{\sigma^2(x) f^1(x)}{f^0(x)} dx \\ &+ \frac{c^4\, n_0^{-4/5}}{4} \mu_2(K)^2 \int \left[m''(x)^2 + \frac{4m'(x)m''(x)\left(f^0\right)'(x)}{f^0(x)} \right. \\ &\left. + \frac{4m'(x)^2 \left(f^0\right)'(x)^2}{f^0(x)^2} \right] f^1(x)\, dx + O(n_0^{-6/5}),\end{aligned}$$

so that

$$\lim_{n_0\to\infty} n_0^{4/5} MISE^a\left(c\, n_0^{-1/5}\right) = R(K)c^{-1}\int \frac{\sigma^2(x)f^1(x)}{f^0(x)}dx + \frac{c^4}{4}\mu_2(K)^2\int\left[m''(x)^2 + \frac{4m'(x)m''(x)\left(f^0\right)'(x)}{f^0(x)} + \frac{4m'(x)^2\left(f^0\right)'(x)^2}{f^0(x)^2}\right]f^1(x)\,dx. \tag{31}$$

Then,

$$\lim_{n_0\to\infty} n_0^{1/5}\cdot h_{MISE^a} = c_0. \tag{32}$$

Proof Given that h_{MISE^a} is the value which minimizes the function $MISE^a$, then

$$n_0^{4/5} MISE^a\left(c_0\cdot n_0^{-1/5}\right) \geq n_0^{4/5} MISE^a\left(h_{MISE^a}\right), \forall n_0 \in \mathbb{N}, \tag{33}$$

which implies that

$$R(K)c_0^{-1}\int \frac{\sigma^2(x)f^1(x)}{f^0(x)}dx + \frac{c_0^4}{4}\mu_2(K)^2\int\left[m''(x)^2 + \frac{4m'(x)m''(x)\left(f^0\right)'(x)}{f^0(x)} + \frac{4m'(x)^2\left(f^0\right)'(x)^2}{f^0(x)^2}\right]f^1(x)\,dx \geq \lim_{n_0\to\infty}\sup n_0^{4/5} MISE^a(h_{MISE^a}) \tag{34}$$

Bringing together expressions (26) and (34), we can conclude that:

$$\lim_{n_0\to\infty}\sup n_0^{1/5} h_{MISE^a} < \infty.$$

Indeed,

$$\lim_{n_0\to\infty}\sup n_0^{4/5} MISE^a(h_{MISE^a}) \geq \lim_{n_0\to\infty}\sup\left[\frac{\left(n_0^{1/5}\, h_{MISE^a}\right)^4}{4}\mu_2(K)^2\int\left[m''(x)^2 + \frac{4m'(x)m''(x)\left(f^0\right)'(x)}{f^0(x)} + \frac{4m'(x)^2\left(f^0\right)'(x)^2}{f^0(x)^2}\right]f^1(x)\,dx\right],$$

and together with expression (34) lead to conclude that

$$k \geq \lim_{n_0 \to \infty} \sup \left[\frac{\left(n_0^{1/5} h_{MISE^a}\right)^4}{4} \mu_2(K)^2 \int \left[m''(x)^2 + \frac{4m'(x)m''(x)\left(f^0\right)'(x)}{f^0(x)} + \frac{4m'(x)^2 \left(f^0\right)'(x)^2}{f^0(x)^2} \right] f^1(x)\, dx \right],$$

being k a real positive number, which implies that:

$$\lim_{n_0 \longrightarrow \infty} \sup \left(n_0^{1/5} h_{MISE^a}\right)^4 \leq k \Leftrightarrow \lim_{n_0 \longrightarrow \infty} \sup \left(n_0^{1/5} h_{MISE^a}\right) \leq k.$$

By means of similar calculations, we can prove that $\lim_{n_0 \to \infty} \inf n_0^{1/5} h_{MISE^a} > 0$. As a matter of fact,

$$\lim_{n_0 \to \infty} \sup n_0^{4/5} MISE^a(h_{MISE^a}) \geq \lim_{n_0 \to \infty} \sup \left[\frac{R(K)}{n_0^{1/5} h_{MISE^a}} \int \frac{\sigma^2(x) f^1(x)}{f^0(x)} dx \right],$$

which implies, together with expression (34), that

$$\lim_{n_0 \to \infty} \sup a_{n_0} = \lim_{n_0 \to \infty} \sup \left[\frac{1}{n_0^{1/5} h_{MISE^a}} \right] = k, \forall k \in \mathbb{R}^+. \tag{35}$$

Given that a_{n_0} is a positive sequence and using expression (35), we can conclude that $\lim_{n_0 \to \infty} \inf n_0^{1/5} h_{MISE^a} > 0$.

From both limit conditions, we can state that there exist two numbers $L, U \in \mathbb{R}^+, L < U$, satisfying:

$$L \leq n_0^{1/5} h_{MISE^a} \leq U, \text{ for almost all } n_0 \in \mathbb{N}. \tag{36}$$

Using expressions (26) and (36), we obtain:

$$\begin{aligned}
& n_0^{4/5} MISE^a(h_{MISE^a}) \\
& = R(K) \left(n_0^{1/5} h_{MISE^a}\right)^{-1} \int \frac{\sigma^2(x) f^1(x)}{f^0(x)} dx + \frac{\left(n_0^{1/5} h_{MISE^a}\right)^4}{4} \mu_2(K)^2 \\
& \int \left[m''(x)^2 + \frac{4m'(x)m''(x)\left(f^0\right)'(x)}{f^0(x)} + \frac{4m'(x)^2\left(f^0\right)'(x)^2}{f^0(x)^2} \right] f^1(x)\, dx \\
& + O\left(\left(n_0^{1/5} h_{MISE^a}\right)^4 h_{MISE^a}^2\right) + O\left(n_0^{-1/5} h_{MISE^a}\right)
\end{aligned}$$

$$= R(K)\left(n_0^{1/5}h_{MISE^a}\right)^{-1}\int\frac{\sigma^2(x)f^1(x)}{f^0(x)}dx+\frac{\left(n_0^{1/5}h_{MISE^a}\right)^4}{4}\mu_2(K)^2$$
$$\int\left[m''(x)^2+\frac{4m'(x)m''(x)\left(f^0\right)'(x)}{f^0(x)}+\frac{4m'(x)^2\left(f^0\right)'(x)^2}{f^0(x)^2}\right]f^1(x)\,dx$$
$$+O\left(h^2_{MISE^a}\right)+O\left(n_0^{-2/5}\right). \quad (37)$$

Consider a subsequence of

$$\left\{n_0^{1/5}h_{MISE^a}\right\}_{n_0\in\mathbb{N}}, \quad (38)$$

which converges to a real number l (which is positive as a consequence of expression (36)). Furthermore, expression (37) assures that the corresponding subsequence of

$$\left\{n_0^{4/5}MISE^a\left(h_{MISE^a}\right)\right\}_{n_0\in\mathbb{N}},$$

converges to

$$R(K)l^{-1}\int\frac{\sigma^2(x)f^1(x)}{f^0(x)}dx+\frac{l^4}{4}\mu_2(K)^2\int\left[m''(x)^2+\frac{4m'(x)m''(x)\left(f^0\right)'(x)}{f^0(x)}\right. \quad (39)$$
$$\left.+\frac{4m'(x)^2\left(f^0\right)'(x)^2}{f^0(x)^2}\right]f^1(x)\,dx+O\left(h^2_{MISE^a}\right)+O\left(n_0^{-2/5}\right).$$

Moreover, expression (34) guarantees that

$$R(K)c_0^{-1}\int\frac{\sigma^2(x)f^1(x)}{f^0(x)}dx \quad (40)$$
$$+\frac{c_0^4}{4}\mu_2(K)^2\int\left[m''(x)^2+\frac{4m'(x)m''(x)\left(f^0\right)'(x)}{f^0(x)}+\frac{4m'(x)^2\left(f^0\right)'(x)^2}{f^0(x)^2}\right]f^1(x)\,dx$$
$$\geq R(K)l^{-1}\int\frac{\sigma^2(x)f^1(x)}{f^0(x)}dx+\frac{l^4}{4}\mu_2(K)^2\int\left[m''(x)^2+\frac{4m'(x)m''(x)\left(f^0\right)'(x)}{f^0(x)}\right.$$
$$\left.+\frac{4m'(x)^2\left(f^0\right)'(x)^2}{f^0(x)^2}\right]f^1(x)\,dx. \quad (41)$$

As a consequence of c_0 being a strict absolute minimum (in $c > 0$) of the second addend of expression (31), the previous inequality can only be satisfied when $l = c_0$. This reasoning establishes that c_0 is the unique adherent point of the sequence given in (38). Accordingly, considering expression (36), the proof is concluded.

Up to now, we have obtained an asymptotic result concerning a first order bandwidth which minimizes $MISE^a$. From now on, we investigate the second order term further.

6[Technical result] Consider the sequence of functions defined below and L, U satisfying expression (36).

$$\Gamma_{n_0}(z) := MISE^a(z\, n_0^{-1/5}), z > 0, n_0 \in \mathbb{N},$$

and

$$\gamma_{n_0} := \arg \min_{L \leq z \leq U} \Gamma_{n_0}(z).$$

It turns out that the relation among these two functions and expression (26) is as follows:

$$h_{MISE^a} = \gamma_{n_0} n_0^{-1/5}, \text{ for almost all } n_0 \in \mathbb{N}. \tag{42}$$

Proof Expression (26) guarantees the existence of L, U verifying expression (36). In addition, the continuity of the kernel function K assures the continuity of $MISE^a(h)$. Therefore, it exists a point, namely γ_{n_0}, within the interval $[L, U]$, in which the function Γ_{n_0} attains its minimum, whether it is unique or not.

Nevertheless, given that inequalities in expression (36) are fulfilled, any minimizer of Γ_{n_0} is attained within the interval $[L, U]$. Indeed, it it were not the case, there would exist a minimizer z_0 outside $[L, U]$ verifying

$$\Gamma_{n_0}(z_0) \leq \Gamma_{n_0}(z), \forall z > 0, \tag{43}$$

and therefore,

$$MISE^a(z_0\, n_0^{-1/5}) \leq MISE^a(h), \forall h > 0, \tag{44}$$

so that $z_0\, n_0^{-1/5}$ would be a minimizer of $MISE^a$, but it would not satisfy inequalities given in (36) for that particular $n_0 \in \mathbb{N}$. Finally, the equivalence between expressions (43) and (44) provides (42), and the proof is concluded.

In particular, the function Γ_{n_0} satisfies

$$\begin{aligned}\Gamma_{n_0}(z) = {} & R(K) n_0^{-4/5} z^{-1} \int \frac{\sigma^2(x) f^1(x)}{f^0(x)} dx \\ & + \frac{z^4\, n_0^{-4/5}}{4} \mu_2(K)^2 \int \left[m''(x)^2 + \frac{4m'(x) m''(x) \left(f^0\right)'(x)}{f^0(x)} \right. \\ & \left. + \frac{4m'(x)^2 \left(f^0\right)'(x)^2}{f^0(x)^2} \right] f^1(x)\, dx + O(z^6\, n_0^{-6/5}) + O\left(z\, n_0^{-6/5}\right), z > 0\end{aligned}$$

If we define the function Λ_{n_0} such that

$$\Lambda_{n_0}(z) := n_0^{4/5} \Gamma_{n_0}(z), z > 0,$$

then,

$$\Lambda_{n_0}(z) = R(K)z^{-1}\int \frac{\sigma^2(x)f^1(x)}{f^0(x)}dx + \frac{z^4}{4}\mu_2(K)^2\int\left[m''(x)^2 + \frac{4m'(x)m''(x)\left(f^0\right)'(x)}{f^0(x)}\right.$$
$$\left.+\frac{4m'(x)^2\left(f^0\right)'(x)^2}{f^0(x)^2}\right]f^1(x)\,dx + O\left(z\,n_0^{-2/5}\right), z > 0.$$

If we restrict to those z within the interval $[L, U]$, then

$$\Lambda_{n_0}(z) = R(K)z^{-1}\int \frac{\sigma^2(x)f^1(x)}{f^0(x)}dx + \frac{z^4}{4}\mu_2(K)^2\int\left[m''(x)^2 + \frac{4m'(x)m''(x)\left(f^0\right)'(x)}{f^0(x)}\right.$$
$$\left.+\frac{4m'(x)^2\left(f^0\right)'(x)^2}{f^0(x)^2}\right]f^1(x)\,dx + O\left(n_0^{-2/5}\right),$$

uniformly in $z \in [L, U]$.

Define now the function

$$T(z) := R(K)z^{-1}\int \frac{\sigma^2(x)f^1(x)}{f^0(x)}dx + \frac{z^4}{4}\mu_2(K)^2\int\left[m''(x)^2 + \frac{4m'(x)m''(x)\left(f^0\right)'(x)}{f^0(x)}\right.$$
$$\left.+\frac{4m'(x)^2\left(f^0\right)'(x)^2}{f^0(x)^2}\right]f^1(x)\,dx, \tag{45}$$

then we have,

$$\sup_{L\le z\le U}\left|\Lambda_{n_0}(z) - T(z)\right| = O\left(n_0^{-2/5}\right). \tag{46}$$

Denote, for the sake of brevity,

$$a := \frac{1}{4}\mu_2(K)^2\int\left[m''(x)^2 + \frac{4m'(x)m''(x)\left(f^0\right)'(x)}{f^0(x)} + \frac{4m'(x)^2\left(f^0\right)'(x)^2}{f^0(x)^2}\right]f^1(x)\,dx$$
$$= \frac{1}{4}\mu_2(K)^2\int\left[m''(x)f^0(x) + 2\,m'(x)\left(f^0\right)'(x)\right]^2 f^1(x)\left(f^0(x)\right)^{-1}dx, \text{ and}$$
$$b := R(K)\int \frac{\sigma^2(x)f^1(x)}{f^0(x)}dx.$$

Then, $T(z) = b\,z^{-1} + z^4\,a$. Due to the fact that the kernel K is a nonnegative function and $\int K(u)\,du = 1$, then $R(K) > 0$ and $\mu_2(K) > 0$. Moreover, assume that K is bounded and $\int u^2K(u)\,du < \infty$, $R(K)$ and $\mu_2(K)$ are finite numbers. In addition, assume that the density function, f^0, is at least one time differentiable and its first derivative is bounded and continuous in point x. As for the regression function m, assume it is two times differentiable and its first and second derivatives are bounded and continuous in point x. Then, we can conclude that $a > 0$ and $b > 0$.

7[Technical result] Consider T the function defined in (45), it turns out that T attains a strict relative minimum in

$$z_0 := \left(\frac{b}{4\,a}\right)^{1/5}. \tag{47}$$

Furthermore, fix $\delta > 0$ a real number,

$$T(z_0+\delta) - T(z_0) > 6\,a\,z_0^2\,\delta^2. \tag{48}$$

If $\delta < z_0$, then we have

$$T(z_0-\delta) - T(z_0) > 6\,a\,z_0^2\,\delta^2 - 4\,a\,z_0\,\delta^3. \tag{49}$$

Proof Function T is differentiable in $(0,+\infty)$ and its derivative is given by:

$$T'(z) = 4\,a\,z^3 + b\,z^{-2}, \forall z > 0.$$

It is straightforward to see that z_0 is the unique point in $(0,+\infty)$ where T' is zero. Moreover, T' takes negative values within the interval $(0,z_0)$ and positive values within the interval $(z_0,+\infty)$. Then, T is strictly decreasing in $(0,z_0)$ and strictly increasing in $(z_0,+\infty)$, leading to (47).

Now, to prove expression (48), fix $\delta > 0$ and consider

$$\begin{aligned}
T(z_0+\delta) - T(z_0) &= b(z_0+\delta)^{-1} + (z_0+\delta)^4\,a - b\,z_0^{-1} - a\,z_0^4 \\
&= b(z_0+\delta)^{-1} + a(z_0^4 + 4\,\delta\,z_0^3 + 6\,\delta^2\,z_0^2 + 4\,\delta^3\,z_0 + \delta^4) - b\,z_0^{-1} - a\,z_0^4 \\
&= b\left((z_0+\delta)^{-1} - z_0^{-1}\right) + a(4\,\delta\,z_0^3 + 6\,\delta^2\,z_0^2 + 4\,\delta^3\,z_0 + \delta^4) \\
&= a(4\,\delta\,z_0^3 + 6\,\delta^2\,z_0^2 + 4\,\delta^3\,z_0 + \delta^4) - b\,\delta\,((z_0+\delta)\,z_0)^{-1} \\
&= \delta(4\,a\,z_0^3 - b(\,z_0^2 + \delta z_0)^{-1}) + 6\,a\,\delta^2 z_0^2 + 4\,a\,\delta^3 z_0 + a\,\delta^4 \\
&> \delta(4\,a\,z_0^3 - b\,z_0^{-2}) + 6\,a\,\delta^2 z_0^2 + 4\,a\,\delta^3 z_0 + a\,\delta^4 \\
&> 6\,a\,\delta^2 z_0^2.
\end{aligned}$$

Analogously, in order to prove expression (49), consider $\delta < z_0$ and,

$$\begin{aligned}
T(z_0-\delta)-T(z_0) &= b(z_0-\delta)^{-1}+(z_0-\delta)^4\, a - b\, z_0^{-1} - a\, z_0^4 \\
&= b(z_0-\delta)^{-1} + a(z_0^4 - 4\,\delta\, z_0^3 + 6\,\delta^2\, z_0^2 - 4\,\delta^3\, z_0 + \delta^4) - b\, z_0^{-1} - a\, z_0^4 \\
&= b\left((z_0-\delta)^{-1} - z_0^{-1}\right) + a(-4\,\delta\, z_0^3 + 6\,\delta^2\, z_0^2 - 4\,\delta^3\, z_0 + \delta^4) \\
&= a(-4\,\delta\, z_0^3 + 6\,\delta^2\, z_0^2 - 4\,\delta^3\, z_0 + \delta^4) - b\,\delta\,((z_0-\delta)\, z_0)^{-1} \\
&= \delta(-4\, a\, z_0^3 + b(\, z_0^2 - \delta z_0)^{-1}) + 6\, a\,\delta^2 z_0^2 - 4\, a\,\delta^3 z_0 + a\,\delta^4 \\
&> \delta(-4\, a\, z_0^3 + b\, z_0^{-2}) + 6\, a\,\delta^2 z_0^2 - 4\, a\,\delta^3 z_0 + a\,\delta^4 \\
&> 6\, a\,\delta^2 z_0^2 - 4\, a\,\delta^3 z_0.
\end{aligned}$$

These two last inequalities prove expressions (48) and (49).

Remark 1. *The minimizer of T, z_0 is precisely c_0. Indeed, T happens to be the dominant part of Λ_{n_0}, whose minimization is (under certain conditions) equivalent to the minimization of $MISE^a$.*

Theorem 3. *Under regularity conditions (B1)-(B3) from the paper, the bandwidth selector which minimizes the function $MISE^a$ has the following asymptotic expression:*

$$h_{MISE^a} = \left(\frac{R(K)\int \sigma^2(x) f^1(x)\left(f^0(x)\right)^{-1} dx}{\mu_2(K)^2 \int \beta(x)\, f^1(x)\, dx}\right)^{1/5} n_0^{-1/5} + O\left(n_0^{-2/5}\right). \tag{50}$$

Proof of Theorem 3 First, assume that $z_0 \in (L, U)$. Indeed, if $z_0 \notin (L, U)$, we would consider a wider interval which would fulfill that condition. Furthermore, expression (46) guarantees the existence of some constant $C > 0$ such that

$$\sup_{L\le z\le U} \left|\Lambda_{n_0}(z) - T(z)\right| \le C\, n_0^{-2/5}, \text{ for almost all } n_0 \in \mathbb{N}. \tag{51}$$

Consider the following sequence of real positive numbers,

$$\delta_{n_0} := \left(C\,(2\,a)^{-1}\right)^{1/2} z_0^{-1}\, n_0^{-1/5}, n_0 \in \mathbb{N}. \tag{52}$$

Using expression (48), we have

$$T(z_0+\delta_{n_0}) - T(z_0) > 6\, a\, z_0^2\, \delta_{n_0}^2 = 3\, C\, n_0^{-2/5}, \forall n_0 \in \mathbb{N}. \tag{53}$$

On the other hand, given that $\delta_{n_0} < z_0$, for almost all natural n_0,

$$\begin{aligned}
T(z_0-\delta_{n_0}) - T(z_0) &> 6\, a\, z_0^2\, \delta_{n_0}^2 - 4\, a\, z_0\, \delta_{n_0}^3 \\
&= 3\, C\, n_0^{-2/5} - 2\, C^{3/2}\, z_0^{-2} (2a)^{-1/2} n^{-3/5}.
\end{aligned} \tag{54}$$

As a consequence,

$$T(z_0 - \delta_{n_0}) - T(z_0) > \qquad 2\,C\,n_0^{-2/5}, \text{ for almost all } n_0 \in \mathbb{N}. \tag{55}$$

Providing that z_0 is an interior point of the interval $[L, U]$ and $\{\delta_{n_0}\}$ tends to zero as n_0 tends to ∞, the sets given below are well defined for almost all $n_0 \in \mathbb{N}$

$$A_{n_0} := \left[L, z_0 - \delta_{n_0}\right] \cup \left[z_0 + \delta_{n_0}, U\right].$$

Consider now $z \in [L, U]$ and expressions (47), (51), (52), (53). For almost $n_0 \in \mathbb{N}$, we have

$$\begin{aligned} z \in \left[L, z_0 - \delta_{n_0}\right] &\Rightarrow z \le z_0 - \delta_{n_0} \Rightarrow \Lambda_{n_0}(z) + C\,n_0^{-2/5} \ge T(z) \ge T(z_0 - \delta_{n_0}) \\ &> T(z_0) + 2\,C\,n_0^{-2/5} \ge \Lambda_{n_0}(z_0) + C\,n_0^{-2/5} \Rightarrow \Lambda_{n_0}(z) > \Lambda_{n_0}(z_0). \\ z \in \left[z_0 + \delta_{n_0}, U\right] &\Rightarrow z \ge z_0 + \delta_{n_0} \Rightarrow \Lambda_{n_0}(z) + C\,n_0^{-2/5} \ge T(z) \ge T(z_0 + \delta_{n_0}) \\ &> T(z_0) + 3\,C\,n_0^{-2/5} \ge \Lambda_{n_0}(z_0) + 2\,C\,n_0^{-2/5} \Rightarrow \Lambda_{n_0}(z) > \Lambda_{n_0}(z_0). \end{aligned}$$

Consequently,

$$z \in A_{n_0} \Rightarrow \Lambda_{n_0}(z) > \Lambda_{n_0}(z_0).$$

Therefore, for all $n_0 \in \mathbb{N}$ except for a finite number of them (at most), any minimizer γ_{n_0} of Λ_{n_0} (and Γ_{n_0}) verifies $\gamma_{n_0} \in [L, U] \setminus A_{n_0}$. In other words, $\left|\gamma_{n_0} - z_0\right| \le \delta_{n_0}$.

According to expression (42), for almost all $n_0 \in \mathbb{N}$, every minimizer h_{MISE^a} of $MISE^a$ satisfies:

$$\left|h_{MISE^a} - c_0\,n_0^{-1/5}\right| \le \delta_{n_0} n_0^{-1/5} = \left(C\,(2\,a)^{-1}\right)^{1/2} z_0^{-1}\,n_0^{-2/5},$$

for almost all $n_0 \in \mathbb{N}$, which implies expression (50).

Proposition 2. *Suppose conditions (C1), (C2) and (C3) from the paper are fulfilled. Consider a sequence of bandwidths* h_{n_0} *such that* $\sum_{n_0} h_{n_0}^{\lambda} < \infty$ *for some* $\lambda > 0$ *and that* $n_0^{\eta} h_{n_0} \longrightarrow \infty$ *for some* $\eta < 1 - s^{-1}$*. Assume* $(n_0\,h)^{-1/2} \log\left(h^{-1}\right) \longrightarrow 0$ *as* $n_0 \longrightarrow \infty$, $h \longrightarrow 0$ *and* $n_0\,h \longrightarrow \infty$*. Then,*

$$ISE(h) = ISE^a(h) + O_P\left(h^6\right) + O_P\left(\frac{h}{n_0}\log\frac{1}{h}\right) + O_P\left(\frac{h^{7/2}}{n_0^{1/2}}\right) + O_P\left(\frac{\log\frac{1}{h}}{n_0^{3/2} h^{3/2}}\right), \tag{56}$$

where $ISE(h) = \int \left(\hat{m}_h(x) - m(x)\right)^2 dF^1(x)$ *and* $ISE^a(h) = \int \left(\tilde{m}_h(x) - m(x)\right)^2 dF^1(x)$*.*

Proof Under conditions (C1), (C2) and assuming that $n_0 \longrightarrow \infty$, $h \longrightarrow 0$ and $n_0\,h \longrightarrow \infty$, we have:

$$\sup_x \left|\hat{f}_h^0(x) - f^0(x)\right| \longrightarrow 0 \text{ almost sure as } n_0 \longrightarrow \infty,$$

and

$$\sup_x \left|\hat{f}_h^0(x) - f^0(x)\right| = O_P\left(h^2 + n_0^{-1/2} h^{-1/2} \left(\log\frac{1}{h}\right)^{1/2}\right). \tag{57}$$

Additionally, thanks to results of Mack and Silverman[1], we have:

$$\sup_J \left|\hat{m}_h^{NW}(x) - m(x)\right| \longrightarrow 0 \text{ almost sure as } n_0 \longrightarrow \infty,$$

and

$$\sup_J \left|\hat{m}_h^{NW}(x) - m(x)\right| = O_P\left(h^2 + n_0^{-1/2} h^{-1/2}\left(\log\frac{1}{h}\right)^{1/2}\right). \tag{58}$$

Consider now $\{\xi_{n_0}\}$ a sequence of random variables defined in the probability space (Ω, A, P) such that $\xi_{n_0} \geq 0$ and $\xi_{n_0} = O_P\left(a_{n_0} + b_{n_0}\right)$, where $\left(a_{n_0}\right), \left(b_{n_0}\right)$ are sequences of real positive numbers. We can show that

$$\xi_{n_0} = O_P\left(a_{n_0} + b_{n_0}\right) \Leftrightarrow \xi_{n_0} = O_P\left(\max\{a_{n_0}, b_{n_0}\}\right) \tag{59}$$

'$\Leftarrow$' $a_{n_0} + b_{n_0} \geq \max\{a_{n_0}, b_{n_0}\} \Rightarrow \dfrac{\xi_{n_0}}{a_{n_0} + b_{n_0}} \leq \dfrac{\xi_{n_0}}{\max\{a_{n_0}, b_{n_0}\}}$.

'$\Rightarrow$' $a_{n_0} + b_{n_0} \leq 2\max\{a_{n_0}, b_{n_0}\} \Rightarrow \dfrac{2\xi_{n_0}}{a_{n_0} + b_{n_0}} \geq \dfrac{\xi_{n_0}}{\max\{a_{n_0,b_{n_0}}\}}$.

It can be shown then that

$$MISE(h) = MISE^a(h) + \mathbb{E}\left[\int 2\,A_1\,A_2\,dF^1(x)\right] + \mathbb{E}\left[\int A_2^2\,dF^1(x)\right],$$

where

$$A_1 = \frac{\hat{\Psi}_h(x)}{f^0(x)} - \frac{m(x)\hat{f}_h^0(x)}{f^0(x)} = \frac{1}{n_0\,f^0(x)}\sum_{i=1}^{n_0} K_h(x - X_i^0)(Y_i^0 - m(x)),$$

$$A_2 = \left(\hat{m}_h^{NW}(x) - m(x)\right)\frac{\left(\hat{f}_h^0(x) - f^0(x)\right)}{f^0(x)},$$

and $MISE^a(h)$ is given in (26).

[1] Mack, Y.P. and Silverman B.W. (1982) Weak and strong uniform consistency of kernel regression estimates *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 61, 405–415.

On the one hand, using expression (59), it turns out:

$$\begin{aligned}
\int_J A_2^2\, dF^1(x) &= \left|\int_J A_2^2\, dF^1(x)\right| = \left|\int_J \left(\hat{m}_h^{NW}(x) - m(x)\right) \frac{\left(\hat{f}_h^0(x) - f^0(x)\right)}{f^0(x)} f^1(x)\, dx\right| \\
&\leq \sup_J \left|\hat{m}_h^{NW}(x) - m(x)\right|^2 \sup_J \left|\hat{f}_h^0(x) - f^0(x)\right|^2 \int_{x\in J} \frac{f^1(x)}{f^0(x)^2}\, dx \\
&= \int_{x\in J} \frac{f^1(x)}{f^0(x)^2}\, dx \left(O_P\left(h^2 + n_0^{-1/2} h^{-1/2}\left(\log\frac{1}{h}\right)^{1/2}\right)\right)^4 \\
&= \int_{x\in J} \frac{f^1(x)}{f^0(x)^2}\, dx \left(O_P\left(\max\left\{h^2, n_0^{-1/2} h^{-1/2}\left(\log\frac{1}{h}\right)^{1/2}\right\}\right)\right)^4 \\
&= \int_{x\in J} \frac{f^1(x)}{f^0(x)^2}\, dx \left(O_P\left(\max\left\{h^2, n_0^{-1/2} h^{-1/2}\left(\log\frac{1}{h}\right)^{1/2}\right\}^4\right)\right) \\
&= \int_{x\in J} \frac{f^1(x)}{f^0(x)^2}\, dx \left(O_P\left(\max\left\{h^8, n_0^{-2} h^{-2}\left(\log\frac{1}{h}\right)^{2}\right\}\right)\right) \\
&= \int_{x\in J} \frac{f^1(x)}{f^0(x)^2}\, dx \left(O_P\left(h^8 + n_0^{-2} h^{-2}\left(\log\frac{1}{h}\right)^{2}\right)\right) \\
&= O_P\left(h^8\right) + O_P\left(n_0^{-2} h^{-2}\left(\log\frac{1}{h}\right)^{2}\right). \qquad (60)
\end{aligned}$$

The first addend in expression (60) is negligible in comparison to the second term in expression (26). Analogously, the second addend in expression (60) becomes insignificant compared to the first term in expression (26). In particular,

$$\begin{aligned}
\left(\int_J A_2^2\, dF^1(x)\right)^{1/2} &= \left(O_P\left(h^8\right) + O_P\left(n_0^{-2} h^{-2}\left(\log\frac{1}{h}\right)^{2}\right)\right)^{1/2} \\
&= \left(O_P\left(h^8 + n_0^{-2} h^{-2}\left(\log\frac{1}{h}\right)^{2}\right)\right)^{1/2} \\
&= \left(O_P\left(\max\left\{h^8, n_0^{-2} h^{-2}\left(\log\frac{1}{h}\right)^{2}\right\}\right)\right)^{1/2} \\
&= O_P\left(\max\left\{h^8, n_0^{-2} h^{-2}\left(\log\frac{1}{h}\right)^{2}\right\}^{1/2}\right) \\
&= O_P\left(\max\left\{h^4, n_0^{-1} h^{-1}\left(\log\frac{1}{h}\right)\right\}\right) \\
&= O_P\left(h^4 + n_0^{-1} h^{-1}\left(\log\frac{1}{h}\right)\right) \\
&= O_P\left(h^4\right) + O_P\left(n_0^{-1} h^{-1}\left(\log\frac{1}{h}\right)\right). \qquad (61)
\end{aligned}$$

Moreover, thanks to expression (26), we know that

$$\mathbb{E}\left[\int_J A_1^2\, dF_1(x)\right] = O\left(n_0^{-1} h^{-1} + h^4\right). \qquad (62)$$

Applying Markov's inequality to expression (62), it turns out:

$$\int_J A_1^2\, dF^1(x) = O_P\left(n_0^{-1}h^{-1} + h^4\right). \tag{63}$$

Thus,

$$\begin{aligned}\left(\int_J A_1^2\, dF^1(x)\right)^{1/2} &= \left(O_P\left(n_0^{-1}h^{-1} + h^4\right)\right)^{1/2} = O_P\left(\max\left\{n_0^{-1}h^{-1}, h^4\right\}^{1/2}\right)\\ &= O_P\left(\max\left\{n_0^{-1/2}h^{-1/2}, h^2\right\}\right) = O_P\left(n_0^{-1/2}h^{-1/2} + h^2\right)\\ &= O_P\left(n_0^{-1/2}h^{-1/2}\right) + O_P\left(h^2\right). \end{aligned}\tag{64}$$

Bringing together expressions (64) and (61) and using Cauchy-Schwarz inequality, we compute:

$$\begin{aligned}\left|\int_J A_1\, A_2\, dF^1(x)\right| &\le 2\left(\int_J A_1^2\, dF^1(x)\right)^{1/2}\left(\int_J A_2^2\, dF^1(x)\right)^{1/2}\\ &= 2\left(O_p\left(h^4\right) + O_P\left(n_0^{-1}h^{-1}\log\frac{1}{h}\right)\right)\\ &\cdot\left(O_p\left(h^2\right) + O_P\left(n_0^{-1/2}h^{-1/2}\log\frac{1}{h}\right)\right)\\ &= O_P\left(h^6\right) + O_P\left(\frac{h^{7/2}}{n_0^{1/2}}\right) + O_P\left(\frac{h}{n_0}\log\frac{1}{h}\right) + O_P\left(\frac{\log\frac{1}{h}}{(n_0\,h)^{3/2}}\right). \end{aligned}\tag{65}$$

The first addend in expression (65) is negligible as compared to the second term in expression (26). Furthermore, the fourth addend in expression (65) becomes insignificant in comparison to the first term in expression (26) if $\left(\frac{\log\frac{1}{h}}{n_0^{1/2}h^{1/2}}\right) \longrightarrow 0$ as $h \longrightarrow 0$, $n_0 \longrightarrow \infty$ and $n_0\,h \longrightarrow \infty$.

It remains to be seen what happens with the second and third terms in expression (65). We begin with the second one. Given that $\left(n_0^{-1}h^{-1} + h^4\right)\cdot n_0 h = 1 + h^5 n_0$ and the bandwidth h is of the form $n^{-\alpha}, \alpha > 0$, then

- If $n_0 h^5 \longrightarrow c$, being c a positive real number, then $n_0^{-1}h^{-1} \sim n_0^{-4/5}$ and $h^4 \sim n_0^{-4/5}$, which implies that $h \sim n_0^{-1/5}$, and

$$\frac{h^{7/2}}{n_0^{1/2}} \sim \frac{\left(n_0^{-1/5}\right)^{7/2}}{n_0^{1/2}} = \frac{n_0^{-7/10}}{n_0^{5/10}} = n_0^{-6/5} \longrightarrow 0,\ \text{as } n_0 \longrightarrow \infty.$$

- If $n_0 h^5 \longrightarrow 0$, then

$$\frac{\frac{h^{7/2}}{n_0^{1/2}}}{\frac{1}{n_0\,h}} \longrightarrow 0 \Leftrightarrow n_0^{1/2}h^{9/2} \longrightarrow 0 \Leftrightarrow n_0 h^9 \longrightarrow 0,$$

  which is true providing that $n_0 h^5 \longrightarrow 0$.

- If $n_0 h^5 \longrightarrow \infty$, then

$$\frac{\frac{h^{7/2}}{n_0^{1/2}}}{h^4} \longrightarrow 0 \Leftrightarrow n_0^{-1/2} h^{-1/2} \longrightarrow 0 \Leftrightarrow n_0 h \longrightarrow \infty,$$

which is true providing that $n_0 h^5 \longrightarrow \infty$.

Therefore, $\frac{h^{7/2}}{n_0^{1/2}} = o\left(\frac{1}{n_0\, h} + h^4\right)$.

Finally, as for the third addend in (65),

- If $n_0 h^5 \longrightarrow c$, being c a positive real number, then

$$\frac{h}{n_0} \log \frac{1}{h} \sim n_0^{-6/5} \log n_0^{1/5} \longrightarrow 0, \text{ as } n_0 \longrightarrow \infty.$$

- If $n_0 h^5 \longrightarrow 0$, then

$$\frac{\frac{h}{n_0} \log \frac{1}{h}}{\frac{1}{n_0\, h}} = h^2 \log \frac{1}{h} \longrightarrow 0 \Leftrightarrow h^2 \longrightarrow 0 \Leftrightarrow n_0 h^9 \longrightarrow 0,$$

which is true providing that $h \longrightarrow 0$.

- If $n_0 h^5 \longrightarrow \infty$, then

$$\frac{\frac{h}{n_0} \log \frac{1}{h}}{h^4} = \frac{\log \frac{1}{h}}{n_0 h^3} \longrightarrow 0 \Leftrightarrow n_0 h^3 \longrightarrow \infty,$$

which is true providing that $n_0 h^5 \longrightarrow \infty$.

Therefore, $\frac{h}{n_0} \log \frac{1}{h} = o\left(\frac{1}{n_0\, h} + h^4\right)$.

Considering this last argument and collecting terms (64), (61) and (65), expression (56) is proven.

1.2.2 | Asymptotic expressions for the bootstrap criterion functions

Lemma 2. *Under regularity conditions (B1)-(B4), the function $MISE^{a*}$ is*

$$MISE^{a*}(h) = \frac{R(K)}{n_0\, h} \hat{A}_g + \frac{h^4}{4} \mu_2(K)^2 \hat{B}_g + O_P\left(h^6\, n_1^{-1}\, g^{-7}\left(g^{-2} + g^{-1} + 1\right)\right)$$
$$+ O_P(h^8\, n_1^{-1} g^{-9}) + O_P\left(h^{-1}\, g^2\, n_1^{-1}\right) + O_P\left(h\, n_1^{-1}\left(1 + g^{-1} + g^{-2}\right)\right). \quad (66)$$

Thus, the dominant part of expression (66), namely $AMISE^{a*}(h)$, is given by:

$$AMISE^{a*}(h) = \frac{R(K)}{n_0\, h} \hat{A}_g + \frac{h^4}{4} \mu_2(K)^2 \hat{B}_g. \quad (67)$$

Proof Consider the smoothed bootstrap version of $MISE^a$. We start by computing $MISE^{a*}(h) := \mathbb{E}^*\left[\int\left(\tilde{m}_h^{NW}(x) - \hat{m}_g(x)\right)\right]$, which is the theoretical analogous to $MASE^*_{\tilde{m}_h^{NW},X^1}$ given in (14). Using a Taylor expansion and a change of variable, we obtain:

$$
\begin{aligned}
&\mathbb{E}^*\left[\int\left(\tilde{m}_h^{NW}(x) - \hat{m}_g(x)\right)^2 d\hat{F}_g^1(x)\right] \\
&= \mathbb{E}^*\left[\int \frac{1}{n_0 \hat{f}_g^0(x)} \sum_{i=1}^{n_0} K_h\left(x - X_i^{0*}\right)\left(Y_i^{0*} - \hat{m}_g(x)\right)\right]^2 d\hat{F}_g^1(x) \\
&= \frac{1}{n_0^2}\int \frac{1}{\hat{f}_g^0(x)^2}\mathbb{E}^*\left[\sum_{i=1}^{n_0} K_h\left(x - X_i^{0*}\right)\left(Y_i^{0*} - \hat{m}_g(x)\right)\right]^2 d\hat{F}_g^1(x) \\
&= \frac{1}{n_0^2}\int \frac{1}{\hat{f}_g^0(x)^2}\left[Var^*\left[\sum_{i=1}^{n_0} K_h\left(x - X_i^{0*}\right)\left(Y_i^{0*} - \hat{m}_g(x)\right)\right]\right. \\
&\left. + \left(\mathbb{E}^*\left[\sum_{i=1}^{n_0} K_h\left(x - X_i^{0*}\right)\left(Y_i^{0*} - \hat{m}_g(x)\right)\right]\right)^2\right] d\hat{F}_g^1(x) \\
&= \frac{1}{n_0^2}\int \frac{1}{\hat{f}_g^0(x)^2}\left[\sum_{i=1}^{n_0} Var^*\left[K_h\left(x - X_i^{0*}\right)\left(Y_i^{0*} - \hat{m}_g(x)\right)\right]\right. \\
&\left. + \left(\sum_{i=1}^{n_0} \mathbb{E}^*\left[K_h\left(x - X_i^{0*}\right)\left(Y_i^{0*} - \hat{m}_g(x)\right)\right]\right)^2\right] d\hat{F}_g^1(x) \\
&= \frac{1}{n_0^2}\int \frac{1}{\hat{f}_g^0(x)^2}\left[n_0 Var^*\left[K_h\left(x - X_1^{0*}\right)\left(Y_1^{0*} - \hat{m}_g(x)\right)\right]\right. \\
&\left. + n_0^2\left(\mathbb{E}^*\left[K_h\left(x - X_1^{0*}\right)\left(Y_1^{0*} - \hat{m}_g(x)\right)\right]\right)^2\right] d\hat{F}_g^1(x) \\
&= \frac{1}{n_0}\int \frac{1}{\hat{f}_g^0(x)^2} Var^*\left[K_h\left(x - X_1^{0*}\right)\left(Y_1^{0*} - \hat{m}_g(x)\right)\right] d\hat{F}_g^1(x) \\
&+ \int \frac{1}{\hat{f}_g^0(x)^2}\left(\mathbb{E}^*\left[K_h\left(x - X_1^{0*}\right)\left(Y_1^{0*} - \hat{m}_g(x)\right)\right]\right)^2 d\hat{F}_g^1(x) \\
&= \frac{1}{n_0}\int \frac{1}{\hat{f}_g^0(x)^2}\left[\mathbb{E}^*\left[K_h\left(x - X_1^{0*}\right)^2\left(Y_1^{0*} - \hat{m}_g(x)\right)^2\right]\right. \\
&\left. - \left(\mathbb{E}^*\left[K_h\left(x - X_1^{0*}\right)\left(Y_1^{0*} - \hat{m}_g(x)\right)\right]\right)^2\right] d\hat{F}_g^1(x) \\
&+ \int \frac{1}{\hat{f}_g^0(x)^2}\left(\mathbb{E}^*\left[K_h\left(x - X_1^{0*}\right)\left(Y_1^{0*} - \hat{m}_g(x)\right)\right]\right)^2 d\hat{F}_g^1(x) \\
&= \frac{1}{n_0}\int \frac{1}{\hat{f}_g^0(x)^2}\mathbb{E}^*\left[K_h\left(x - X_1^{0*}\right)^2\left(Y_1^{0*} - \hat{m}_g(x)\right)^2\right] d\hat{F}_g^1(x) \\
&+ \frac{n_0 - 1}{n_0}\int \frac{1}{\hat{f}_g^0(x)^2}\left(\mathbb{E}^*\left[K_h\left(x - X_1^{0*}\right)\left(Y_1^{0*} - \hat{m}_g(x)\right)\right]\right)^2 d\hat{F}_g^1(x) \\
&= \frac{1}{n_0}\int \frac{1}{\hat{f}_g^0(x)^2}\mathbb{E}^*\left[\mathbb{E}^*\left[K_h\left(x - X_1^{0*}\right)^2\left(Y_1^{0*} - \hat{m}_g(x)\right)^2 \mid X_1^{0*}\right]\right] d\hat{F}_g^1(x)
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{n_0-1}{n_0} \int \frac{1}{\hat{f}_g^0(x)^2} \left(\mathbb{E}^* \left[\mathbb{E}^* \left[K_h \left(x - X_1^{0*} \right) \left(Y_1^{0*} - \hat{m}_g(x) \right) \mid X_1^{0*} \right] \right] \right)^2 d\hat{F}_g^1(x) \\
&= \frac{1}{n_0} \int \frac{1}{\hat{f}_g^0(x)^2} \mathbb{E}^* \left[K_h \left(x - X_1^{0*} \right)^2 \mathbb{E}^* \left[\left(Y_1^{0*} - \hat{m}_g(x) \right)^2 \mid X_1^{0*} \right] \right] d\hat{F}_g^1(x) \\
&+ \frac{n_0-1}{n_0} \int \frac{1}{\hat{f}_g^0(x)^2} \left(\mathbb{E}^* \left[K_h \left(x - X_1^{0*} \right) \left(\mathbb{E}^* \left[Y_1^{0*} \mid X_1^{0*} \right] - \hat{m}_g(x) \right) \right] \right)^2 d\hat{F}_g^1(x) \\
&= \frac{1}{n_0} \int \frac{1}{\hat{f}_g^0(x)^2} \mathbb{E}^* \left[K_h \left(x - X_1^{0*} \right)^2 \left(Var^* \left(Y_1^{0*} - \hat{m}_g(x) \mid X_1^{0*} \right) \right. \right. \\
&\left. \left. + \left(\mathbb{E}^* \left[Y_1^{0*} - \hat{m}_g(x) \mid X_1^{0*} \right] \right)^2 \right) \right] d\hat{F}_g^1(x) \\
&+ \frac{n_0-1}{n_0} \int \frac{1}{\hat{f}_g^0(x)^2} \left(\mathbb{E}^* \left[K_h \left(x - X_1^{0*} \right) \left(\hat{m}_g(X_1^{0*}) - \hat{m}_g(x) \right) \right] \right)^2 d\hat{F}_g^1(x) \\
&= \frac{1}{n_0} \int \frac{1}{\hat{f}_g^0(x)^2} \mathbb{E}^* \left[K_h \left(x - X_1^{0*} \right)^2 \left(Var^* \left(Y_1^{0*} \mid X_1^{0*} \right) \right. \right. \\
&\left. \left. + \left(\mathbb{E}^* \left[Y_1^{0*} \mid X_1^{0*} \right] - \hat{m}_g(x) \right)^2 \right) \right] d\hat{F}_g^1(x) \\
&+ \frac{n_0-1}{n_0} \int \frac{1}{\hat{f}_g^0(x)^2} \left(\mathbb{E}^* \left[K_h \left(x - X_1^{0*} \right) \left(\hat{m}_g(X_1^{0*}) - \hat{m}_g(x) \right) \right] \right)^2 d\hat{F}_g^1(x) \\
&= \frac{1}{n_0} \int \frac{1}{\hat{f}_g^0(x)^2} \mathbb{E}^* \left[K_h \left(x - X_1^{0*} \right)^2 \left(\hat{\sigma}_g^2(X_1^{0*}) + g^2 \mu_2(K) \right. \right. \\
&\left. \left. + \left(\hat{m}_g(X_1^{0*}) - \hat{m}_g(x) \right)^2 \right) \right] d\hat{F}_g^1(x) \\
&+ \frac{n_0-1}{n_0} \int \frac{1}{\hat{f}_g^0(x)^2} \left(\mathbb{E}^* \left[K_h \left(x - X_1^{0*} \right) \left(\hat{m}_g(X_1^{0*}) - \hat{m}_g(x) \right) \right] \right)^2 d\hat{F}_g^1(x) \\
&= \frac{1}{n_0 h^2} \int \frac{1}{\hat{f}_g^0(x)^2} \mathbb{E}^* \left[K \left(\frac{x - X_1^{0*}}{h} \right)^2 \left(\hat{\sigma}_g^2(X_1^{0*}) + g^2 \mu_2(K) \right. \right. \\
&\left. \left. + \left(\hat{m}_g(X_1^{0*}) - \hat{m}_g(x) \right)^2 \right) \right] d\hat{F}_g^1(x) \\
&+ \frac{n_0-1}{n_0 h^2} \int \frac{1}{\hat{f}_g^0(x)^2} \left(\mathbb{E}^* \left[K \left(\frac{x - X_1^{0*}}{h} \right) \left(\hat{m}_g(X_1^{0*}) - \hat{m}_g(x) \right) \right] \right)^2 d\hat{F}_g^1(x) \\
&= \frac{1}{n_0 h^2} \int \frac{1}{\hat{f}_g^0(x)^2} \int K \left(\frac{y-x}{h} \right)^2 \\
&\left(\hat{\sigma}_g^2(y) + g^2 \mu_2(K) + \left(\hat{m}_g(y) - \hat{m}_g(x) \right)^2 \right) \hat{f}_g^0(y)\, dy\, d\hat{F}_g^1(x) \\
&+ \frac{n_0-1}{n_0 h^2} \int \frac{1}{\hat{f}_g^0(x)^2} \left(\int K \left(\frac{y-x}{h} \right) \left(\hat{m}_g(y) - \hat{m}_g(x) \right) \hat{f}_g^0(y)\, dy \right)^2 d\hat{F}_g^1(x) \\
&= \frac{1}{n_0 h} \int \frac{1}{\hat{f}_g^0(x)^2} \int K(u)^2 \left(\hat{\sigma}_g^2(x+hu) + g^2 \mu_2(K) + \left(\hat{m}_g(x+hu) - \hat{m}_g(x) \right)^2 \right) \\
&\hat{f}_g^0(x+hu)\, du\, d\hat{F}_g^1(x) + \frac{n_0-1}{n_0} \int \frac{1}{\hat{f}_g^0(x)^2} \\
&\left(\int K(u) \left(\hat{m}_g(x+hu) - \hat{m}_g(x) \right) \hat{f}_g^0(x+hu)\, du \right)^2 d\hat{F}_g^1(x).
\end{aligned} \tag{68}
$$

Let us denote $\hat{\psi}_{g,x}(y) = (\hat{m}_g(y) - \hat{m}_g(x))\, \hat{f}_g^0(y)$. For the sake of simplicity, we will denote $\hat{\psi}_g(y) = \hat{\psi}_{g,x}(y)$ in the following. Then,

$$
\begin{aligned}
\hat{\psi}_g(x) &= 0, \\
\hat{\psi}_g'(y) &= \hat{m}_g'(y)\hat{f}_g^0(y) + (\hat{m}_g(y) - \hat{m}_g(x))\left(\hat{f}_g^0\right)'(y), \\
\hat{\psi}_g''(y) &= \hat{m}_g''(y)\hat{f}_g^0(y) + 2\hat{m}_g'(y)\left(\hat{f}_g^0\right)'(y) + (\hat{m}_g(y) - \hat{m}_g(x))\left(\hat{f}_g^0\right)''(y), \\
\hat{\psi}_g'''(y) &= \hat{m}_g'''(y)\hat{f}_g^0(y) + 3\hat{m}_g''(y)\left(\hat{f}_g^0\right)'(y) + 3\hat{m}_g'(y)\left(\hat{f}_g^0\right)''(y), \\
&\quad + \left(\hat{m}_g(y) - \hat{m}_g(x)\right)\left(\hat{f}_g^0\right)'''(y), \\
\hat{\psi}_g^{(4)}(y) &= \hat{m}_g^{(4)}(y)\hat{f}_g^0(y) + 4\hat{m}_g'''(y)\left(\hat{f}_g^0\right)'(y) + 6\hat{m}_g''(y)\left(\hat{f}_g^0\right)''(y), \\
&\quad + 4\hat{m}_g'(y)\left(\hat{f}_g^0\right)'''(y)\left(\hat{m}_g(y) - \hat{m}_g(x)\right)\left(\hat{f}_g^0\right)^{(4)}(y), \\
\hat{\psi}_g''(x) &= \hat{m}_g''(x)\hat{f}_g^0(x) + 2\hat{m}_g'(x)\left(\hat{f}_g^0\right)'(x), \\
\hat{\psi}_g^{(4)}(x) &= \hat{m}_g^{(4)}(x)\hat{f}_g^0(x) + 4\hat{m}_g'''(x)\left(\hat{f}_g^0\right)'(x), \text{ and} \\
&\quad + 6\hat{m}_g''(x)\left(\hat{f}_g^0\right)''(x) + 4\hat{m}_g'(x)\left(\hat{f}_g^0\right)'''(x).
\end{aligned}
$$

Using Taylor expansion:

$$
\begin{aligned}
&\int K(u)\hat{\psi}_g(x + hu)\,du \\
&= \int K(u)\hat{\psi}_g(x)\,du + h\int u\,K(u)\,\hat{\psi}_g'(x)\,du + \frac{h^2}{2}\int u^2K(u)\hat{\psi}_g''(x)\,du \\
&+ \frac{h^3}{3!}\int u^3K(u)\hat{\psi}_g'''(x)\,du + \frac{h^4}{4!}\int u^4K(u)\hat{\psi}_g^{(4)}(x)\,du \\
&+ \frac{h^5}{5!}\int u^5K(u)\hat{\psi}_g^{(5)}(x)\,du + O_P(h^6) \\
&= \frac{h^2}{2}\hat{\psi}_g''(x)\mu_2(K) + \frac{h^4}{24}\hat{\psi}_g^{(4)}(x)\mu_4(K) + O_P(h^6), \text{ and} \\
&\left(\int K(u)\hat{\psi}_g(x + hu)\,du\right)^2 \\
&= \frac{h^4}{4}\hat{\psi}_g''(x)^2\mu_2(K)^2 + \frac{h^6}{24}\hat{\psi}_g''(x)\hat{\psi}_g^{(4)}(x)\mu_2(K)\mu_4(K) \\
&+ \frac{h^8}{576}\hat{\psi}_g^{(4)}(x)^2\mu_4(K) + O_P(h^{12}) \\
&= \frac{h^4}{4}\hat{\psi}_g''(x)^2\mu_2(K)^2 + O_P(h^6n_0^{-2}g^{-8}\left(g^{-2} + g^{-1} + 1\right)) + O_P(h^8n_0^{-2}g^{-10}),
\end{aligned}
$$

where

$$\begin{aligned}
\hat{\psi}_g''(x)^2 &= \hat{m}_g''(x)^2 \hat{f}_g^0(x)^2 \\
&+ 4\hat{m}_g'(x)^2 \left(\hat{f}_g^0\right)'(x)^2 + 4\hat{m}_g''(x)\hat{f}_g^0(x)\hat{m}_g'(x)\left(\hat{f}_g^0\right)'(x), \\
\hat{\psi}_g^{(4)}(x)^2 &= \hat{m}_g^{(4)}(x)^2 \hat{f}_g^0(x)^2 + 8\hat{m}_g^{(4)}(x)\hat{m}_g'''(x)\hat{f}_g^0(x)\left(\hat{f}_g^0\right)'(x) \\
&+ 12\hat{m}_g^{(4)}(x)\hat{m}_g''(x)\hat{f}_g^0(x)\left(\hat{f}_g^0\right)''(x) \\
&+ 8\hat{m}_g^{(4)}(x)\hat{m}_g'(x)\hat{f}_g^0(x)\left(\hat{f}_g^0\right)'''(x) + 16\hat{m}_g'''(x)^2\left(\hat{f}_g^0\right)'(x)^2 \\
&+ 48\hat{m}_g'''(x)\hat{m}_g''(x)\left(\hat{f}_g^0\right)'(x)\left(\hat{f}_g^0\right)''(x) \\
&+ 32\hat{m}_g'''(x)\hat{m}_g'(x)\left(\hat{f}_g^0\right)'''(x)\left(\hat{f}_g^0\right)'(x) + 36\hat{m}_g''(x)^2\left(\hat{f}_g^0\right)''(x)^2 \\
&+ 48\hat{m}_g'(x)\hat{m}_g''(x)\left(\hat{f}_g^0\right)''(x)\left(\hat{f}_g^0\right)'''(x) + 16\hat{m}_g'(x)^2\left(\hat{f}_g^0\right)'''(x)^2, \\
\hat{\psi}_g''(x)\hat{\psi}_g^{(4)}(x) &= \hat{m}_g''(x)^2\hat{m}_g^{(4)}(x)\hat{f}_g^0(x)^2 + 4\hat{m}_g'''(x)\hat{m}_g''(x)\hat{f}_g^0(x)\left(\hat{f}_g^0\right)'(x) \\
&+ 6\hat{m}_g''(x)^2\hat{f}_g^0(x)\left(\hat{f}_g^0\right)'''(x) + 2\hat{m}_g'(x)\hat{m}_g^{(4)}(x)\hat{f}_g^0(x)\left(\hat{f}_g^0\right)'(x) \\
&+ 8\hat{m}_g'(x)\hat{m}_g'''(x)\left(\hat{f}_g^0\right)'(x)^2 + 12\hat{m}_g'(x)\hat{m}_g''(x)\left(\hat{f}_g^0\right)'(x)\left(\hat{f}_g^0\right)''(x) \\
&+ 8\hat{m}_g'(x)^2\left(\hat{f}_g^0\right)'(x)\left(\hat{f}_g^0\right)'''(x), \\
\left(\hat{f}_g^0\right)^{(r)}(x) &= n_0^{-1} g^{-r-1}\sum_{i=1}^{n_0} K^{(r)}\left(\frac{x-X_i^0}{g}\right), \\
\hat{m}_g'(x) &= g^{-1}\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^{-2}\cdot\left(\sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right)Y_i^0\right. \\
&\left.\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right) - \sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right)\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)Y_i^0\right), \\
\hat{m}_g''(x) &= \left[\frac{\sum_{i=1}^{n_0} K''\left(\frac{x-X_i^0}{g}\right)Y_i^0}{\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)} + \frac{2\left(\sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right)\right)^2\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)Y_i^0}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^3}\right. \\
&-\frac{\sum_{i=1}^{n_0} K''\left(\frac{x-X_i^0}{g}\right)\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)Y_i^0}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^2} \\
&\left.-\frac{2\sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right)\sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right)Y_i^0}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^2}\right]\cdot g^{-2}, \\
\hat{m}_g'''(x) &= g^{-3}\left[\frac{\sum_{i=1}^{n_0} K'''\left(\frac{x-X_i^0}{g}\right)Y_i^0}{\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)} - \frac{\sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right)\sum_{i=1}^{n_0} K''\left(\frac{x-X_i^0}{g}\right)Y_i^0}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^2}\right.
\end{aligned}$$

$$+\frac{4\sum_{i=1}^{n_0}K'\left(\frac{x-X_i^0}{g}\right)\sum_{i=1}^{n_0}K''\left(\frac{x-X_i^0}{g}\right)\sum_{i=1}^{n_0}K\left(\frac{x-X_i^0}{g}\right)Y_i^0}{\left(\sum_{i=1}^{n_0}K\left(\frac{x-X_i^0}{g}\right)\right)^3}$$

$$-\frac{3\sum_{i=1}^{n_0}K''\left(\frac{x-X_i^0}{g}\right)\sum_{i=1}^{n_0}K'\left(\frac{x-X_i^0}{g}\right)Y_i^0}{\left(\sum_{i=1}^{n_0}K\left(\frac{x-X_i^0}{g}\right)\right)^2}$$

$$-\frac{2\sum_{i=1}^{n_0}K'\left(\frac{x-X_i^0}{g}\right)\sum_{i=1}^{n_0}K''\left(\frac{x-X_i^0}{g}\right)Y_i^0}{\left(\sum_{i=1}^{n_0}K\left(\frac{x-X_i^0}{g}\right)\right)^2}$$

$$+\frac{6\left(\sum_{i=1}^{n_0}K'\left(\frac{x-X_i^0}{g}\right)\right)^2\sum_{i=1}^{n_0}K'\left(\frac{x-X_i^0}{g}\right)Y_i^0}{\left(\sum_{i=1}^{n_0}K\left(\frac{x-X_i^0}{g}\right)\right)^3}$$

$$\left.+\frac{6\left(\sum_{i=1}^{n_0}K'\left(\frac{x-X_i^0}{g}\right)\right)^3\sum_{i=1}^{n_0}K\left(\frac{x-X_i^0}{g}\right)Y_i^0}{\left(\sum_{i=1}^{n_0}K\left(\frac{x-X_i^0}{g}\right)\right)^4}\right], \text{ and}$$

$$\hat{m}_g^{(4)}(x)=g^{-4}\left[\frac{\sum_{i=1}^{n_0}K^{(4)}\left(\frac{x-X_i^0}{g}\right)Y_i^0}{\sum_{i=1}^{n_0}K\left(\frac{x-X_i^0}{g}\right)}\right.$$

$$+\frac{22\sum_{i=1}^{n_0}K'\left(\frac{x-X_i^0}{g}\right)\sum_{i=1}^{n_0}K''\left(\frac{x-X_i^0}{g}\right)\sum_{i=1}^{n_0}K'\left(\frac{x-X_i^0}{g}\right)Y_i^0}{\left(\sum_{i=1}^{n_0}K\left(\frac{x-X_i^0}{g}\right)\right)^3}$$

$$+\frac{4\left(\sum_{i=1}^{n_0}K''\left(\frac{x-X_i^0}{g}\right)\right)^2\sum_{i=1}^{n_0}K\left(\frac{x-X_i^0}{g}\right)Y_i^0}{\left(\sum_{i=1}^{n_0}K\left(\frac{x-X_i^0}{g}\right)\right)^3}$$

$$+\frac{12\left(\sum_{i=1}^{n_0}K'\left(\frac{x-X_i^0}{g}\right)\right)^2\sum_{i=1}^{n_0}K''\left(\frac{x-X_i^0}{g}\right)Y_i^0}{\left(\sum_{i=1}^{n_0}K\left(\frac{x-X_i^0}{g}\right)\right)^3}$$

$$
\begin{aligned}
&+\frac{4\sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right)\sum_{i=1}^{n_0} K'''\left(\frac{x-X_i^0}{g}\right)\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)Y_i^0}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^3}\\
&-\frac{3\sum_{i=1}^{n_0} K''''\left(\frac{x-X_i^0}{g}\right)\sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right)Y_i^0}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^2}\\
&-\frac{6\sum_{i=1}^{n_0} K''\left(\frac{x-X_i^0}{g}\right)\sum_{i=1}^{n_0} K''\left(\frac{x-X_i^0}{g}\right)Y_i^0}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^2}\\
&-\frac{4\sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right)\sum_{i=1}^{n_0} K'''\left(\frac{x-X_i^0}{g}\right)Y_i^0}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^2}\\
&-\frac{12\left(\sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right)\right)^3\sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right)Y_i^0}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^4}\\
&-\frac{24\left(\sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right)\right)^4\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)Y_i^0}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^5}\\
&\left.+\frac{6\left(\sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right)\right)^2\sum_{i=1}^{n_0} K''\left(\frac{x-X_i^0}{g}\right)\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)Y_i^0}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^4}\right].
\end{aligned}
$$

Carrying on with calculations leads to:

$$
\begin{aligned}
&\frac{n_0-1}{n_0}\int \frac{1}{\hat{f}_g^0(x)^2}\left(\int K(u)\,\hat{\psi}_g(x+hu)\,du\right)^2 d\hat{F}_g^1(x)\\
&=\frac{n_0-1}{n_0}\int \frac{1}{\hat{f}_g^0(x)^2}\frac{h^4}{4}\hat{\psi}_g''(x)^2\mu_2(K)^2 d\hat{F}_g^1(x)\\
&+O_P(h^6\,n_1^{-1}\,g^{-7}\left(g^{-2}+g^{-1}+1\right))+O_P(h^8\,n_1^{-1}g^{-9})\\
&=\frac{n_0-1}{n_0}\frac{h^4}{4}\mu_2(K)^2\\
&\int\left[\hat{m}_g''(x)^2+\frac{4\hat{m}_g'(x)\hat{m}_g''(x)\left(\hat{f}_g^0\right)'(x)}{\hat{f}_g^0(x)}+\frac{4\hat{m}_g'(x)^2\left(\hat{f}_g^0\right)'(x)^2}{\hat{f}_g^0(x)^2}\right]\hat{f}_g^1(x)dx\\
&+O_P(h^6\,n_1^{-1}\,g^{-7}\left(g^{-2}+g^{-1}+1\right))+O_P(h^8\,n_1^{-1}g^{-9}).
\end{aligned}\tag{69}
$$

On the other hand, denote $\hat{\phi}_{g,x}(y)=\hat{\phi}_g(y)=\left[\hat{\sigma}_g^2(y)+g^2\,\mu_2(K)+(\hat{m}_g(y)-\hat{m}_g(x))^2\right]\hat{f}_g^0(y)$. Then,

$$
\begin{aligned}
\hat{\phi}_g(x)&=\hat{\sigma}_g^2(x)\hat{f}_g^0(x)+g^2\,\mu_2(K)\,\hat{f}_g^0(x),\\
\hat{\phi}_g'(y)&=\left(\hat{\sigma}_g^2\right)'(y)\hat{f}_g^0(y)+\hat{\sigma}_g^2(y)\left(\hat{f}_g^0\right)'(y)+g^2\mu_2(K)\left(\hat{f}_g^0\right)'(y),\\
&+2\hat{m}_g(y)\hat{m}_g'(y)\hat{f}_g^0(y)+\hat{m}_g(y)^2\left(\hat{f}_g^0\right)'(y)+\hat{m}_g(x)^2\left(\hat{f}_g^0\right)'(y)\\
&-2\hat{m}_g'(y)\hat{m}_g(x)\hat{f}_g^0(y)-2\hat{m}_g(y)\hat{m}_g(x)\left(\hat{f}_g^0\right)'(y),\\
\hat{\phi}_g''(y)&=\left(\hat{\sigma}_g^2\right)''(y)\hat{f}_g^0(y)+2\left(\hat{\sigma}_g^2\right)'(y)\left(\hat{f}_g^0\right)'(y)+\hat{\sigma}_g^2(y)\left(\hat{f}_g^0\right)''(y)\\
&+g^2\mu_2(K)\left(\hat{f}_g^0\right)''(y)+2\hat{m}_g'(y)^2\hat{f}_g^0(y)+2\hat{m}_g(y)\hat{m}_g'(y)\hat{f}_g^0(y)\\
&+2\hat{m}_g(y)^2\left(\hat{f}_g^0\right)'(y)+2\hat{m}_g(y)\hat{m}_g'(y)\left(\hat{f}_g^0\right)'(y)\\
&+\hat{m}_g(x)^2(y)\left(\hat{f}_g^0\right)''(y)+\hat{m}_g^2(x)\left(\hat{f}_g^0\right)''(y)-2\hat{m}_g''(y)\hat{f}_g^0(y)\hat{m}_g(x)\\
&-2\hat{m}_g'(y)\left(\hat{f}_g^0\right)'(y)\hat{m}_g(x)-2\hat{m}_g'(y)\hat{m}_g(x)\left(\hat{f}_g^0\right)'(y)\\
&-2\hat{m}_g(y)\hat{m}_g(x)\left(\hat{f}_g^0\right)''(y),\text{ and}\\
\hat{\phi}_g''(x)&=\left(\hat{\sigma}_g^2\right)''(x)\hat{f}_g^0(x)+2\left(\hat{\sigma}_g^2\right)'(x)\left(\hat{f}_g^0\right)'(x)+\hat{\sigma}_g^2(x)\left(\hat{f}_g^0\right)''(x)\\
&+g^2\mu_2(K)\left(\hat{f}_g^0\right)''(x)+2\hat{m}_g'(x)^2\hat{f}_g^0(x)+2\hat{m}_g(x)\hat{m}_g'(x)\hat{f}_g^0(x)\\
&+2\hat{m}_g(x)^2\left(\hat{f}_g^0\right)'(x)+2\hat{m}_g(x)\hat{m}_g'(x)\left(\hat{f}_g^0\right)'(x)\\
&-2\hat{m}_g''(x)\hat{f}_g^0(x)\hat{m}_g(x)-4\hat{m}_g'(x)\left(\hat{f}_g^0\right)'(x)\hat{m}_g(x),
\end{aligned}
$$

where

$$\hat{\sigma}_g^2(x) = \frac{1}{n_0 \hat{f}_g^0(x)} \sum_{i=1}^{n_0} K_g(x - X_i^0)\left(Y_i^0\right)^2 - \left[\frac{1}{n_0 \hat{f}_g^0(x)} \sum_{i=1}^{n_0} K_g(x - X_i^0) Y_i^0\right]^2$$
$$= \left[\frac{\sum_{i=1}^{n_0} K\left(\frac{x - X_i^0}{g}\right)\left(Y_i^0\right)^2}{\sum_{i=1}^{n_0} K\left(\frac{x - X_i^0}{g}\right)}\right] - \left[\frac{\sum_{i=1}^{n_0} K\left(\frac{x - X_i^0}{g}\right) Y_i^0}{\sum_{i=1}^{n_0} K\left(\frac{x - X_i^0}{g}\right)}\right]^2,$$

$$\left(\hat{\sigma}_g^2\right)'(x) = g^{-1}\left(\left[\frac{\sum_{i=1}^{n_0} K'\left(\frac{x - X_i^0}{g}\right)\left(Y_i^0\right)^2 \sum_{i=1}^{n_0} K\left(\frac{x - X_i^0}{g}\right)}{\left(\sum_{i=1}^{n_0} K\left(\frac{x - X_i^0}{g}\right)\right)^2} - \frac{\sum_{i=1}^{n_0} K'\left(\frac{x - X_i^0}{g}\right) \sum_{i=1}^{n_0} K\left(\frac{x - X_i^0}{g}\right)\left(Y_i^0\right)^2}{\left(\sum_{i=1}^{n_0} K\left(\frac{x - X_i^0}{g}\right)\right)^2}\right] - 2\left[\frac{\sum_{i=1}^{n_0} K'\left(\frac{x - X_i^0}{g}\right) Y_i^0 \sum_{i=1}^{n_0} K\left(\frac{x - X_i^0}{g}\right) Y_i^0}{\left(\sum_{i=1}^{n_0} K\left(\frac{x - X_i^0}{g}\right)\right)^2} - \frac{\sum_{i=1}^{n_0} K'\left(\frac{x - X_i^0}{g}\right)\left(\sum_{i=1}^{n_0} K\left(\frac{x - X_i^0}{g}\right) Y_i^0\right)^2}{\left(\sum_{i=1}^{n_0} K\left(\frac{x - X_i^0}{g}\right)\right)^3}\right]\right), \text{ and}$$

$$\left(\hat{\sigma}_g^2\right)''(x) = g^{-2}\left(\frac{\sum_{i=1}^{n_0} K\left(\frac{x - X_i^0}{g}\right) \sum_{i=1}^{n_0} K''\left(\frac{x - X_i^0}{g}\right)\left(Y_i^0\right)^2}{\left(\sum_{i=1}^{n_0} K\left(\frac{x - X_i^0}{g}\right)\right)^2} + \frac{\sum_{i=1}^{n_0} K'\left(\frac{x - X_i^0}{g}\right) \sum_{i=1}^{n_0} K'\left(\frac{x - X_i^0}{g}\right)\left(Y_i^0\right)^2}{\left(\sum_{i=1}^{n_0} K\left(\frac{x - X_i^0}{g}\right)\right)^2} - \frac{2\sum_{i=1}^{n_0} K'\left(\frac{x - X_i^0}{g}\right) \sum_{i=1}^{n_0} K'\left(\frac{x - X_i^0}{g}\right)\left(Y_i^0\right)^2}{\left(\sum_{i=1}^{n_0} K\left(\frac{x - X_i^0}{g}\right)\right)^2} + \frac{2\left(\sum_{i=1}^{n_0} K'\left(\frac{x - X_i^0}{g}\right)\right)^2 \sum_{i=1}^{n_0} K\left(\frac{x - X_i^0}{g}\right)\left(Y_i^0\right)^2}{\left(\sum_{i=1}^{n_0} K\left(\frac{x - X_i^0}{g}\right)\right)^3}\right.$$

$$
\begin{aligned}
&-2\left[\frac{\sum_{i=1}^{n_0} K''\left(\frac{x-X_i^0}{g}\right) Y_i^0 \sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right) Y_i^0}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^2}\right. \\
&+\frac{\left(\sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right) Y_i^0\right)^2}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^2} \\
&-\frac{2\sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right)\left(\sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right) Y_i^0 \sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right) Y_i^0\right)}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^3} \\
&-\frac{\sum_{i=1}^{n_0} K''\left(\frac{x-X_i^0}{g}\right)\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right) Y_i^0\right)^2}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^3} \\
&+\frac{3\left(\sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right)\right)^2\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right) Y_i^0\right)^2}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^5} \\
&\left.-\frac{2\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right) Y_i^0 \sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right) Y_i^0 \sum_{i=1}^{n_0} K'\left(\frac{x-X_i^0}{g}\right)}{\left(\sum_{i=1}^{n_0} K\left(\frac{x-X_i^0}{g}\right)\right)^3}\right]\Bigg).
\end{aligned}
$$

Applying a Taylor expansion:

$$
\begin{aligned}
\int K(u)^2 \hat{\phi}_g(x+hu)\, du &= R(K)\hat{\phi}_g(x) + \frac{h^2}{2}\hat{\phi}_g''(x)\int u^2 K(u)^2\, du + O_P\left(h^4\right) \\
&= R(K)\hat{\phi}_g(x) + O_P\left(h^2\, n_0^{-1} g^{-1}\left(1+g^{-1}+g^{-2}\right)\right).
\end{aligned}
$$

Carrying on with calculations leads to:

$$
\begin{aligned}
&\frac{1}{n_0\,h}\int \frac{1}{\hat{f}_g^0(x)^2}\int K(u)^2\hat{\phi}_g(x+hu)\,du\,d\hat{F}_g^1(x) \\
&= \frac{R(K)}{n_0\,h}\int \frac{\left(\hat{\sigma}_g^2(x)+g^2\,\mu_2(K)\right)\,\hat{f}_g^1(x)}{\hat{f}_g^0(x)}dx + O_P\left(h\,n_1^{-1}\left(1+g^{-1}+g^{-2}\right)\right) \\
&= \frac{R(K)}{n_0\,h}\int \frac{\hat{\sigma}_g^2(x)\,\hat{f}_g^1(x)}{\hat{f}_g^0(x)}dx + \frac{R(K)\,g^2\,\mu_2(K)}{n_0\,h}\int \frac{\hat{f}_g^1(x)}{\hat{f}_g^0(x)}dx \\
&+ O_P\left(h\,n_1^{-1}\left(1+g^{-1}+g^{-2}\right)\right) \\
&= \frac{R(K)}{n_0\,h}\int \frac{\hat{\sigma}_g^2(x)\,\hat{f}_g^1(x)}{\hat{f}_g^0(x)}dx \\
&+ O_P\left(h\,n_1^{-1}\left(1+g^{-1}+g^{-2}\right)\right) + O_P\left(h^{-1}\,g^2\,n_1^{-1}\right).
\end{aligned} \tag{70}
$$

Assembling terms (69) and (70), and inserting them in expression (68), Lemma 2 holds.

Proposition 3. *Assume conditions (C1), (C2), (C3). Suppose, additionally, that $n_0 \longrightarrow \infty$, $h \longrightarrow 0$ and $n_0\,h \longrightarrow \infty$. Consider h_{n_0} a sequence of bandwidths such that $\sum_{n_0} h_{n_0}^{\lambda} < \infty$ for some $\lambda > 0$ and that $n_0^{\eta} h_{n_0} \longrightarrow \infty$ for some $\eta < 1 - s^{-1}$. Assume further that $(n_0\,h)^{-1/2}\log\left(h^{-1}\right) \longrightarrow 0$ as $n_0 \longrightarrow \infty$, $h \longrightarrow 0$ and $n_0\,h \longrightarrow \infty$. Then,*

$$
ISE^*(h) = ISE^{a*}(h) + O_{P^*}\left(h^6\right) + O_{P^*}\left(\frac{h}{n_0}\log\frac{1}{h}\right) + O_{P^*}\left(\frac{h^{7/2}}{n_0^{1/2}}\right) + O_{P^*}\left(\frac{\log\frac{1}{h}}{n_0^{3/2}h^{3/2}}\right), \tag{71}
$$

almost sure with respect to P, where

$$
ISE^*(h) = \int \left(\hat{m}_h^{NW*}(x) - \hat{m}_g(x)\right)^2 d\hat{F}_g^1(x), \text{ and } ISE^{a*}(h) = \int \left(\tilde{m}_h^{NW*}(x) - \hat{m}_g(x)\right)^2 d\hat{F}_g^1(x).
$$

Proof Given that $n_0 \longrightarrow \infty$, $h \longrightarrow 0$ and $n_0\,h \longrightarrow \infty$, we have:

$$
\sup_x \left|\hat{f}_h^{0*}(x) - \hat{f}_g^0(x)\right| \longrightarrow 0 \text{ in bootstrap probability as } n_0 \longrightarrow \infty, \tag{72}
$$

and

$$
\sup_x \left|\hat{f}_h^{0*}(x) - \hat{f}_g^0(x)\right| = O_{P^*}\left(h^2 + n_0^{-1/2}h^{-1/2}\left(\log\frac{1}{h}\right)^{1/2}\right). \tag{73}
$$

Moreover,

$$
\sup_J \left|\hat{m}_h^{NW*}(x) - \hat{m}_g(x)\right| \longrightarrow 0 \text{ in bootstrap probability as } n_0 \longrightarrow \infty, \tag{74}
$$

and

$$
\sup_J \left|\hat{m}_h^{NW*}(x) - \hat{m}_g(x)\right| = O_{P^*}\left(h^2 + n_0^{-1/2}h^{-1/2}\left(\log\frac{1}{h}\right)^{1/2}\right). \tag{75}
$$

Focusing now on $MISE^*(h)$, it is straightforward that

$$MISE^*(h) = MISE^{a*}(h) + \mathbb{E}^*\left[\int 2\, A_1^*\, A_2^*\, d\hat{F}_g^1(x)\right] + \mathbb{E}^*\left[\int \left(A_2^*\right)^2\, d\hat{F}_g^1(x)\right],$$

where

$$A_1^* = \frac{\hat{\Psi}_h^*(x)}{\hat{f}_g^0(x)} - \frac{\hat{m}_g(x)\hat{f}_h^{0*}(x)}{\hat{f}_g^0(x)} = \frac{1}{n_0\, \hat{f}_g^0(x)} \sum_{i=1}^{n_0} K_h(x - X_i^{0*})(Y_i^{0*} - \hat{m}_g(x)),$$
$$A_2^* = \left(\hat{m}_h^{NW*}(x) - \hat{m}_g(x)\right) \frac{\left(\hat{f}_h^{0*}(x) - \hat{f}_g^0(x)\right)}{\hat{f}_g^0(x)},$$

and $MISE^{a*}(h)$ is given in (66). On the one hand, using expression (59) as well as (C1)-(C3), it turns out:

$$\begin{aligned}
\int_J \left(A_2^*\right)^2\, d\hat{F}_g^1(x) &= \left|\int_J \left(A_2^*\right)^2\, d\hat{F}_g^1(x)\right| \\
&= \left|\int_J \left(\hat{m}_h^{NW*}(x) - \hat{m}_g(x)\right) \frac{\left(\hat{f}_h^0(x) - \hat{f}_g^0(x)\right)}{\hat{f}_g^0(x)} \hat{f}_g^1(x)\, dx\right| \\
&\leq \sup_J \left|\hat{m}_h^{NW*}(x) - \hat{m}_g(x)\right|^2 \sup_J \left|\hat{f}_h^{0*}(x) - \hat{f}_g^0(x)\right|^2 \int_{x\in J} \frac{\hat{f}_g^1(x)}{\hat{f}_g^0(x)^2}\, dx \\
&= \int_{x\in J} \frac{\hat{f}_g^1(x)}{\hat{f}_g^0(x)^2}\, dx \left(O_{P^*}\left(h^2 + n_0^{-1/2}h^{-1/2}\left(\log\frac{1}{h}\right)^{1/2}\right)\right)^4 \\
&= \int_{x\in J} \frac{\hat{f}_g^1(x)}{\hat{f}_g^0(x)^2}\, dx \left(O_{P^*}\left(\max\left\{h^2, n_0^{-1/2}h^{-1/2}\left(\log\frac{1}{h}\right)^{1/2}\right\}\right)\right)^4 \\
&= \int_{x\in J} \frac{\hat{f}_g^1(x)}{\hat{f}_g^0(x)^2}\, dx \left(O_{P^*}\left(\max\left\{h^2, n_0^{-1/2}h^{-1/2}\left(\log\frac{1}{h}\right)^{1/2}\right\}^4\right)\right) \\
&= \int_{x\in J} \frac{\hat{f}_g^1(x)}{\hat{f}_g^0(x)^2}\, dx \left(O_{P^*}\left(\max\left\{h^8, n_0^{-2}h^{-2}\left(\log\frac{1}{h}\right)^2\right\}\right)\right) \\
&= \int_{x\in J} \frac{\hat{f}_g^1(x)}{\hat{f}_g^0(x)^2}\, dx \left(O_{P^*}\left(h^8 + n_0^{-2}h^{-2}\left(\log\frac{1}{h}\right)^2\right)\right) \\
&= O_{P^*}\left(h^8\right) + O_{P^*}\left(n_0^{-2}h^{-2}\left(\log\frac{1}{h}\right)^2\right).
\end{aligned} \tag{76}$$

The first addend in expression (76) is negligible in comparison to the second term in expression (66). Analogously, the second addend in expression (76) becomes insignificant compared to the first term in expression (66). In particular,

$$\begin{aligned}
\left(\int_J (A_2^*)^2 \, d\hat{F}_g^1(x)\right)^{1/2} &= \left(O_{P^*}\left(h^8\right) + O_{P^*}\left(n_0^{-2}h^{-2}\left(\log\frac{1}{h}\right)^2\right)\right)^{1/2} \\
&= \left(O_{P^*}\left(h^8 + n_0^{-2}h^{-2}\left(\log\frac{1}{h}\right)^2\right)\right)^{1/2} \\
&= \left(O_{P^*}\left(\max\left\{h^8, n_0^{-2}h^{-2}\left(\log\frac{1}{h}\right)^2\right\}\right)\right)^{1/2} \\
&= O_{P^*}\left(\max\left\{h^8, n_0^{-2}h^{-2}\left(\log\frac{1}{h}\right)^2\right\}^{1/2}\right) \\
&= O_{P^*}\left(\max\left\{h^4, n_0^{-1}h^{-1}\left(\log\frac{1}{h}\right)\right\}\right) \\
&= O_{P^*}\left(h^4 + n_0^{-1}h^{-1}\left(\log\frac{1}{h}\right)\right) \\
&= O_{P^*}\left(h^4\right) + O_{P^*}\left(n_0^{-1}h^{-1}\left(\log\frac{1}{h}\right)\right). \qquad (77)
\end{aligned}$$

Moreover, thanks to expression (66), we have

$$\mathbb{E}\left[\int_J (A_1^*)^2 \, dF_1(x)\right] = O\left(n_0^{-1}h^{-1} + h^4\right). \qquad (78)$$

Applying Markov's inequality to expression (78), it turns out:

$$\int_J (A_1^*)^2 \, d\hat{F}_g^1(x) = O_{P^*}\left(n_0^{-1}h^{-1} + h^4\right). \qquad (79)$$

Thus,

$$\begin{aligned}
\left(\int_J (A_1^*)^2 \, d\hat{F}_g^1(x)\right)^{1/2} &= \left(O_{P^*}\left(n_0^{-1}h^{-1} + h^4\right)\right)^{1/2} = O_{P^*}\left(\max\left\{n_0^{-1}h^{-1}, h^4\right\}^{1/2}\right) \\
&= O_{P^*}\left(\max\left\{n_0^{-1/2}h^{-1/2}, h^2\right\}\right) = O_{P^*}\left(n_0^{-1/2}h^{-1/2} + h^2\right) \\
&= O_{P^*}\left(n_0^{-1/2}h^{-1/2}\right) + O_{P^*}\left(h^2\right). \qquad (80)
\end{aligned}$$

Bringing together expressions (80) and (77) and using Cauchy-Schwarz inequality, we compute:

$$\left|\int_J A_1\, A_2\, d\hat{F}_g^1(x)\right| \leq 2\left(\int_J \left(A_1^*\right)^2\, d\hat{F}_g^1(x)\right)^{1/2}\left(\int_J \left(A_2^*\right)^2\, d\hat{F}_g^1(x)\right)^{1/2} \tag{81}$$

$$= 2\left(O_p\left(h^4\right) + O_{P^*}\left(n_0^{-1}h^{-1}\log\frac{1}{h}\right)\right)$$

$$\cdot\left(O_p\left(h^2\right) + O_{P^*}\left(n_0^{-1/2}h^{-1/2}\log\frac{1}{h}\right)\right)$$

$$= O_{P^*}\left(h^6\right) + O_{P^*}\left(\frac{h^{7/2}}{n_0^{1/2}}\right) + O_{P^*}\left(\frac{h}{n_0}\log\frac{1}{h}\right)$$

$$+ O_{P^*}\left(\frac{\log\frac{1}{h}}{(n_0\, h)^{3/2}}\right). \tag{82}$$

The first addend in expression (81) is negligible as compared to the second term in expression (66). Furthermore, the fourth addend in expression (81) becomes insignificant in comparison to the first term in expression (66) if

$$\left(\frac{\log\frac{1}{h}}{n_0^{1/2}h^{1/2}}\right) \longrightarrow 0,$$

as $h \longrightarrow 0$, $n_0 \longrightarrow \infty$ and $n_0\, h \longrightarrow \infty$. It remains to be seen what happens with the second and third addends in expression (81). We begin with the second one. Given that $\left(n_0^{-1}h^{-1} + h^4\right)\cdot n_0 h = 1 + h^5 n_0$ and the bandwidth h is of the form $n^{-\alpha}, \alpha > 0$, then

- If $n_0 h^5 \longrightarrow c$, being c a positive real number, then $n_0^{-1}h^{-1} \sim n_0^{-4/5}$ and $h^4 \sim n_0^{-4/5}$, which implies that $h \sim n_0^{-1/5}$, and

$$\frac{h^{7/2}}{n_0^{1/2}} \sim \frac{\left(n_0^{-1/5}\right)^{7/2}}{n_0^{1/2}} = \frac{n_0^{-7/10}}{n_0^{5/10}} = n_0^{-6/5} \longrightarrow 0, \text{ as } n_0 \longrightarrow \infty.$$

- If $n_0 h^5 \longrightarrow 0$, then

$$\frac{\frac{h^{7/2}}{n_0^{1/2}}}{\frac{1}{n_0\, h}} \longrightarrow 0 \Leftrightarrow n_0^{1/2}h^{9/2} \longrightarrow 0 \Leftrightarrow n_0 h^9 \longrightarrow 0,$$

  which is true providing that $n_0 h^5 \longrightarrow 0$.
- If $n_0 h^5 \longrightarrow \infty$, then

$$\frac{\frac{h^{7/2}}{n_0^{1/2}}}{h^4} \longrightarrow 0 \Leftrightarrow n_0^{-1/2}h^{-1/2} \longrightarrow 0 \Leftrightarrow n_0 h \longrightarrow \infty,$$

  which is true providing that $n_0 h^5 \longrightarrow \infty$.

Therefore, $\frac{h^{7/2}}{n_0^{1/2}} = o\left(\frac{1}{n_0\, h} + h^4\right)$.

Finally, as for the third addend in (81),

- If $n_0 h^5 \longrightarrow c$, being c a positive real number, then

$$\frac{h}{n_0} \log \frac{1}{h} \sim n_0^{-6/5} \log n_0^{1/5} \longrightarrow 0, \text{ as } n_0 \longrightarrow \infty.$$

- If $n_0 h^5 \longrightarrow 0$, then

$$\frac{\frac{h}{n_0} \log \frac{1}{h}}{\frac{1}{n_0\, h}} = h^2 \log \frac{1}{h} \longrightarrow 0 \Leftrightarrow h^2 \longrightarrow 0 \Leftrightarrow n_0 h^9 \longrightarrow 0,$$

  which is true providing that $h \longrightarrow 0$.
- If $n_0 h^5 \longrightarrow \infty$, then

$$\frac{\frac{h}{n_0} \log \frac{1}{h}}{h^4} = \frac{\log \frac{1}{h}}{n_0 h^3} \longrightarrow 0 \Leftrightarrow n_0 h^3 \longrightarrow \infty,$$

  which is true providing that $n_0 h^5 \longrightarrow \infty$.

Therefore, $\frac{h}{n_0} \log \frac{1}{h} = o\left(\frac{1}{n_0\, h} + h^4\right)$.

Considering this last reasoning and collecting terms (80), (77) and (81), expression (71) is proven.

Theorem 4. *Consider h_{MISE^a}, its bootstrap version, $h^*_{MISE^a}$, and h_{AMISE^a}, which are the minimizers of expressions (26), (66) and (67), respectively. Under regularity conditions (B1)-(B4) and assuming that g is of order $n_0^{-1/2}$, it holds that*

$$h^*_{MISE^a} - h_{MISE^a} = O_P\left(n_0^{-7/10}\right), \text{ and } \frac{h^*_{MISE^a} - h_{MISE^a}}{h_{MISE^a}} = O_P\left(n_0^{-1/2}\right). \tag{83}$$

Proof Consider h_{MISE^a}, its bootstrap version, $h^*_{MISE^a}$, and h_{AMISE^a}, which are the minimizers of expressions (26), (66) and (67), respectively. Hence, it is obvious that $AMISE^{a*'}\left(h^*_{AMISE^a}\right) = 0$ as well as $MISE^{a'}\left(h_{MISE^a}\right) = 0$. Moreover, applying Taylor expansion to $MISE^{a*'}\left(h^*_{MISE^a}\right)$ towards the point h_{MISE^a}, leads to:

$$\begin{aligned} MISE^{a*'}\left(h^*_{MISE^a}\right) &= MISE^{a*'}\left(h_{MISE^a}\right) - AMISE^{a'}\left(h_{AMISE^a}\right) \\ &+ \left(h^*_{MISE^a} - h_{MISE^a}\right) \cdot MISE^{a*''}\left(h_{MISE^a}\right) + \\ &+ \frac{1}{2}\left(h^*_{MISE^a} - h_{MISE^a}\right)^2 \\ &\cdot MISE^{a*'''}\left(\widetilde{h_{MISE^a}}\right), \end{aligned} \tag{84}$$

where $\widetilde{h_{MISE^a}}$ is an intermediate value between $h^*_{MISE^a}$ and h_{MISE^a}. As a consequence of expression (84), it follows that:

$$
\begin{aligned}
h^*_{MISE^a} - h_{MISE^a} &= -\frac{MISE^{a*\prime}\left(h_{MISE^a}\right) - AMISE^{a\prime}\left(h_{AMISE^a}\right)}{MISE^{a*\prime\prime}\left(h_{MISE^a}\right)} \\
&\cdot \left(1 + o_P(1)\right) \\
&= -\frac{MISE^{a*\prime}\left(h_{AMISE^a}\right) - AMISE^{a\prime}\left(h_{AMISE^a}\right)}{MISE^{a*\prime\prime}\left(h_{AMISE^a}\right)} \\
&\cdot \left(1 + o_P(1)\right) \\
&= -\frac{AMISE^{a*\prime}\left(h_{AMISE^a}\right) - AMISE^{a\prime}\left(h_{AMISE^a}\right)}{AMISE^{a*\prime\prime}\left(h_{AMISE^a}\right)} \\
&\cdot \left(1 + o_P(1)\right). \qquad (85)
\end{aligned}
$$

Thanks to expressions (12) and (35) in the paper, we have:

$$
\begin{aligned}
AMISE^{a\prime\prime}\left(h_{AMISE^a}\right) &= \frac{R(K)}{n_0\, h^3_{AMISE^a}} A + 3\, h^2_{AMISE^a}\, \mu_2(K)^2\, B \\
&= n_0^{-2/5} \cdot \left[R(K)\, A + 3\, \mu_2(K)^2\, B\right], \qquad (86)
\end{aligned}
$$

and,

$$
\begin{aligned}
& AMISE^{a*\prime}\left(h_{AMISE^a}\right) - AMISE^{a\prime}\left(h_{AMISE^a}\right) \\
&= -\frac{R(K)}{n_0\, h^2_{AMISE^a}} \left(\hat{A}_g - A\right) + h^3_{AMISE^a}\, \mu_2(K)^2\, \left(\hat{B}_g - B\right) \\
&= -n_0^{-3/5} \cdot \left[R(K)\left(\hat{A}_g - A\right) + \mu_2(K)^2\, \left(\hat{B}_g - B\right)\right] \\
&= -n_0^{-3/5} \cdot n_0^{-1/2} \cdot \left[R(K) + \mu_2(K)^2\right] \qquad (87) \\
&= -n_0^{-11/10} \cdot \left[R(K) + \mu_2(K)^2\right].
\end{aligned}
$$

Considering now expressions (35) in the paper and (86) above, it turns out:

$$
\begin{aligned}
AMISE^{a*\prime\prime}\left(h_{AMISE^a}\right) &= AMISE^{a*\prime\prime}\left(h_{AMISE^a}\right) - AMISE^{a\prime\prime}\left(h_{AMISE^a}\right) \\
&+ AMISE^{a\prime\prime}\left(h_{AMISE^a}\right) \\
&= AMISE^{a\prime\prime}\left(h_{AMISE^a}\right) + O_P\left(n_0^{-9/10}\right). \qquad (88)
\end{aligned}
$$

Therefore, combining expressions (85) and (88),

$$
\begin{aligned}
h^*_{MISE^a} - h_{MISE^a} &= -\frac{AMISE^{a*\prime}\left(h_{AMISE^a}\right) - AMISE^{a\prime}\left(h_{AMISE^a}\right)}{AMISE^{a\prime\prime}\left(h_{AMISE^a}\right)} \\
&\cdot \left(1 + o_P(1)\right).
\end{aligned}
$$

Finally, collecting terms (87) and (88), and plugging them into (85) leads to proof the result in Theorem 4.

2 | PROOF OF THE RESULTS PRESENTED IN THE APPENDIX

Lemma 3. *Consider the expressions for $\hat{A}_g$ and A from the paper. Then,*

$$\hat{A}_g - A = \sum_{i=1}^{k_0} a_i C^{[s_i]}_{v_i,\ell_i,r_i} + A_1, \tag{89}$$

where $k_0 = 6$, $a_1 = 1$, $a_2 = -1$, $a_3 = 1$, $a_4 = -1$, $a_5 = -2$, $a_6 = -2$, $v_1(x) = \frac{\sigma^2(x)}{f^0(x)}$, $v_2(x) = \frac{\sigma^2(x) f^1(x)}{f^0(x)^2}$, $v_3(x) = \frac{f^1(x)}{f^0(x)^2}$, $v_4(x) = \frac{f^1(x)\Psi_2(x)}{f^0(x)^3}$, $v_5(x) = \frac{f^1(x)\Psi_1(x)}{f^0(x)^3}$,
$v_6(x) = \frac{f^1(x)\Psi_1^2(x)}{f^0(x)^4}$, $\ell_1 = 0$, $\ell_2 = 0$, $\ell_3 = 2$, $\ell_4 = 0$, $\ell_5 = 1$, $\ell_6 = 0$, $r_1 = 0$, $r_2 = 0$, $r_3 = 0$, $r_4 = 0$, $r_5 = 0$, $r_6 = 0$, $[s_1] = 1$, $[s_2] = 0$, $[s_3] = 0$, $[s_4] = 0$, $[s_5] = 0$, $[s_6] = 0$ *and* $A_1 = O\left(r_{0,n_0}\right)$, *with*

$$\begin{aligned} r_{0,n_0} = &\int \left(\hat{f}^0_g(x) - f^0(x)\right)^2 dx + \int \left(\hat{f}^0_g(x) - f^0(x)\right) \cdot \left(\hat{\Psi}_{2,g}(x) - \Psi_2(x)\right) dx \\ &+ \int \left(\hat{f}^0_g(x) - f^0(x)\right) \cdot \left(\hat{\sigma}^2_g(x)\hat{f}^1_g(x) - \sigma^2(x) f^1(x)\right) dx \\ &+ \int \left(\hat{f}^0_g(x)^2 - f^0(x)^2\right) \cdot \left(\hat{\Psi}^2_{1,g}(x) - \Psi^2_1(x)\right) dx + \int \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right)^2 dx \\ &+ \int \left(\hat{\sigma}^2_g(x) - \sigma^2(x)\right) \cdot \left(\hat{f}^1_g(x) - f^1(x)\right) dx + \int \left(\hat{f}^0_g(x)^2 - f^0(x)^2\right)^2 dx. \end{aligned} \tag{90}$$

Proof Our aim is to obtain upper bounds for $\hat{A}_g - A$. On the one hand,

$$\begin{aligned}
\hat{A}_g - A &= \int \frac{\hat{\sigma}_g^2(x)\hat{f}_g^1(x)}{\hat{f}_g^0(x)}\,dx - \int \frac{\sigma^2(x)f^1(x)}{f^0(x)}\,dx \\
&= \int \frac{\hat{\sigma}_g^2(x)\hat{f}_g^1(x) - \sigma^2(x)f^1(x)}{f^0(x)}\,dx - \int \frac{\sigma^2(x)f^1(x)\left(\hat{f}_g^0(x) - f^0(x)\right)}{f^0(x)^2}\,dx \\
&+ O\left(\int \left(\hat{f}_g^0(x) - f^0(x)\right)^2 dx \right.\\
&\left. + \int \left(\hat{f}_g^0(x) - f^0(x)\right) \cdot \left(\hat{\sigma}_g^2(x)\hat{f}_g^1(x) - \sigma^2(x)f^1(x)\right) dx\right) \\
&= \int \frac{\left(\hat{\sigma}_g^2(x) - \sigma^2(x)\right)\hat{f}_g^1(x)}{f^0(x)}\,dx + \int \frac{\sigma^2(x)}{f^0(x)}\left[\hat{f}_g^1(x) - f^1(x)\right] dx \\
&- \int \frac{\sigma^2(x)f^1(x)}{f^0(x)^2}\left[\hat{f}_g^0(x) - f^0(x)\right] dx \\
&+ O\left(\int \left(\hat{f}_g^0(x) - f^0(x)\right)^2 dx \right.\\
&\left. + \int \left(\hat{f}_g^0(x) - f^0(x)\right) \cdot \left(\hat{\sigma}_g^2(x)\hat{f}_g^1(x) - \sigma^2(x)f^1(x)\right) dx\right) \\
&= \int \frac{\left(\hat{\sigma}_g^2(x) - \sigma^2(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right)}{f^0(x)}\,dx + \int \frac{f^1(x)}{f^0(x)}\left[\hat{\sigma}_g^2(x) - \sigma^2(x)\right] dx \\
&+ \int \frac{\sigma^2(x)}{f^0(x)}\left[\hat{f}_g^1(x) - f^1(x)\right] dx - \int \frac{\sigma^2(x)f^1(x)}{f^0(x)^2}\left[\hat{f}_g^0(x) - f^0(x)\right] dx \\
&+ O\left(\int \left(\hat{f}_g^0(x) - f^0(x)\right)^2 dx \right.\\
&\left. + \int \left(\hat{f}_g^0(x) - f^0(x)\right) \cdot \left(\hat{\sigma}_g^2(x)\hat{f}_g^1(x) - \sigma^2(x)f^1(x)\right) dx\right) \\
&= \int \frac{f^1(x)}{f^0(x)}\left[\hat{\sigma}_g^2(x) - \sigma^2(x)\right] dx + \int \frac{\sigma^2(x)}{f^0(x)}\left[\hat{f}_g^1(x) - f^1(x)\right] dx \\
&- \int \frac{\sigma^2(x)f^1(x)}{f^0(x)^2}\left[\hat{f}_g^0(x) - f^0(x)\right] dx \\
&+ O\left(\int \left(\hat{f}_g^0(x) - f^0(x)\right)^2 dx \right.\\
&\left. + \int \left(\hat{f}_g^0(x) - f^0(x)\right) \cdot \left(\hat{\sigma}_g^2(x)\hat{f}_g^1(x) - \sigma^2(x)f^1(x)\right) dx\right) \\
&+ O\left(\int \left(\hat{\sigma}_g^2(x) - \sigma^2(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right). \qquad (91)
\end{aligned}$$

Furthermore,

$$
\begin{aligned}
&\int \frac{f^1(x)}{f^0(x)} \left[\hat{\sigma}_g^2(x) - \sigma^2(x)\right] dx \\
&= \int \frac{f^1(x)}{f^0(x)} \left[\frac{\hat{\Psi}_{2,g}(x)}{\hat{f}_g^0(x)} - \frac{\Psi_2(x)}{f^0(x)}\right] dx - \int \frac{f^1(x)}{f^0(x)} \left[\frac{\hat{\Psi}_{1,g}^2(x)}{\hat{f}_g^0(x)^2} - \frac{\Psi_1^2(x)}{f^0(x)^2}\right] dx \\
&= \int \frac{f^1(x)}{f^0(x)} \left[\frac{\hat{\Psi}_{2,g}(x) - \Psi_2(x)}{f^0(x)} - \frac{\Psi_2(x)\left(\hat{f}_g^0(x) - f^0(x)\right)}{f^0(x)^2}\right] dx \\
&- \int \frac{f^1(x)}{f^0(x)} \left[\frac{\hat{\Psi}_{1,g}^2(x) - \Psi_1^2(x)}{f^0(x)^2} - \frac{\Psi_1^2(x)\left(\hat{f}_g^0(x)^2 - f^0(x)^2\right)}{f^0(x)^4}\right] dx \\
&+ O\left(\int \left(\hat{f}_g^0(x) - f^0(x)\right)^2 dx + \int \left(\hat{f}_g^0(x) - f^0(x)\right)\right. \\
&\left.\cdot \left(\hat{\Psi}_{2,g}(x) - \Psi_2(x)\right) dx\right) \\
&+ O\left(\int \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right)^2 dx + \int \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right)\right. \\
&\left.\cdot \left(\hat{\Psi}_{1,g}^2(x) - \Psi_1^2(x)\right) dx\right) \\
&= \int \frac{f^1(x)}{f^0(x)^2} \left[\hat{\Psi}_{2,g}(x) - \Psi_2(x)\right] dx - \int \frac{f^1(x)\Psi_2(x)}{f^0(x)^3} \left[\hat{f}_g^0(x) - f^0(x)\right] dx \\
&- \int \frac{f^1(x)}{f^0(x)^3} \left[\hat{\Psi}_{1,g}^2(x) - \Psi_1^2(x)\right] dx - \int \frac{f^1(x)\Psi_1^2(x)}{f^0(x)^5} \left[\hat{f}_g^0(x)^2 - f^0(x)^2\right] dx \\
&+ O\left(\int \left(\hat{f}_g^0(x) - f^0(x)\right)^2 dx\right. \\
&\left.+ \int \left(\hat{f}_g^0(x) - f^0(x)\right) \cdot \left(\hat{\Psi}_{2,g}(x) - \Psi_2(x)\right) dx\right) \\
&+ O\left(\int \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right)^2 dx\right. \\
&\left.+ \int \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right) \cdot \left(\hat{\Psi}_{1,g}^2(x) - \Psi_1^2(x)\right) dx\right) \\
&= \int \frac{f^1(x)}{f^0(x)^2} \left[\hat{\Psi}_{2,g}(x) - \Psi_2(x)\right] dx - \int \frac{f^1(x)\Psi_2(x)}{f^0(x)^3} \left[\hat{f}_g^0(x) - f^0(x)\right] dx \\
&- 2\int \frac{f^1(x)\Psi_1(x)}{f^0(x)^3} \left[\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right] dx \\
&- 2\int \frac{f^1(x)\Psi_1^2(x)}{f^0(x)^4} \left[\hat{f}_g^0(x) - f^0(x)\right] dx \\
&+ O\left(\int \left(\hat{f}_g^0(x) - f^0(x)\right)^2 dx + \int \left(\hat{f}_g^0(x) - f^0(x)\right) \cdot \left(\hat{\Psi}_{2,g}(x) - \Psi_2(x)\right) dx\right) \\
&+ O\left(\int \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right)^2 dx\right) \\
&+ O\left(\int \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right)^2 dx\right. \\
&\left.+ \int \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right) \cdot \left(\hat{\Psi}_{1,g}^2(x) - \Psi_1^2(x)\right) dx\right).
\end{aligned}
\tag{92}
$$

In the following remark (expression (93)) is collected the proof used in last equality of (92).

Remark 2. *Let $\hat{\Psi}_2$ be the estimator of Ψ_2. Then:*

$$\begin{aligned}
\hat{\Psi}_2^2 - \Psi_2^2 &= \left(\hat{\Psi}_2 + \Psi_2\right) \cdot \left(\hat{\Psi}_2 - \Psi_2\right) = \left(\hat{\Psi}_2 - \Psi_2 + \Psi_2 + \Psi_2\right) \cdot \left(\hat{\Psi}_2 - \Psi_2\right) \\
&= 2\Psi_2 \left[\hat{\Psi}_2 - \Psi_2\right] + O\left(\left(\hat{\Psi}_2 - \Psi_2\right)^2\right). \\
\hat{\Psi}_2^3 - \Psi_2^3 &= \left(\hat{\Psi}_2^2 + \hat{\Psi}_2\Psi_2 + \Psi_2^2\right) \cdot \left(\hat{\Psi}_2 - \Psi_2\right) \\
&= \left(\hat{\Psi}_2^2 - \Psi_2^2 + \Psi_2^2 + \hat{\Psi}_2\Psi_2 - \Psi_2^2 + \Psi_2^2 + \Psi_2^2\right) \cdot \left(\hat{\Psi}_2 - \Psi_2\right) \\
&= \left(3\Psi_2^2 + (\hat{\Psi}_2 + \Psi_2) \cdot (\hat{\Psi}_2 - \Psi_2) + \Psi_2(\hat{\Psi}_2 - \Psi_2)\right) \cdot \left(\hat{\Psi}_2 - \Psi_2\right) \\
&= 3\Psi_2^2 \left[\hat{\Psi}_2 - \Psi_2\right] + O\left(\left(\hat{\Psi}_2 - \Psi_2\right)^2 + \left(\hat{\Psi}_2 + \Psi_2\right) \cdot \left(\hat{\Psi}_2 - \Psi_2\right)\right). \\
\hat{\Psi}_2^4 - \Psi_2^4 &= \left(\hat{\Psi}_2^3 + \hat{\Psi}_2^2\Psi_2 + \Psi_2^3 + \Psi_2^2\hat{\Psi}_2\right) \cdot \left(\hat{\Psi}_2 - \Psi_2\right) \\
&= \left(\left(\hat{\Psi}_2^3 - \Psi_2^3\right) + \Psi_2^3 + \left(\hat{\Psi}_2^2\Psi_2 - \Psi_2^3\right) + \Psi_2^3 + \Psi_2^3\right. \\
&\quad \left. + \left(\Psi_2^2\hat{\Psi}_2 - \Psi_2^3\right) + \Psi_2^3\right) \cdot \left(\hat{\Psi}_2 - \Psi_2\right) \\
&= \left(4\Psi_2^3 + \left(\hat{\Psi}_2^3 - \Psi_2^3\right) + \Psi_2\left(\hat{\Psi}_2^2 - \Psi_2^2\right) + \Psi_2^2\left(\hat{\Psi}_2 - \Psi_2\right)\right) \cdot \left(\hat{\Psi}_2 - \Psi_2\right) \\
&= 4\Psi_2^3 \left[\hat{\Psi}_2 - \Psi_2\right] + O\left(\left(\hat{\Psi}_2 - \Psi_2\right)^2\right. \\
&\quad \left. + \left(\hat{\Psi}_2 - \Psi_2\right) \cdot \left(\hat{\Psi}_2^2 - \Psi_2^2\right) + \left(\hat{\Psi}_2 - \Psi_2\right) \cdot \left(\hat{\Psi}_2^3 - \Psi_2^3\right)\right). \\
\hat{\Psi}_2^5 - \Psi_2^5 &= \left(\hat{\Psi}_2^4 + \Psi_2^4 + \hat{\Psi}_2\Psi_2^3 + \Psi_2\hat{\Psi}_2^3\right) \cdot \left(\hat{\Psi}_2 - \Psi_2\right) \\
&= \left(\left(\hat{\Psi}_2^4 - \Psi_2^4\right) + \Psi_2^4\right. \\
&\quad \left. + \left(\hat{\Psi}_2\Psi_2^3 - \Psi_2^4\right) + \Psi_2^4 + \Psi_2^4 + \left(\Psi_2\hat{\Psi}_2^3 - \Psi_2^4\right) + \Psi_2^4\right) \cdot \left(\hat{\Psi}_2 - \Psi_2\right) \\
&= \left(\left(\hat{\Psi}_2^4 - \Psi_2^4\right) + 4\Psi_2^4 + \Psi_2^3\left(\hat{\Psi}_2 - \Psi_2\right) + \Psi_2\left(\hat{\Psi}_2^3 - \Psi_2^3\right)\right) \cdot \left(\hat{\Psi}_2 - \Psi_2\right) \\
&= 4\Psi_2^4 \left[\hat{\Psi}_2 - \Psi_2\right] + O\left(\left(\hat{\Psi}_2 - \Psi_2\right) \cdot \left(\hat{\Psi}_2^4 - \Psi_2^4\right) + \left(\hat{\Psi}_2 - \Psi_2\right)^2\right) \\
&\quad + O\left(\left(\hat{\Psi}_2 - \Psi_2\right) \cdot \left(\hat{\Psi}_2^3 - \Psi_2^3\right)\right).
\end{aligned}$$

Then,

$$\hat{\Psi}_2^2 - \Psi_2^2 = 2\Psi_2\left[\hat{\Psi}_2 - \Psi_2\right] + O\left(\left(\hat{\Psi}_2 - \Psi_2\right)^2\right), \tag{93}$$

$$\begin{aligned}\hat{\Psi}_2^3 - \Psi_2^3 &= 3\Psi_2^2\left[\hat{\Psi}_2 - \Psi_2\right] \\ &+ O\left(\left(\hat{\Psi}_2 - \Psi_2\right)^2 + \left(\hat{\Psi}_2 + \Psi_2\right)\cdot\left(\hat{\Psi}_2 - \Psi_2\right)\right),\end{aligned} \tag{94}$$

$$\begin{aligned}\hat{\Psi}_2^4 - \Psi_2^4 &= 4\Psi_2^3\left[\hat{\Psi}_2 - \Psi_2\right] + O\left(\left(\hat{\Psi}_2 - \Psi_2\right)^2 + \left(\hat{\Psi}_2 - \Psi_2\right)\cdot\left(\hat{\Psi}_2^2 - \Psi_2^2\right)\right) \\ &+ O\left(\left(\hat{\Psi}_2 - \Psi_2\right)\cdot\left(\hat{\Psi}_2^3 - \Psi_2^3\right)\right), \text{ and}\end{aligned} \tag{95}$$

$$\begin{aligned}\hat{\Psi}_2^5 - \Psi_2^5 &= 4\Psi_2^4\left[\hat{\Psi}_2 - \Psi_2\right] + O\left(\left(\hat{\Psi}_2 - \Psi_2\right)\cdot\left(\hat{\Psi}_2^4 - \Psi_2^4\right) + \left(\hat{\Psi}_2 - \Psi_2\right)^2\right) \\ &+ O\left(\left(\hat{\Psi}_2 - \Psi_2\right)\cdot\left(\hat{\Psi}_2^3 - \Psi_2^3\right)\right).\end{aligned} \tag{96}$$

Combining expressions (92) and (91), Lemma 3 is concluded.

Lemma 4. *Given the expressions for $\hat{B}_g$ and B (from the paper), then $\hat{B}_g - B$ consists of a sum of 60 terms similar to those in expression (89). Specifically,*

$$\hat{B}_g - B = \sum_{i=7}^{k_1} a_i C_{\nu_i,\ell_i,r_i}^{[s_i]} + B_1, \tag{97}$$

where $k_1 = 66$. The functions $\nu(x)$, and the values of r, ℓ, a and $[s]$ are collected in Tables 1-3. Additionally, term B_1 is of order $O\left(r_{1,n_0}\right)$ with r_{1,n_0} being also given in the tables.

Proof Focusing now on expression $\hat{B}_g - B$,

$$\begin{aligned}\hat{B}_g - B = &\int\left[\hat{m}_g''(x)^2\hat{f}_g^1(x) - m''(x)^2 f^1(x)\right]dx \\ &+ 4\int\left[\frac{\hat{m}_g'(x)\hat{m}_g''(x)\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x)}{\hat{f}_g^0(x)} - \frac{m'(x)m''(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)}\right]dx \\ &+ 4\int\left[\frac{\hat{m}_g'(x)^2\left(\hat{f}_g^0\right)'(x)^2\hat{f}_g^1(x)}{\hat{f}_g^0(x)^2} - \frac{m'(x)^2\left(f^0\right)'(x)^2 f^1(x)}{f^0(x)^2}\right]dx,\end{aligned} \tag{98}$$

where

$$B_1 := \int\left[\hat{m}_g''(x)^2\hat{f}_g^1(x) - m''(x)^2 f^1(x)\right]dx,$$

$$B_2 := 4\int\left[\frac{\hat{m}_g'(x)\hat{m}_g''(x)\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x)}{\hat{f}_g^0(x)} - \frac{m'(x)m''(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)}\right]dx, \text{ and}$$

$$B_3 := 4\int\left[\frac{\hat{m}_g'(x)^2\left(\hat{f}_g^0\right)'(x)^2\hat{f}_g^1(x)}{\hat{f}_g^0(x)^2} - \frac{m'(x)^2\left(f^0\right)'(x)^2 f^1(x)}{f^0(x)^2}\right]dx.$$

i	$v_i(x)$	$[s]_i$	ℓ_i	r_i	a_i
7	$\frac{\left(f^0\right)'(x)^2 f^1(x) m'(x)}{f^0(x)^3}$	0	1	1	8
8	$\frac{\left(f^0\right)'(x)^2 f^1(x) m'(x) \Psi_1'(x)}{f^0(x)^4}$	0	0	0	−8
9	$\frac{\left(f^0\right)'(x)^2 f^1(x) m'(x) \left(f^0\right)'(x)}{f^0(x)^4}$	0	1	0	−8
10	$\frac{m'(x)^2 \left(f^0\right)'(x)^2}{f^0(x)^2}$	1	0	0	4
11	$\frac{m'(x)^2 f^1(x) \left(f^0\right)'(x)}{f^0(x)^2}$	0	0	1	8
12	$\frac{\left(f^0\right)'(x)^2 f^1(x) m'(x) \left(f^0\right)'(x) \Psi_1(x)}{f^0(x)^4}$	0	0	0	−16
13	$\frac{\left(f^0\right)'(x)^2 f^1(x) m'(x) \Psi_1(x)}{f^0(x)^4}$	0	0	1	−8
14	$\frac{m'(x)^2 \left(f^0\right)'(x)^2 f^1(x)}{f^0(x)^3}$	0	0	0	−8
15	$\frac{f^1(x) m''(x)}{f^0(x)}$	0	1	2	2
16	$\frac{f^1(x) m''(x) \Psi_1''(x)}{f^0(x)^2}$	0	0	0	−2
17	$\frac{f^1(x) m''(x) \Psi_1(x)}{f^0(x)^2}$	0	0	2	−2
18	$\frac{f^1(x) m''(x) \left(f^0\right)''(x)}{f^0(x)^2}$	0	1	0	−2
19	$\frac{f^1(x) m''(x) \Psi_1(x) \left(f^0\right)''(x)}{f^0(x)}$	0	0	0	8
20	$\frac{f^1(x) m''(x) \Psi_1'(x)}{f^0(x)^2}$	0	0	1	−4
21	$\frac{f^1(x) m''(x) \left(f^0\right)'(x)}{f^0(x)^2}$	0	1	1	−4
22	$\frac{f^1(x) m''(x) \Psi_1(x) \left(f^0\right)'(x)}{f^0(x)^3}$	0	0	0	8
23	$\frac{f^1(x) m''(x) \Psi_1(x) \left(f^0\right)'(x)}{f^0(x)^3}$	0	0	1	8
24	$\frac{f^1(x) m''(x) \left(f^0\right)'(x)^2}{f^0(x)^3}$	0	1	0	4
25	$\frac{f^1(x) m''(x) \Psi_1(x) \left(f^0\right)'(x)^2}{f^0(x)^4}$	0	0	0	−12
26	$m''(x)^2$	1	0	0	1
27	$\frac{\Psi_1'(x) \Psi_1''(x) \left(f^0\right)'(x)}{f^0(x)^3}$	1	0	0	4

TABLE 1 Values of $v_i(x)$, r_i, ℓ_i, a_i and $[s]_i$ in expression (97), $i \in \{7, \ldots, 27\}$.

i	$v_i(x)$	$[s_i]$	ℓ_i	r_i	a_i
28	$\frac{\Psi_1'(x)\Psi_1''(x)f^1(x)}{f^0(x)^3}$	0	0	1	4
29	$\frac{\Psi_1'(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^3}$	0	1	2	4
30	$\frac{\Psi_1''(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^3}$	0	1	1	4
31	$\frac{m'(x)m''(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^2}$	0	0	0	−4
32	$\frac{\Psi_1'(x)\Psi_1''(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^4}$	0	0	0	−8
33	$\frac{\Psi_1'(x)\left(f^0\right)'(x)^3\Psi_1(x)}{f^0(x)^5}$	1	0	0	8
34	$\frac{\Psi_1'(x)\left(f^0\right)'(x)^3f^1(x)}{f^0(x)^5}$	0	1	0	8
35	$\frac{\Psi_1'(x)\Psi_1(x)f^1(x)\left(f^0\right)'(x)^2}{f^0(x)^5}$	0	0	1	24
36	$\frac{\left(f^0\right)'(x)^3\Psi_1(x)f^1(x)}{f^0(x)^5}$	0	1	1	8
37	$\frac{\Psi_1'(x)\left(f^0\right)'(x)^3\Psi_1(x)f^1(x)}{f^0(x)^5}$	0	0	0	−32
38	$\frac{\Psi_1'(x)\left(f^0\right)''(x)\Psi_1(x)\left(f^0\right)'(x)}{f^0(x)^4}$	1	0	0	−4
39	$\frac{\Psi_1'(x)\left(f^0\right)''(x)\Psi_1(x)f^1(x)}{f^0(x)^4}$	0	0	1	−4
40	$\frac{\Psi_1'(x)\left(f^0\right)''(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^4}$	0	1	0	−4
41	$\frac{\Psi_1'(x)\Psi_1(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^4}$	0	0	2	−4
42	$\frac{\left(f^0\right)''(x)\Psi_1(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^4}$	0	1	1	−4
43	$\frac{\Psi_1'(x)\left(f^0\right)''(x)\Psi_1(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^5}$	0	0	0	−12
44	$\frac{\Psi_1'(x)^2\left(f^0\right)'(x)^2}{f^0(x)^4}$	1	0	0	−8
45	$\frac{\Psi_1'(x)^2f^1(x)\left(f^0\right)'(x)}{f^0(x)^4}$	0	0	1	−16
46	$\frac{\left(f^0\right)'(x)^2f^1(x)\Psi_1'(x)}{f^0(x)^4}$	0	1	1	−16
47	$\frac{\Psi_1'(x)^2\left(f^0\right)'(x)^2f^1(x)}{f^0(x)^5}$	0	0	0	24

TABLE 2 Values of $v_i(x)$, r_i, ℓ_i, a_i and $[s_i]$ in expression (97), $i \in \{28, \ldots, 47\}$.

i	$v_i(x)$	$[s_i]$	ℓ_i	r_i	a_i
48	$\frac{\left(f^0\right)'(x)^2\Psi_1(x)\Psi_1''(x)}{f^0(x)^4}$	1	0	0	−4
49	$\frac{\left(f^0\right)'(x)^2\Psi_1(x)f^1(x)}{f^0(x)^4}$	0	1	2	−4
50	$\frac{\left(f^0\right)'(x)^2\Psi_1''(x)f^1(x)}{f^0(x)^4}$	0	1	0	−4
51	$\frac{\Psi_1(x)\Psi_1''(x)f^1(x)\left(f^0\right)'(x)}{f^0(x)^4}$	0	0	1	−8
52	$\frac{\left(f^0\right)'(x)^2\Psi_1(x)\Psi_1''(x)f^1(x)}{f^0(x)^5}$	0	0	0	12
53	$\frac{\left(f^0\right)'(x)^4\Psi_1^2(x)}{f^0(x)^6}$	1	0	0	−8
54	$\frac{\left(f^0\right)'(x)^4f^1(x)\Psi_1(x)}{f^0(x)^6}$	0	1	0	−16
55	$\frac{\Psi_1^2(x)f^1(x)}{f^0(x)^3}$	0	0	1	−32
56	$\frac{\left(f^0\right)'(x)^4\Psi_1^2(x)f^1(x)}{f^0(x)^7}$	0	0	0	32
57	$\frac{\left(f^0\right)'(x)^2\Psi_1(x)^2\left(f^0\right)''(x)}{f^0(x)^5}$	1	0	0	4
58	$\frac{\left(f^0\right)'(x)^2\Psi_1(x)^2f^1(x)}{f^0(x)^5}$	0	0	2	4
59	$\frac{\left(f^0\right)'(x)^2\left(f^0\right)''(x)f^1(x)\Psi_1(x)}{f^0(x)^5}$	0	1	0	8
60	$\frac{\Psi_1(x)^2\left(f^0\right)''(x)f^1(x)\left(f^0\right)'(x)}{f^0(x)^5}$	0	0	1	8
61	$\frac{\left(f^0\right)'(x)^2\Psi_1(x)^2\left(f^0\right)''(x)f^1(x)}{f^0(x)^6}$	0	0	0	−16
62	$\frac{\left(f^0\right)'(x)^3\Psi_1(x)\Psi_1'(x)}{f^0(x)^5}$	1	0	0	8
63	$\frac{\left(f^0\right)'(x)^3\Psi_1(x)f^1(x)}{f^0(x)^5}$	0	1	1	8
64	$\frac{\left(f^0\right)'(x)^3\Psi_1'(x)f^1(x)}{f^0(x)^5}$	0	1	0	8
65	$\frac{\Psi_1(x)\Psi_1'(x)f^1(x)\left(f^0\right)'(x)^3}{f^0(x)^5}$	0	0	1	24
66	$\frac{\left(f^0\right)'(x)^3\Psi_1(x)\Psi_1'(x)f^1(x)}{f^0(x)^6}$	0	0	0	−32

TABLE 3 Values of $v_i(x)$, r_i, ℓ_i, a_i and $[s_i]$ in expression (97), $i \in \{48, \ldots, 66\}$.

We will focus on term B_1 in the first place. Carrying on with calculations, considering expression (93), it leads to:

$$\begin{aligned}
B_1 &:= \int \left[\hat{m}_g''(x)^2 \hat{f}_g^1(x) - m''(x)^2 f^1(x)\right] dx \\
&= \int \hat{m}_g''(x)^2 \hat{f}_g^1(x)\, dx - \int m''(x)^2 f^1(x)\, dx \\
&= \int \hat{m}_g''(x)^2 \left(\hat{f}_g^1(x) - f^1(x) + f^1(x)\right) dx - \int m''(x)^2 f^1(x)\, dx \\
&= \int f^1(x)\left[\hat{m}_g''(x)^2 - m''(x)^2\right] dx + \int \hat{m}_g''(x)^2 \left[\hat{f}_g^1(x) - f^1(x)\right] dx \\
&= \int f^1(x)\left[\hat{m}_g''(x)^2 - m''(x)^2\right] dx + \int m''(x)^2 \left[\hat{f}_g^1(x) - f^1(x)\right] dx \\
&+ \int \left(\hat{m}_g''(x)^2 - m''(x)^2\right)\cdot\left(\hat{f}_g^1(x) - f^1(x)\right) dx \\
&= \int f^1(x)\left[\hat{m}_g''(x)^2 - m''(x)^2\right] dx + \int m''(x)^2 \left[\hat{f}_g^1(x) - f^1(x)\right] dx \\
&+ O\left(\int \left(\hat{m}_g''(x)^2 - m''(x)^2\right)\cdot\left(\hat{f}_g^1(x) - f^1(x)\right) dx\right) \\
&= 2\int f^1(x) m''(x)\left[\hat{m}_g''(x) - m''(x)\right] dx + \int m''(x)^2 \left[\hat{f}_g^1(x) - f^1(x)\right] dx \\
&+ O\left(\int \left(\hat{m}_g''(x)^2 - m''(x)^2\right)\cdot\left(\hat{f}_g^1(x) - f^1(x)\right) dx\right) \\
&+ O\left(\int \left(\hat{m}_g''(x) - m''(x)\right)^2 dx\right).
\end{aligned} \tag{99}$$

Furthermore, considering expressions (93), (94) and (95), it turns out:

$$\begin{aligned}
&2\int f^1(x) m''(x)\left[\hat{m}_g''(x) - m''(x)\right] dx \\
&= 2\int f^1(x) m''(x)\left[\frac{\hat{\Psi}_{1,g}''(x)}{\hat{f}_g^0(x)} - \frac{\Psi_1''(x)}{f^0(x)}\right] dx \\
&- 2\int f^1(x) m''(x)\left[\frac{\hat{\Psi}_{1,g}(x)\left(\hat{f}_g^0\right)''(x)}{\hat{f}_g^0(x)^2} - \frac{\Psi_1(x)\left(f^0\right)''(x)}{f^0(x)^2}\right] dx \\
&- 4\int f^1(x) m''(x)\left[\frac{\hat{\Psi}_{1,g}'(x)\left(\hat{f}_g^0\right)'(x)}{\hat{f}_g^0(x)^2} - \frac{\Psi_1'(x)\left(f^0\right)'(x)}{f^0(x)^2}\right] dx \\
&+ 4\int f^1(x) m''(x)\left[\frac{\hat{\Psi}_{1,g}(x)\left(\hat{f}_g^0\right)'(x)^2}{\hat{f}_g^0(x)^3} - \frac{\Psi_1(x)\left(f^0\right)'(x)^2}{f^0(x)^3}\right] dx \\
&= 2\int \frac{f^1(x) m''(x)}{f^0(x)}\left[\hat{\Psi}_{1,g}''(x) - \Psi_1''(x)\right] dx \\
&- 2\int \frac{f^1(x) m''(x)\Psi_1''(x)}{f^0(x)^2}\left[\hat{f}_g^0(x) - f^0(x)\right] dx
\end{aligned}$$

$$
\begin{aligned}
&+2\int \frac{f^1(x)m''(x)\Psi_1(x)\left(f^0\right)''(x)}{f^0(x)^4}\left[\hat{f}_g^0(x)^4-f^0(x)^4\right]dx\\
&+4\int \frac{f^1(x)m''(x)\Psi_1(x)\left(f^0\right)'(x)}{f^0(x)^4}\left[\hat{f}_g^0(x)^2-f^0(x)^2\right]dx\\
&-2\int \frac{f^1(x)m''(x)}{f^0(x)^2}\left[\hat{\Psi}_{1,g}(x)\left(\hat{f}_g^0\right)''(x)-\Psi_1(x)\left(f^0\right)''(x)\right]dx\\
&-4\int \frac{f^1(x)m''(x)}{f^0(x)^2}\left[\hat{\Psi}'_{1,g}(x)\left(\hat{f}_g^0\right)'(x)-\Psi'_1(x)\left(f^0\right)'(x)\right]dx\\
&+4\int \frac{f^1(x)m''(x)}{f^0(x)^3}\left[\hat{\Psi}_{1,g}(x)\left(\hat{f}_g^0\right)'(x)^2-\Psi_1(x)\left(f^0\right)'(x)^2\right]dx\\
&-4\int \frac{f^1(x)m''(x)\Psi_1(x)\left(f^0\right)'(x)^2}{f^0(x)^6}\left[\hat{f}_g^0(x)^3-f^0(x)^3\right]dx\\
&+O\left(\int\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)^2dx\right)+O\left(\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)^2\right)\\
&+O\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{\Psi}''_{1,g}(x)-\Psi''_1(x)\right)dx\right)\\
&+O\left(\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)\left(\hat{f}_g^0\right)''(x)-\Psi_1(x)\left(f^0\right)''(x)\right)dx\right)\\
&+O\left(\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)\cdot\left(\hat{\Psi}'_{1,g}(x)\left(\hat{f}_g^0\right)'(x)-\Psi'_1(x)\left(f^0\right)'(x)\right)dx\right)\\
&+O\left(\int\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}(x)\left(\hat{f}_g^0\right)'(x)^2-\Psi_1(x)\left(f^0\right)'(x)^2\right)dx\right)\\
&=2\int \frac{f^1(x)m''(x)}{f^0(x)}\left[\hat{\Psi}''_{1,g}(x)-\Psi''_1(x)\right]dx\\
&-2\int \frac{f^1(x)m''(x)\Psi''_1(x)}{f^0(x)^2}\left[\hat{f}_g^0(x)-f^0(x)\right]dx\\
&-2\int \frac{f^1(x)m''(x)\Psi_1(x)}{f^0(x)^2}\left[\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right]dx\\
&-2\int \frac{f^1(x)m''(x)}{f^0(x)^2}\left[\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^0\right)''(x)\right]dx\\
&+8\int \frac{f^1(x)m''(x)\Psi_1(x)\left(f^0\right)''(x)}{f^0(x)}\left[\hat{f}_g^0(x)-f^0(x)\right]dx\\
&-4\int \frac{f^1(x)m''(x)\Psi'_1(x)}{f^0(x)^2}\left[\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right]dx\\
&-4\int \frac{f^1(x)m''(x)}{f^0(x)^2}\left[\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\hat{f}_g^0\right)'(x)\right]dx\\
&+8\int \frac{f^1(x)m''(x)\Psi_1(x)\left(f^0\right)'(x)}{f^0(x)^3}\left[\hat{f}_g^0(x)-f^0(x)\right]dx\\
&+4\int \frac{f^1(x)m''(x)\Psi_1(x)}{f^0(x)^3}\left[\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right]dx\\
&+4\int \frac{f^1(x)m''(x)}{f^0(x)^3}\left[\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^0\right)'(x)^2\right]dx
\end{aligned}
$$

$$
\begin{aligned}
&- 12 \int \frac{f^1(x) m''(x) \Psi_1(x) \left(f^0\right)'(x)^2}{f^0(x)^4} \left[\hat{f}_g^0(x) - f^0(x)\right] dx \\
&+ O\left(\int \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right) dx\right) + O\left(\int \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right)^2 dx\right) \\
&+ O\left(\int \left(\hat{f}_g^0(x) - f^0(x)\right) \cdot \left(\hat{\Psi}_{1,g}''(x) - \Psi_1''(x)\right) dx\right) \\
&+ O\left(\int \left(\hat{f}_g^0(x) - f^0(x)\right) \cdot \left(\hat{f}_g^0(x)^3 - f^0(x)^3\right) dx\right) \\
&+ O\left(\int \left(\hat{f}_g^0(x) - f^0(x)\right) \cdot \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right) dx\right) \\
&+ O\left(\int \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right) \cdot \left(\hat{\Psi}_{1,g}(x) \left(\hat{f}_g^0\right)''(x) - \Psi_1(x) \left(f^0\right)''(x)\right) dx\right) \\
&+ O\left(\int \left(\hat{f}_g^0(x)^3 - f^0(x)^3\right)^2 dx\right) + O\left(\int \left(\hat{f}_g^0(x) - f^0(x)\right)^2 dx\right) \\
&+ O\left(\int \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right) \cdot \left(\hat{\Psi}_{1,g}'(x) \left(\hat{f}_g^0\right)'(x) - \Psi_1'(x) \left(f^0\right)'(x)\right) dx\right) \\
&+ O\left(\int \left(\hat{f}_g^0(x)^3 - f^0(x)^3\right) \cdot \left(\hat{\Psi}_{1,g}(x) \left(\hat{f}_g^0\right)'(x)^2 - \Psi_1(x) \left(f^0\right)'(x)^2\right) dx\right) \\
&= 2 \int \frac{f^1(x) m''(x)}{f^0(x)} \left[\hat{\Psi}_{1,g}''(x) - \Psi_1''(x)\right] dx \\
&- 2 \int \frac{f^1(x) m''(x) \Psi_1''(x)}{f^0(x)^2} \left[\hat{f}_g^0(x) - f^0(x)\right] dx \\
&- 2 \int \frac{f^1(x) m''(x) \Psi_1(x)}{f^0(x)^2} \left[\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right] dx \\
&- 2 \int \frac{f^1(x) m''(x) \left(f^0\right)''(x)}{f^0(x)^2} \left[\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right] dx \\
&+ 8 \int \frac{f^1(x) m''(x) \Psi_1(x) \left(f^0\right)''(x)}{f^0(x)} \left[\hat{f}_g^0(x) - f^0(x)\right] dx \\
&- 4 \int \frac{f^1(x) m''(x) \Psi_1'(x)}{f^0(x)^2} \left[\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right] dx \\
&- 4 \int \frac{f^1(x) m''(x) \left(f^0\right)'(x)}{f^0(x)^2} \left[\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right] dx \\
&+ 8 \int \frac{f^1(x) m''(x) \Psi_1(x) \left(f^0\right)'(x)}{f^0(x)^3} \left[\hat{f}_g^0(x) - f^0(x)\right] dx \\
&+ 8 \int \frac{f^1(x) m''(x) \Psi_1(x) \left(f^0\right)'(x)}{f^0(x)^3} \left[\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right] dx \\
&+ 4 \int \frac{f^1(x) m''(x) \left(f^0\right)'(x)^2}{f^0(x)^3} \left[\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right] dx \\
&- 12 \int \frac{f^1(x) m''(x) \Psi_1(x) \left(f^0\right)'(x)^2}{f^0(x)^4} \left[\hat{f}_g^0(x) - f^0(x)\right] dx \\
&+ O\left(\int \hat{f}_g^0(x)^2 - f^0(x)^2 \, dx\right) \\
&+ O\left(\int \left(\hat{f}_g^0(x) - f^0(x)\right) \cdot \left(\hat{\Psi}_{1,g}''(x) - \Psi_1''(x)\right) dx\right) \\
&+ O\left(\int \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right)^2 dx\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ O\left(\int \left(\hat{f}_g^0(x)^3 - f^0(x)^3\right)^2 dx\right) \\
&+ O\left(\int \left(\hat{f}_g^0(x) - f^0(x)\right) \cdot \left(\hat{f}_g^0(x)^3 - f^0(x)^3\right) dx\right) \\
&+ O\left(\int \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right) \cdot \left(\hat{\Psi}_{1,g}(x)\left(\hat{f}_g^0\right)''(x) - \Psi_1(x)\left(f^0\right)''(x)\right) dx\right) \\
&+ O\left(\int \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right) \cdot \left(\hat{\Psi}_{1,g}'(x)\left(\hat{f}_g^0\right)'(x) - \Psi_1'(x)\left(f^0\right)'(x)\right) dx\right) \\
&+ O\left(\int \left(\hat{f}_g^0(x) - f^0(x)\right) \cdot \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right) dx\right) \\
&+ O\left(\int \left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right) \cdot \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right) dx\right) \\
&+ O\left(\int \left(\hat{f}_g^0(x)^3 - f^0(x)^3\right) \cdot \left(\hat{\Psi}_{1,g}(x)\left(\hat{f}_g^0\right)'(x)^2 - \Psi_1(x)\left(f^0\right)'(x)^2\right) dx\right) \\
&+ O\left(\int \left(\hat{f}_g^0(x) - f^0(x)\right)^2 dx\right) \\
&+ O\left(\int \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right) dx\right) \\
&+ O\left(\int \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right)^2 dx\right) \\
&+ O\left(\int \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\left(\hat{f}_g^0\right)'(x)^2 - \left(f^0\right)'(x)^2\right) dx\right). \qquad (100)
\end{aligned}
$$

Plugging expression (100) in (99) leads to the following expression for B_1:

$$
\begin{aligned}
B_1 := & \, 2\int \frac{f^1(x)m''(x)}{f^0(x)}\left[\hat{\Psi}''_{1,g}(x)-\Psi''_1(x)\right]dx+\int m''(x)^2\left[\hat{f}^1_g(x)-f^1(x)\right]dx \\
& -2\int \frac{f^1(x)m''(x)\Psi''_1(x)}{f^0(x)^2}\left[\hat{f}^0_g(x)-f^0(x)\right]dx \\
& -2\int \frac{f^1(x)m''(x)\Psi_1(x)}{f^0(x)^2}\left[\left(\hat{f}^0_g\right)''(x)-\left(f^0\right)''(x)\right]dx \\
& -2\int \frac{f^1(x)m''(x)\left(f^0\right)''(x)}{f^0(x)^2}\left[\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right]dx \\
& +8\int \frac{f^1(x)m''(x)\Psi_1(x)\left(f^0\right)''(x)}{f^0(x)}\left[\hat{f}^0_g(x)-f^0(x)\right]dx \\
& -4\int \frac{f^1(x)m''(x)\Psi'_1(x)}{f^0(x)^2}\left[\left(\hat{f}^0_g\right)'(x)-\left(f^0\right)'(x)\right]dx \\
& -4\int \frac{f^1(x)m''(x)\left(f^0\right)'(x)}{f^0(x)^2}\left[\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right]dx \\
& +8\int \frac{f^1(x)m''(x)\Psi_1(x)\left(f^0\right)'(x)}{f^0(x)^3}\left[\hat{f}^0_g(x)-f^0(x)\right]dx \\
& +8\int \frac{f^1(x)m''(x)\Psi_1(x)\left(f^0\right)'(x)}{f^0(x)^3}\left[\left(\hat{f}^0_g\right)'(x)-\left(f^0\right)'(x)\right]dx \\
& +4\int \frac{f^1(x)m''(x)\left(f^0\right)'(x)^2}{f^0(x)^3}\left[\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right]dx \\
& -12\int \frac{f^1(x)m''(x)\Psi_1(x)\left(f^0\right)'(x)^2}{f^0(x)^4}\left[\hat{f}^0_g(x)-f^0(x)\right]dx
\end{aligned}
$$

$$
\begin{aligned}
&+O\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)dx\right)\\
&+O\left(\int\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)^2dx\right)+O\left(\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)^2dx\right)\\
&+O\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)dx\right)\\
&+O\left(\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)\left(\hat{f}_g^0\right)''(x)-\Psi_1(x)\left(f^0\right)''(x)\right)dx\right)\\
&+O\left(\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}'(x)\left(\hat{f}_g^0\right)'(x)-\Psi_1'(x)\left(f^0\right)'(x)\right)dx\right)\\
&+O\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)dx\right)\\
&+O\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)dx\right)\\
&+O\left(\int\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}(x)\left(\hat{f}_g^0\right)'(x)^2-\Psi_1(x)\left(f^0\right)'(x)^2\right)dx\right)\\
&+O\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2dx\right)+O\left(\int\left(\hat{m}_g''(x)-m''(x)\right)^2dx\right)\\
&+O\left(\int\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)dx\right)\\
&+O\left(\int\left(\hat{m}_g''(x)^2-m''(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+O\left(\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)dx\right)+O\left(\int\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)^2dx\right)\\
&+O\left(\int\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)dx\right).
\end{aligned}
\tag{101}
$$

We now move on to achieve a deeper insight of term B_2, considering expressions (93), (94), (95) and (96), it follows that:

$$
\begin{aligned}
B_2 :=\ & 4\int\left[\frac{\hat{m}_g'(x)\hat{m}_g''(x)\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x)}{\hat{f}_g^0(x)}-\frac{m'(x)m''(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)}\right]dx\\
=\ & 4\int\frac{1}{f^0(x)}\left[\hat{m}_g'(x)\hat{m}_g''(x)\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x)-m'(x)m''(x)\left(f^0\right)'(x)f^1(x)\right]dx\\
&-4\int\frac{m'(x)m''(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^2}\left[\hat{f}_g^0(x)-f^0(x)\right]dx\\
&+O\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2dx\right)+O\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)\right.\\
&\left.\cdot\left(\hat{m}_g'(x)\hat{m}_g''(x)\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x)-m'(x)m''(x)\left(f^0\right)'(x)f^1(x)\right)dx\right).
\end{aligned}
\tag{102}
$$

Computing further calculations with the first term in expression (102) using (93), (94), (95) and (96), we have:

$$
\begin{aligned}
&4\int \frac{1}{f^0(x)}\left[\hat{m}_g'(x)\hat{m}_g''(x)\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x) - m'(x)m''(x)\left(f^0\right)'(x)f^1(x)\right]dx \\
&= 4\int \frac{1}{f^0(x)}\left[\frac{\hat{\Psi}_{1,g}'(x)\hat{\Psi}_{1,g}''(x)\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x)}{\hat{f}_g^0(x)^2} - \frac{\Psi_1'(x)\Psi_1''(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^2}\right]dx \\
&+ 8\int \frac{1}{f^0(x)}\left[\frac{\hat{\Psi}_{1,g}'(x)\left(\hat{f}_g^0\right)'(x)^3\hat{\Psi}_{1,g}(x)\hat{f}_g^1(x)}{\hat{f}_g^0(x)^4} - \frac{\Psi_1'(x)\left(f^0\right)'(x)^3\Psi_1(x)f^1(x)}{f^0(x)^4}\right]dx \\
&- 4\int \frac{1}{f^0(x)}\left[\frac{\hat{\Psi}_{1,g}'(x)\left(\hat{f}_g^0\right)''(x)\hat{\Psi}_{1,g}(x)\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x)}{\hat{f}_g^0(x)^3}\right. \\
&\left. - \frac{\Psi_1'(x)\left(f^0\right)''(x)\Psi_1(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^3}\right]dx \\
&- 8\int \frac{1}{f^0(x)}\left[\frac{\hat{\Psi}_{1,g}'(x)^2\left(\hat{f}_g^0\right)'(x)^2\hat{f}_g^1(x)}{\hat{f}_g^0(x)^3} - \frac{\Psi_1'(x)^2\left(f^0\right)'(x)^2f^1(x)}{f^0(x)^3}\right]dx \\
&- 4\int \frac{1}{f^0(x)}\left[\frac{\left(\hat{f}_g^0\right)'(x)^2\hat{\Psi}_{1,g}(x)\hat{\Psi}_{1,g}''(x)\hat{f}_g^1(x)}{\hat{f}_g^0(x)^3} - \frac{\left(f^0\right)'(x)^2\Psi_1(x)\Psi_1''(x)f^1(x)}{f^0(x)^3}\right]dx \\
&- 8\int \frac{1}{f^0(x)}\left[\frac{\left(\hat{f}_g^0\right)'(x)^4\hat{\Psi}_{1,g}^2(x)\hat{f}_g^1(x)}{\hat{f}_g^0(x)^5} - \frac{\left(f^0\right)'(x)^4\Psi_1^2(x)f^1(x)}{f^0(x)^5}\right]dx \\
&+ 4\int \frac{1}{f^0(x)}\left[\frac{\left(\hat{f}_g^0\right)'(x)^2\hat{\Psi}_{1,g}(x)^2\left(\hat{f}_g^0\right)''(x)\hat{f}_g^1(x)}{\hat{f}_g^0(x)^4}\right. \\
&\left. - \frac{\left(f^0\right)'(x)^2\Psi_1(x)^2\left(f^0\right)''(x)f^1(x)}{f^0(x)^4}\right]dx \\
&+ 8\int \frac{1}{f^0(x)}\left[\frac{\left(\hat{f}_g^0\right)'(x)^3\hat{\Psi}_{1,g}(x)\hat{\Psi}_{1,g}'(x)\hat{f}_g^1(x)}{\hat{f}_g^0(x)^4} - \frac{\left(f^0\right)'(x)^3\Psi_1(x)\Psi_1'(x)f^1(x)}{f^0(x)^4}\right]dx. \qquad (103)
\end{aligned}
$$

The first addend in expression (103), using (93), (94), (95) and (96), happens to be:

$$
\begin{aligned}
&4\int \frac{1}{f^0(x)}\left[\frac{\hat\Psi'_{1,g}(x)\hat\Psi''_{1,g}(x)\left(\hat f_g^0\right)'(x)\hat f_g^1(x)}{\hat f_g^0(x)^2}-\frac{\Psi'_1(x)\Psi''_1(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^2}\right]dx\\
&=4\int \frac{1}{f^0(x)^3}\left[\hat\Psi'_{1,g}(x)\hat\Psi''_{1,g}(x)\left(\hat f_g^0\right)'(x)\hat f_g^1(x)-\Psi'_1(x)\Psi''_1(x)\left(f^0\right)'(x)f^1(x)\right]dx\\
&-4\int \frac{\Psi'_1(x)\Psi''_1(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^5}\left[\hat f_g^0(x)^2-f^0(x)^2\right]dx\\
&+O\left(\int\left(\hat f_g^0(x)^2-f^0(x)^2\right)^2dx\right)+O\left(\int\left(\hat f_g^0(x)^2-f^0(x)^2\right)\right.\\
&\left.\cdot\left(\hat\Psi'_{1,g}(x)\hat\Psi''_{1,g}(x)\left(\hat f_g^0\right)'(x)\hat f_g^1(x)-\Psi'_1(x)\Psi''_1(x)\left(f^0\right)'(x)f^1(x)\right)dx\right)\\
&=4\int \frac{1}{f^0(x)^3}\left[\hat\Psi'_{1,g}(x)\hat\Psi''_{1,g}(x)\left(\hat f_g^0\right)'(x)\hat f_g^1(x)-\Psi'_1(x)\Psi''_1(x)\left(f^0\right)'(x)f^1(x)\right]dx\\
&-8\int \frac{\Psi'_1(x)\Psi''_1(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^4}\left[\hat f_g^0(x)-f^0(x)\right]dx\\
&+O\left(\int\left(\hat f_g^0(x)^2-f^0(x)^2\right)^2dx\right)+O\left(\int\left(\hat f_g^0(x)^2-f^0(x)^2\right)\right.\\
&\left.\cdot\left(\hat\Psi'_{1,g}(x)\hat\Psi''_{1,g}(x)\left(\hat f_g^0\right)'(x)\hat f_g^1(x)-\Psi'_1(x)\Psi''_1(x)\left(f^0\right)'(x)f^1(x)\right)dx\right)\\
&+O\left(\int\left(\hat f_g^0(x)-f^0(x)\right)^2dx\right).
\end{aligned}
\tag{104}
$$

Working out calculations with the first term in expression (104) leads to:

$$
\begin{aligned}
&4\int \frac{1}{f^0(x)^3}\left[\hat\Psi'_{1,g}(x)\hat\Psi''_{1,g}(x)\left(\hat f_g^0\right)'(x)\hat f_g^1(x)-\Psi'_1(x)\Psi''_1(x)\left(f^0\right)'(x)f^1(x)\right]dx\\
&=4\int \frac{\Psi'_1(x)}{f^0(x)^3}\left[\hat\Psi''_{1,g}(x)\left(\hat f_g^0\right)'(x)\hat f_g^1(x)-\Psi''_1(x)\left(f^0\right)'(x)f^1(x)\right]dx\\
&+4\int \frac{1}{f^0(x)^3}\left[\left(\hat\Psi'_{1,g}(x)-\Psi'_1(x)\right)\hat\Psi''_{1,g}(x)\left(\hat f_g^0\right)'(x)\hat f_g^1(x)\right]dx\\
&=4\int \frac{\Psi'_1(x)\Psi''_1(x)}{f^0(x)^3}\left[\left(\hat f_g^0\right)'(x)\hat f_g^1(x)-\left(f^0\right)'(x)f^1(x)\right]dx\\
&+4\int \frac{\Psi'_1(x)}{f^0(x)^3}\left[\left(\hat\Psi''_{1,g}(x)-\Psi''_1(x)\right)\left(\hat f_g^0\right)'(x)\hat f_g^1(x)\right]dx\\
&+4\int \frac{\Psi''_1(x)}{f^0(x)^3}\left[\left(\hat\Psi'_{1,g}(x)-\Psi'_1(x)\right)\left(\hat f_g^0\right)'(x)\hat f_g^1(x)\right]dx\\
&+4\int \frac{1}{f^0(x)^3}\left[\left(\hat\Psi'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\hat\Psi''_{1,g}(x)-\Psi''_1(x)\right)\left(\hat f_g^0\right)'(x)\hat f_g^1(x)\right]dx
\end{aligned}
$$

$$
\begin{aligned}
&= 4\int \frac{\Psi_1'(x)\Psi_1''(x)\left(f^0\right)'(x)}{f^0(x)^3}\left[\hat f_g^1(x) - f^1(x)\right]dx \\
&+ 4\int \frac{\Psi_1'(x)\Psi_1''(x)}{f^0(x)^3}\left[\left(\left(\hat f_g^0\right)'(x) - \left(f^0\right)'(x)\right)\cdot \hat f_g^1(x)\right]dx \\
&+ 4\int \frac{\Psi_1'(x)\left(f^0\right)'(x)}{f^0(x)^3}\left[\left(\hat\Psi_{1,g}''(x) - \Psi_1''(x)\right)\cdot \hat f_g^1(x)\right]dx \\
&+ 4\int \frac{\Psi_1''(x)\left(f^0\right)'(x)}{f^0(x)^3}\left[\left(\hat\Psi_{1,g}'(x) - \Psi_1'(x)\right)\hat f_g^1(x)\right]dx \\
&+ 4\int \frac{\Psi_1'(x)}{f^0(x)^3}\left[\left(\hat\Psi_{1,g}''(x) - \Psi_1''(x)\right)\cdot\left(\left(\hat f_g^0\right)'(x) - \left(f^0\right)'(x)\right)\hat f_g^1(x)\right]dx \\
&+ 4\int \frac{\Psi_1''(x)}{f^0(x)^3}\left[\left(\hat\Psi_{1,g}'(x) - \Psi_1'(x)\right)\cdot\left(\left(\hat f_g^0\right)'(x) - \left(f^0\right)'(x)\right)\hat f_g^1(x)\right]dx \\
&+ 4\int \frac{\left(f^0\right)'(x)}{f^0(x)^3}\left[\left(\hat\Psi_{1,g}'(x) - \Psi_1'(x)\right)\cdot\left(\hat\Psi_{1,g}''(x) - \Psi_1''(x)\right)\hat f_g^1(x)\right]dx \\
&+ 4\int \frac{1}{f^0(x)^3}\left[\left(\hat\Psi_{1,g}'(x) - \Psi_1'(x)\right)\cdot\left(\hat\Psi_{1,g}''(x) - \Psi_1''(x)\right)\right. \\
&\left.\cdot\left(\left(\hat f_g^0\right)'(x) - \left(f^0\right)'(x)\right)\hat f_g^1(x)\right]dx \\
&= 4\int \frac{\Psi_1'(x)\Psi_1''(x)\left(f^0\right)'(x)}{f^0(x)^3}\left[\hat f_g^1(x) - f^1(x)\right]dx \\
&+ 4\int \frac{\Psi_1'(x)\Psi_1''(x)f^1(x)}{f^0(x)^3}\left[\left(\hat f_g^0\right)'(x) - \left(f^0\right)'(x)\right]dx \\
&+ 4\int \frac{\Psi_1'(x)\Psi_1''(x)}{f^0(x)^3}\left[\left(\left(\hat f_g^0\right)'(x) - \left(f^0\right)'(x)\right)\cdot\left(\hat f_g^1(x) - f^1(x)\right)\right]dx \\
&+ 4\int \frac{\Psi_1'(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^3}\left[\hat\Psi_{1,g}''(x) - \Psi_1''(x)\right]dx \\
&+ 4\int \frac{\Psi_1''(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^3}\left[\hat\Psi_{1,g}'(x) - \Psi_1'(x)\right]dx \\
&+ 4\int \frac{\Psi_1'(x)\left(f^0\right)'(x)}{f^0(x)^3}\left[\left(\hat\Psi_{1,g}''(x) - \Psi_1''(x)\right)\cdot\left(\hat f_g^1(x) - f^1(x)\right)\right]dx \\
&+ 4\int \frac{\Psi_1'(x)f^1(x)}{f^0(x)^3}\left[\left(\hat\Psi_{1,g}''(x) - \Psi_1''(x)\right)\cdot\left(\left(\hat f_g^0\right)'(x) - \left(f^0\right)'(x)\right)\right]dx \\
&+ 4\int \frac{\Psi_1'(x)}{f^0(x)^3}\left[\left(\hat\Psi_{1,g}''(x) - \Psi_1''(x)\right)\cdot\left(\left(\hat f_g^0\right)'(x) - \left(f^0\right)'(x)\right)\right. \\
&\left.\cdot\left(\hat f_g^1(x) - f^1(x)\right)\right]dx \\
&+ 4\int \frac{\Psi_1''(x)\left(f^0\right)'(x)}{f^0(x)^3}\left[\left(\hat\Psi_{1,g}'(x) - \Psi_1'(x)\right)\cdot\left(\hat f_g^1(x) - f^1(x)\right)\right]dx \\
&+ 4\int \frac{\Psi_1''(x)f^1(x)}{f^0(x)^3}\left[\left(\hat\Psi_{1,g}'(x) - \Psi_1'(x)\right)\cdot\left(\left(\hat f_g^0\right)'(x) - \left(f^0\right)'(x)\right)\right]dx
\end{aligned}
$$

$$
\begin{aligned}
&+4\int \frac{\Psi_1''(x)}{f^0(x)^3}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\right.\\
&\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx\\
&+4\int \frac{\left(f^0\right)'(x)f^1(x)}{f^0(x)^3}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\right]dx\\
&+4\int \frac{\left(f^0\right)'(x)}{f^0(x)^3}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\right.\\
&\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx\\
&+4\int \frac{f^1(x)}{f^0(x)^3}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\right.\\
&\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\right]dx\\
&+4\int \frac{1}{f^0(x)^3}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\right.\\
&\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx\\
&=4\int \frac{\Psi_1'(x)\Psi_1''(x)\left(f^0\right)'(x)}{f^0(x)^3}\left[\hat{f}_g^1(x)-f^1(x)\right]dx\\
&+4\int \frac{\Psi_1'(x)\Psi_1''(x)f^1(x)}{f^0(x)^3}\left[\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right]dx\\
&+4\int \frac{\Psi_1'(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^3}\left[\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right]dx\\
&+4\int \frac{\Psi_1''(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^3}\left[\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right]dx\\
&+O\left(\int\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+O\left(\int\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+O\left(\int\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)dx\right)\\
&+O\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+O\left(\int\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\right.\\
&\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+O\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)dx\right)\\
&+O\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)dx\right)\\
&+O\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+O\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+O\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)dx\right)\\
&+O\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\right.\\
&\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right).
\end{aligned}
\tag{105}
$$

Plugging expression (105) in (104), the first addend in expression (103) happens to be:

$$\begin{aligned}
&4\int \frac{\Psi_1'(x)\Psi_1''(x)\left(f^0\right)'(x)}{f^0(x)^3}\left[\hat{f}_g^1(x)-f^1(x)\right]dx\\
&+4\int \frac{\Psi_1'(x)\Psi_1''(x)f^1(x)}{f^0(x)^3}\left[\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right]dx\\
&+4\int \frac{\Psi_1'(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^3}\left[\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right]dx\\
&+4\int \frac{\Psi_1''(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^3}\left[\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right]dx\\
&-8\int \frac{\Psi_1'(x)\Psi_1''(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^4}\left[\hat{f}_g^0(x)-f^0(x)\right]dx\\
&+O\left(\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)^2dx\right)+O\left(\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)\right.\\
&\cdot\left.\left(\hat{\Psi}_{1,g}'(x)\hat{\Psi}_{1,g}''(x)\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x)-\Psi_1'(x)\Psi_1''(x)\left(f^0\right)'(x)f^1(x)\right)dx\right)\\
&+O\left(\int\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+O\left(\int\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+O\left(\int\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)dx\right)\\
&+O\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+O\left(\int\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+O\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2dx\right)\\
&+O\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)dx\right)\\
&+O\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)dx\right)\\
&+O\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+O\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+O\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)dx\right)\\
&+O\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\right.\\
&\cdot\left.\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right).
\end{aligned}\tag{106}$$

The second term in expression (103) turns out to be:

$$
\begin{aligned}
&8\int \frac{1}{f^0(x)}\left[\frac{\hat{\Psi}_{1,g}'(x)\left(\hat{f}_g^0\right)'(x)^3\hat{\Psi}_{1,g}(x)\hat{f}_g^1(x)}{\hat{f}_g^0(x)^4}-\frac{\Psi_1'(x)\left(f^0\right)'(x)^3\Psi_1(x)f^1(x)}{f^0(x)^4}\right]dx\\
&=8\int \frac{1}{f^0(x)^5}\left[\hat{\Psi}_{1,g}'(x)\left(\hat{f}_g^0\right)'(x)^3\hat{\Psi}_{1,g}(x)\hat{f}_g^1(x)-\Psi_1'(x)\left(f^0\right)'(x)^3\Psi_1(x)f^1(x)\right]dx\\
&-8\int \frac{\Psi_1'(x)\left(f^0\right)'(x)^3\Psi_1(x)f^1(x)}{f^0(x)^9}\left[\hat{f}_g^0(x)^4-f^0(x)^4\right]dx\\
&+O\left(\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)^2dx\right)+O\left(\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)\right.\\
&\left.\cdot\left(\hat{\Psi}_{1,g}'(x)\left(\hat{f}_g^0\right)'(x)^3\hat{\Psi}_{1,g}(x)\hat{f}_g^1(x)-\Psi_1'(x)\left(f^0\right)'(x)^3\Psi_1(x)f^1(x)\right)dx\right)\\
&=8\int \frac{1}{f^0(x)^5}\left[\hat{\Psi}_{1,g}'(x)\left(\hat{f}_g^0\right)'(x)^3\hat{\Psi}_{1,g}(x)\hat{f}_g^1(x)-\Psi_1'(x)\left(f^0\right)'(x)^3\Psi_1(x)f^1(x)\right]dx\\
&-32\int \frac{\Psi_1'(x)\left(f^0\right)'(x)^3\Psi_1(x)f^1(x)}{f^0(x)^5}\left[\hat{f}_g^0(x)-f^0(x)\right]dx\\
&+O\left(\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)^2dx\right)+O\left(\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)\right.\\
&\left.\cdot\left(\hat{\Psi}_{1,g}'(x)\left(\hat{f}_g^0\right)'(x)^3\hat{\Psi}_{1,g}(x)\hat{f}_g^1(x)-\Psi_1'(x)\left(f^0\right)'(x)^3\Psi_1(x)f^1(x)\right)dx\right)\\
&+O\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)dx+\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2dx\right)\\
&+O\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)dx\right).
\end{aligned}
\tag{107}
$$

Carrying on with calculations considering the first term in expression (107) leads to:

$$
\begin{aligned}
& 8\int \frac{1}{f^0(x)^5}\left[\hat{\Psi}'_{1,g}(x)\left(\hat{f}^0_g\right)'(x)^3\hat{\Psi}_{1,g}(x)\hat{f}^1_g(x)-\Psi'_1(x)\left(f^0\right)'(x)^3\Psi_1(x)f^1(x)\right]dx \\
&= 8\int \frac{\Psi'_1(x)}{f^0(x)^5}\left[\left(\hat{f}^0_g\right)'(x)^3\hat{\Psi}_{1,g}(x)\hat{f}^1_g(x)-\left(f^0\right)'(x)^3\Psi_1(x)f^1(x)\right]dx \\
&+ 8\int \frac{1}{f^0(x)^5}\left[\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\hat{f}^0_g\right)'(x)^3\hat{\Psi}_{1,g}(x)\hat{f}^1_g(x)\right]dx \\
&= 8\int \frac{\Psi'_1(x)\left(f^0\right)'(x)^3}{f^0(x)^5}\left[\hat{\Psi}_{1,g}(x)\hat{f}^1_g(x)-\Psi_1(x)f^1(x)\right]dx \\
&+ 8\int \frac{\Psi'_1(x)}{f^0(x)^5}\left[\left(\left(\hat{f}^0_g\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\hat{\Psi}_{1,g}(x)\hat{f}^1_g(x)\right]dx \\
&+ 8\int \frac{\left(f^0\right)'(x)^3}{f^0(x)^5}\left[\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\hat{\Psi}_{1,g}(x)\hat{f}^1_g(x)\right]dx \\
&+ 8\int \frac{1}{f^0(x)^5}\left[\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\left(\hat{f}^0_g\right)'(x)^3-\left(f^0\right)'(x)^3\right)\right. \\
&\left.\cdot\hat{\Psi}_{1,g}(x)\hat{f}^1_g(x)\right]dx \\
&= 8\int \frac{\Psi'_1(x)\left(f^0\right)'(x)^3\Psi_1(x)}{f^0(x)^5}\left[\hat{f}^1_g(x)-f^1(x)\right]dx \\
&+ 8\int \frac{\Psi'_1(x)\left(f^0\right)'(x)^3}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\hat{f}^1_g(x)\right]dx \\
&+ 8\int \frac{\Psi'_1(x)\Psi_1(x)}{f^0(x)^5}\left[\left(\left(\hat{f}^0_g\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\hat{f}^1_g(x)\right]dx \\
&+ 8\int \frac{\Psi'_1(x)}{f^0(x)^5}\left[\left(\left(\hat{f}^0_g\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\hat{f}^1_g(x)\right]dx \\
&+ 8\int \frac{\left(f^0\right)'(x)^3\Psi_1(x)}{f^0(x)^5}\left[\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\hat{f}^1_g(x)\right]dx \\
&+ 8\int \frac{\left(f^0\right)'(x)^3}{f^0(x)^5}\left[\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\hat{f}^1_g(x)\right]dx \\
&+ 8\int \frac{\Psi_1(x)}{f^0(x)^5}\left[\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\left(\hat{f}^0_g\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\hat{f}^1_g(x)\right]dx \\
&+ 8\int \frac{1}{f^0(x)^5}\left[\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\left(\hat{f}^0_g\right)'(x)^3-\left(f^0\right)'(x)^3\right)\right. \\
&\left.\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\hat{f}^1_g(x)\right]dx \\
&= 8\int \frac{\Psi'_1(x)\left(f^0\right)'(x)^3\Psi_1(x)}{f^0(x)^5}\left[\hat{f}^1_g(x)-f^1(x)\right]dx \\
&+ 8\int \frac{\Psi'_1(x)\left(f^0\right)'(x)^3f^1(x)}{f^0(x)^5}\left[\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right]dx \\
&+ 8\int \frac{\Psi'_1(x)\left(f^0\right)'(x)^3}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}^1_g(x)-f^1(x)\right)\right]dx
\end{aligned}
$$

$$+8\int \frac{\Psi_1'(x)\Psi_1(x)f^1(x)}{f^0(x)^5}\left[\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right]dx$$

$$+8\int \frac{\Psi_1'(x)\Psi_1(x)}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx$$

$$+8\int \frac{\Psi_1'(x)f^1(x)}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right]dx$$

$$+8\int \frac{\Psi_1'(x)}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right.$$

$$\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx+8\int \frac{\left(f^0\right)'(x)^3\Psi_1(x)f^1(x)}{f^0(x)^5}\left[\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right]dx$$

$$+8\int \frac{\left(f^0\right)'(x)^3\Psi_1(x)}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx$$

$$+8\int \frac{\left(f^0\right)'(x)^3f^1(x)}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right]dx$$

$$+8\int \frac{\left(f^0\right)'(x)^3}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx$$

$$+8\int \frac{\Psi_1(x)f^1(x)}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\right]dx$$

$$+8\int \frac{\Psi_1(x)}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\right.$$

$$\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx$$

$$+8\int \frac{f^1(x)}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\right.$$

$$\left.\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right]dx$$

$$+8\int \frac{1}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\right.$$

$$\left.\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx$$

$$=8\int \frac{\Psi_1'(x)\left(f^0\right)'(x)^3\Psi_1(x)}{f^0(x)^5}\left[\hat{f}_g^1(x)-f^1(x)\right]dx$$

$$+8\int \frac{\Psi_1'(x)\left(f^0\right)'(x)^3f^1(x)}{f^0(x)^5}\left[\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right]dx$$

$$+24\int \frac{\Psi_1'(x)\Psi_1(x)f^1(x)\left(f^0\right)'(x)^2}{f^0(x)^5}\left[\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right]dx$$

$$+8\int \frac{\left(f^0\right)'(x)^3\Psi_1(x)f^1(x)}{f^0(x)^5}\left[\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right]dx$$

$$+O\left(\int\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)$$

$$+O\left(\int\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)$$

$$+O\left(\int\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)dx\right)$$

$$+O\left(\int\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)^2dx\right)$$

$$
\begin{aligned}
&+ O\left(\int \left(\left(\hat{f}_g^0\right)'(x)^3 - \left(f^0\right)'(x)^3\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right) \\
&+ O\left(\int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right) \\
&+ O\left(\int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x)\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) dx\right) \\
&+ O\left(\int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x)\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right) \\
&+ O\left(\int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x)\right) \cdot \left(\left(\hat{f}_g^0\right)'(x)^3 - \left(f^0\right)'(x)^3\right) dx\right) \\
&+ O\left(\int \left(\left(\hat{f}_g^0\right)'(x)^2 - \left(f^0\right)'(x)^2\right) dx\right) \\
&+ O\left(\int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x)\right) \cdot \left(\left(\hat{f}_g^0\right)'(x)^3 - \left(f^0\right)'(x)^3\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right) \\
&+ O\left(\int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x)\right) \cdot \left(\left(\hat{f}_g^0\right)'(x)^3 - \left(f^0\right)'(x)^3\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) dx\right) \\
&+ O\left(\int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x)\right) \cdot \left(\left(\hat{f}_g^0\right)'(x)^3 - \left(f^0\right)'(x)^3\right)\right. \\
&\left. \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right). \qquad (108)
\end{aligned}
$$

Plugging expression (108) in (107), the second addend of expression (103) turns out ot be:

$$
\begin{aligned}
& 8\int \frac{\Psi_1'(x)\left(f^0\right)'(x)^3\Psi_1(x)}{f^0(x)^5}\left[\hat{f}_g^1(x)-f^1(x)\right]dx \\
& +8\int \frac{\Psi_1'(x)\left(f^0\right)'(x)^3 f^1(x)}{f^0(x)^5}\left[\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right]dx \\
& +24\int \frac{\Psi_1'(x)\Psi_1(x)f^1(x)\left(f^0\right)'(x)^2}{f^0(x)^5}\left[\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right]dx \\
& +8\int \frac{\left(f^0\right)'(x)^3\Psi_1(x)f^1(x)}{f^0(x)^5}\left[\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right]dx \\
& -32\int \frac{\Psi_1'(x)\left(f^0\right)'(x)^3\Psi_1(x)f^1(x)}{f^0(x)^5}\left[\hat{f}_g^0(x)-f^0(x)\right]dx \\
& +O\left(\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)^2dx\right)+O\left(\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)\right. \\
& \left.\cdot\left(\hat{\Psi}_{1,g}'(x)\left(\hat{f}_g^0\right)'(x)^3\hat{\Psi}_{1,g}(x)\hat{f}_g^1(x)-\Psi_1'(x)\left(f^0\right)'(x)^3\Psi_1(x)f^1(x)\right)dx\right) \\
& +O\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)+\left(\hat{f}_g^0(x)-f^0(x)\right)^2dx\right) \\
& +O\left(\int\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)^2dx\right) \\
& +O\left(\int\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right) \\
& +O\left(\int\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right) \\
& +O\left(\int\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)dx\right) \\
& +O\left(\int\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)dx\right)+O\left(\int\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\right. \\
& \left.\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right) \\
& +O\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right) \\
& +O\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)dx\right) \\
& +O\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right) \\
& +O\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)dx\right) \\
& +O\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)dx\right) \\
& +O\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right) \\
& +O\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)dx\right) \\
& +O\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right. \\
& \left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right).
\end{aligned}
\tag{109}
$$

Carrying on with calculations with the third addend in expression (103), using (93), (94), (95) and (96), it leads to:

$$
\begin{aligned}
&-4\int \frac{1}{f^0(x)}\left[\frac{\hat{\Psi}'_{1,g}(x)\left(\hat{f}_g^0\right)''(x)\hat{\Psi}_{1,g}(x)\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x)}{\hat{f}_g^0(x)^3}\right.\\
&\left.-\frac{\Psi'_1(x)\left(f^0\right)''(x)\Psi_1(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^3}\right]dx\\
&=-4\int \frac{1}{f^0(x)^4}\left[\hat{\Psi}'_{1,g}(x)\left(\hat{f}_g^0\right)''(x)\hat{\Psi}_{1,g}(x)\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x)\right.\\
&\left.-\Psi'_1(x)\left(f^0\right)''(x)\Psi_1(x)\left(f^0\right)'(x)f^1(x)\right]dx\\
&-4\int \frac{\Psi'_1(x)\left(f^0\right)''(x)\Psi_1(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^7}\left[\hat{f}_g^0(x)^3-f^0(x)^3\right]dx\\
&+O\left(\int\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)^2dx\right)\\
&+O\left(\int\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)\cdot\left(\hat{\Psi}'_{1,g}(x)\left(\hat{f}_g^0\right)''(x)\hat{\Psi}_{1,g}(x)\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x)\right.\right.\\
&\left.\left.-\Psi'_1(x)\left(f^0\right)''(x)\Psi_1(x)\left(f^0\right)'(x)f^1(x)\right)dx\right)\\
&=-4\int \frac{1}{f^0(x)^4}\left[\hat{\Psi}'_{1,g}(x)\left(\hat{f}_g^0\right)''(x)\hat{\Psi}_{1,g}(x)\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x)\right.\\
&\left.-\Psi'_1(x)\left(f^0\right)''(x)\Psi_1(x)\left(f^0\right)'(x)f^1(x)\right]dx\\
&-12\int \frac{\Psi'_1(x)\left(f^0\right)''(x)\Psi_1(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^5}\left[\hat{f}_g^0(x)-f^0(x)\right]dx\\
&+O\left(\int\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)^2dx\right)+O\left(\int\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)\right.\\
&\cdot\left(\hat{\Psi}'_{1,g}(x)\left(\hat{f}_g^0\right)''(x)\hat{\Psi}_{1,g}(x)\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x)-\Psi'_1(x)\left(f^0\right)''(x)\Psi_1(x)\left(f^0\right)'(x)\right.\\
&\left.\left.f^1(x)\right)dx\right)+O\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2+\left(\hat{f}_g^0(x)-f^0(x)\right)\right.\\
&\left.\cdot\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)dx\right).
\end{aligned}
\tag{110}
$$

Working out further calculation with the first term in expression (110), using (93), (94), (95) and (96), it turns out:

$$
\begin{aligned}
& - 4 \int \frac{1}{f^0(x)^4} \left[\hat{\Psi}'_{1,g}(x) \left(\hat{f}^0_g\right)''(x) \hat{\Psi}_{1,g}(x) \left(\hat{f}^0_g\right)'(x) \hat{f}^1_g(x) \right. \\
& \left. - \Psi'_1(x) \left(f^0\right)''(x) \Psi_1(x) \left(f^0\right)'(x) f^1(x) \right] dx \\
& = -4 \int \frac{\Psi'_1(x)}{f^0(x)^4} \left[\left(\hat{f}^0_g\right)''(x) \hat{\Psi}_{1,g}(x) \left(\hat{f}^0_g\right)'(x) \hat{f}^1_g(x) \right. \\
& \left. - \left(f^0\right)''(x) \Psi_1(x) \left(f^0\right)'(x) f^1(x) \right] dx \\
& - 4 \int \frac{1}{f^0(x)^4} \left[\left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x) \right) \cdot \left(\hat{f}^0_g\right)''(x) \hat{\Psi}_{1,g}(x) \left(\hat{f}^0_g\right)'(x) \hat{f}^1_g(x) \right] dx \\
& = -4 \int \frac{\Psi'_1(x) \left(f^0\right)''(x)}{f^0(x)^4} \left[\hat{\Psi}_{1,g}(x) \left(\hat{f}^0_g\right)'(x) \hat{f}^1_g(x) - \Psi_1(x) \left(f^0\right)'(x) f^1(x) \right] dx \\
& - 4 \int \frac{\Psi'_1(x)}{f^0(x)^4} \left[\left(\left(\hat{f}^0_g\right)''(x) - \left(f^0\right)''(x) \right) \cdot \hat{\Psi}_{1,g}(x) \left(\hat{f}^0_g\right)'(x) \hat{f}^1_g(x) \right] dx \\
& - 4 \int \frac{\left(f^0\right)''(x)}{f^0(x)^4} \left[\left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x) \right) \cdot \hat{\Psi}_{1,g}(x) \left(\hat{f}^0_g\right)'(x) \hat{f}^1_g(x) \right] dx \\
& - 4 \int \frac{1}{f^0(x)^4} \left[\left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x) \right) \cdot \left(\left(\hat{f}^0_g\right)''(x) - \left(f^0\right)''(x) \right) \right. \\
& \left. \cdot \hat{\Psi}_{1,g}(x) \left(\hat{f}^0_g\right)'(x) \hat{f}^1_g(x) \right] dx \\
& = -4 \int \frac{\Psi'_1(x) \left(f^0\right)''(x) \Psi_1(x)}{f^0(x)^4} \left[\left(\hat{f}^0_g\right)'(x) \hat{f}^1_g(x) - \left(f^0\right)'(x) f^1(x) \right] dx \\
& - 4 \int \frac{\Psi'_1(x) \left(f^0\right)''(x)}{f^0(x)^4} \left[\left(\hat{\Psi}_{1,g}(x) - \Psi_1(x) \right) \cdot \left(\hat{f}^0_g\right)'(x) \hat{f}^1_g(x) \right] dx \\
& - 4 \int \frac{\Psi'_1(x) \Psi_1(x)}{f^0(x)^4} \left[\left(\left(\hat{f}^0_g\right)''(x) - \left(f^0\right)''(x) \right) \cdot \left(\hat{f}^0_g\right)'(x) \hat{f}^1_g(x) \right] dx \\
& - 4 \int \frac{\Psi'_1(x)}{f^0(x)^4} \left[\left(\left(\hat{f}^0_g\right)''(x) - \left(f^0\right)''(x) \right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x) \right) \right. \\
& \left. \cdot \left(\hat{f}^0_g\right)'(x) \hat{f}^1_g(x) \right] dx \\
& - 4 \int \frac{\left(f^0\right)''(x) \Psi_1(x)}{f^0(x)^4} \left[\left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x) \right) \cdot \left(\hat{f}^0_g\right)'(x) \hat{f}^1_g(x) \right] dx \\
& - 4 \int \frac{\left(f^0\right)''(x)}{f^0(x)^4} \left[\left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x) \right) \right. \\
& \left. \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x) \right) \cdot \left(\hat{f}^0_g\right)'(x) \hat{f}^1_g(x) \right] dx
\end{aligned}
$$

$$-4\int \frac{\Psi_1(x)}{f^0(x)^4}\left[\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\right.$$

$$\left.\cdot\left(\left(\hat{f}^0_g\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{f}^0_g\right)'(x)\hat{f}^1_g(x)\right]dx$$

$$-4\int \frac{1}{f^0(x)^4}\left[\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\left(\hat{f}^0_g\right)''(x)-\left(f^0\right)''(x)\right)\right.$$

$$\left.\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}^0_g\right)'(x)\hat{f}^1_g(x)\right]dx$$

$$=-4\int \frac{\Psi'_1(x)\left(f^0\right)''(x)\Psi_1(x)\left(f^0\right)'(x)}{f^0(x)^4}\left[\hat{f}^1_g(x)-f^1(x)\right]dx$$

$$-4\int \frac{\Psi'_1(x)\left(f^0\right)''(x)\Psi_1(x)}{f^0(x)^4}\left[\left(\left(\hat{f}^0_g\right)'(x)-\left(f^0\right)'(x)\right)\cdot\hat{f}^1_g(x)\right]dx$$

$$-4\int \frac{\Psi'_1(x)\left(f^0\right)''(x)\left(f^0\right)'(x)}{f^0(x)^4}\left[\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\hat{f}^1_g(x)\right]dx$$

$$-4\int \frac{\Psi'_1(x)\left(f^0\right)''(x)}{f^0(x)^4}\left[\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right.$$

$$\left.\cdot\left(\left(\hat{f}^0_g\right)'(x)-\left(f^0\right)'(x)\right)\cdot\hat{f}^1_g(x)\right]dx$$

$$-4\int \frac{\Psi'_1(x)\Psi_1(x)\left(f^0\right)'(x)}{f^0(x)^4}\left[\left(\left(\hat{f}^0_g\right)''(x)-\left(f^0\right)''(x)\right)\cdot\hat{f}^1_g(x)\right]dx$$

$$-4\int \frac{\Psi'_1(x)\Psi_1(x)}{f^0(x)^4}\left[\left(\left(\hat{f}^0_g\right)''(x)-\left(f^0\right)''(x)\right)\right.$$

$$\left.\cdot\left(\left(\hat{f}^0_g\right)'(x)-\left(f^0\right)'(x)\right)\cdot\hat{f}^1_g(x)\right]dx-4\int \frac{\Psi'_1(x)\left(f^0\right)'(x)}{f^0(x)^4}$$

$$\left[\left(\left(\hat{f}^0_g\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\hat{f}^1_g(x)\right]dx$$

$$-4\int \frac{\Psi'_1(x)}{f^0(x)^4}\left[\left(\left(\hat{f}^0_g\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right.$$

$$\left.\cdot\left(\left(\hat{f}^0_g\right)'(x)-\left(f^0\right)'(x)\right)\cdot\hat{f}^1_g(x)\right]dx$$

$$-4\int \frac{\left(f^0\right)''(x)\Psi_1(x)\left(f^0\right)'(x)}{f^0(x)^4}\left[\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\hat{f}^1_g(x)\right]dx$$

$$-4\int \frac{\left(f^0\right)^{''}(x)\Psi_1(x)}{f^0(x)^4}\left[\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\left(\hat{f}^0_g\right)^{'}(x)-\left(f^0\right)^{'}(x)\right)\cdot\hat{f}^1_g(x)\right]dx$$

$$-4\int \frac{\left(f^0\right)^{''}(x)\left(f^0\right)^{'}(x)}{f^0(x)^4}\left[\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\hat{f}^1_g(x)\right]dx$$

$$-4\int \frac{\left(f^0\right)^{''}(x)}{f^0(x)^4}\left[\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right.$$

$$\left.\cdot\left(\left(\hat{f}^0_g\right)^{'}(x)-\left(f^0\right)^{'}(x)\right)\cdot\hat{f}^1_g(x)\right]dx-4\int\frac{\Psi_1(x)\left(f^0\right)^{'}(x)}{f^0(x)^4}$$

$$\left[\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\left(\hat{f}^0_g\right)^{''}(x)-\left(f^0\right)^{''}(x)\right)\cdot\hat{f}^1_g(x)\right]dx$$

$$-4\int \frac{\Psi_1(x)}{f^0(x)^4}\left[\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\left(\hat{f}^0_g\right)^{''}(x)-\left(f^0\right)^{''}(x)\right)\right.$$

$$\left.\cdot\left(\left(\hat{f}^0_g\right)^{'}(x)-\left(f^0\right)^{'}(x)\right)\cdot\hat{f}^1_g(x)\right]dx$$

$$-4\int \frac{\left(f^0\right)^{'}(x)}{f^0(x)^4}\left[\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\left(\hat{f}^0_g\right)^{''}(x)-\left(f^0\right)^{''}(x)\right)\right.$$

$$\left.\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\hat{f}^1_g(x)\right]dx-4\int\frac{1}{f^0(x)^4}\left[\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\right.$$

$$\cdot\left(\left(\hat{f}^0_g\right)^{''}(x)-\left(f^0\right)^{''}(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\left(\hat{f}^0_g\right)^{'}(x)-\left(f^0\right)^{'}(x)\right)$$

$$\left.\cdot\hat{f}^1_g(x)\right]dx$$

$$=-4\int \frac{\Psi'_1(x)\left(f^0\right)^{''}(x)\Psi_1(x)\left(f^0\right)^{'}(x)}{f^0(x)^4}\left[\hat{f}^1_g(x)-f^1(x)\right]dx$$

$$-4\int \frac{\Psi'_1(x)\left(f^0\right)^{''}(x)\Psi_1(x)f^1(x)}{f^0(x)^4}\left[\left(\hat{f}^0_g\right)^{'}(x)-\left(f^0\right)^{'}(x)\right]dx$$

$$-4\int \frac{\Psi'_1(x)\left(f^0\right)^{''}(x)\Psi_1(x)}{f^0(x)^4}\left[\left(\left(\hat{f}^0_g\right)^{'}(x)-\left(f^0\right)^{'}(x)\right)\cdot\left(\hat{f}^1_g(x)-f^1(x)\right)\right]dx$$

$$-4\int \frac{\Psi'_1(x)\left(f^0\right)^{''}(x)\left(f^0\right)^{'}(x)f^1(x)}{f^0(x)^4}\left[\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right]dx$$

$$-4\int \frac{\Psi'_1(x)\left(f^0\right)^{''}(x)\left(f^0\right)^{'}(x)}{f^0(x)^4}\left[\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}^1_g(x)-f^1(x)\right)\right]dx$$

$$-4\int \frac{\Psi_1'(x)\left(f^0\right)''(x)f^1(x)}{f^0(x)^4}\left[\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\right]dx$$

$$-4\int \frac{\Psi_1'(x)\left(f^0\right)''(x)}{f^0(x)^4}\left[\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\right.$$
$$\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx$$

$$-4\int \frac{\Psi_1'(x)\Psi_1(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^4}\left[\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right]dx$$

$$-4\int \frac{\Psi_1'(x)\Psi_1(x)\left(f^0\right)'(x)}{f^0(x)^4}\left[\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx$$

$$-4\int \frac{\Psi_1'(x)\Psi_1(x)f^1(x)}{f^0(x)^4}\left[\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\right.$$
$$\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\right]dx-4\int \frac{\Psi_1'(x)\Psi_1(x)}{f^0(x)^4}$$
$$\left[\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx$$

$$-4\int \frac{\Psi_1'(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^4}\left[\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right]dx$$

$$-4\int \frac{\Psi_1'(x)\left(f^0\right)'(x)}{f^0(x)^4}\left[\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right.$$
$$\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx-4\int \frac{\Psi_1'(x)f^1(x)}{f^0(x)^4}$$
$$\left[\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\right]dx$$

$$-4\int \frac{\Psi_1'(x)}{f^0(x)^4}\left[\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right.$$
$$\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx$$

$$-4\int \frac{\left(f^0\right)''(x)\Psi_1(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^4}\left[\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right]dx$$

$$-4\int \frac{\left(f^0\right)''(x)\Psi_1(x)\left(f^0\right)'(x)}{f^0(x)^4}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx$$

$$-4\int \frac{\left(f^0\right)''(x)\Psi_1(x)f^1(x)}{f^0(x)^4}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\right]dx$$

$$-4\int \frac{\left(f^0\right)''(x)\Psi_1(x)}{f^0(x)^4}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\right.$$
$$\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx$$

$$-4\int \frac{\left(f^0\right)''(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^4}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right]dx$$

$$-4\int \frac{\left(f^0\right)''(x)\left(f^0\right)'(x)}{f^0(x)^4}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right.$$

$$\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\Big]\,dx-4\int\frac{\left(f^0\right)''(x)f^1(x)}{f^0(x)^4}$$
$$\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\right]dx$$
$$-4\int\frac{\left(f^0\right)''(x)}{f^0(x)^4}\Big[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)$$
$$\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\Big]\,dx-4\int\frac{\Psi_1(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^4}$$
$$\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\right]dx$$
$$-4\int\frac{\Psi_1(x)\left(f^0\right)'(x)}{f^0(x)^4}\Big[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)$$
$$\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\Big]\,dx-4\int\frac{\Psi_1(x)f^1(x)}{f^0(x)^4}$$
$$\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\right]dx$$
$$-4\int\frac{\Psi_1(x)}{f^0(x)^4}\Big[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)$$
$$\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\Big]\,dx-4\int\frac{\left(f^0\right)'(x)f^1(x)}{f^0(x)^4}$$
$$\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right]dx$$
$$-4\int\frac{\left(f^0\right)'(x)}{f^0(x)^4}\Big[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)$$
$$\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\Big]\,dx-4\int\frac{f^1(x)}{f^0(x)^4}$$
$$\Big[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)$$
$$\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\Big]\,dx$$
$$-4\int\frac{1}{f^0(x)^4}\Big[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)$$
$$\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\Big]\,dx$$
$$=-4\int\frac{\Psi_1'(x)\left(f^0\right)''(x)\Psi_1(x)\left(f^0\right)'(x)}{f^0(x)^4}\left[\hat{f}_g^1(x)-f^1(x)\right]dx$$
$$-4\int\frac{\Psi_1'(x)\left(f^0\right)''(x)\Psi_1(x)f^1(x)}{f^0(x)^4}\left[\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right]dx$$

$$
\begin{aligned}
&- 4 \int \frac{\Psi_1'(x) \left(f^0\right)''(x) \left(f^0\right)'(x) f^1(x)}{f^0(x)^4} \left[\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right] dx \\
&- 4 \int \frac{\Psi_1'(x)\Psi_1(x) \left(f^0\right)'(x) f^1(x)}{f^0(x)^4} \left[\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right] dx \\
&- 4 \int \frac{\left(f^0\right)''(x)\Psi_1(x) \left(f^0\right)'(x) f^1(x)}{f^0(x)^4} \left[\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right] dx \\
&+ O\left(\int \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right) \\
&+ O\left(\int \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right) \\
&+ O\left(\int \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right) dx\right) \\
&+ O\left(\int \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right) \\
&+ O\left(\int \left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right) \\
&+ O\left(\int \left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right) \cdot \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right) dx\right) \\
&+ O\left(\int \left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right) \cdot \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right)\right. \\
&\left.\cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right) + O\left(\int \left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right)\right. \\
&\left.\cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right) \\
&+ O\left(\int \left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) dx\right) \\
&+ O\left(\int \left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right)\right. \\
&\left.\cdot \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right) dx\right) + O\left(\int \left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right)\right. \\
&\left.\cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right) \\
&+ O\left(\int \left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right) \\
&+ O\left(\int \left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right) \cdot \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right) dx\right) \\
&+ O\left(\int \left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right) \cdot \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right) \\
&+ O\left(\int \left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) dx\right) \\
&+ O\left(\int \left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right) \\
&+ O\left(\int \left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right) dx\right) \\
&+ O\left(\int \left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right)\right. \\
&\left.\cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ O\left(\int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x)\right) \cdot \left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right) dx\right) \\
&+ O\left(\int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x)\right) \cdot \left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right) \\
&+ O\left(\int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x)\right) \cdot \left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right)\right. \\
&\left.\cdot \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right) dx\right) + O\left(\int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x)\right)\right. \\
&\left.\cdot \left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right) \cdot \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right) \\
&+ O\left(\int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x)\right) \cdot \left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) dx\right) \\
&+ O\left(\int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x)\right) \cdot \left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right)\right. \\
&\left.\cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right) + O\left(\int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x)\right)\right. \\
&\left.\cdot \left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right) dx\right) \\
&+ O\left(\int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x)\right) \cdot \left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right)\right. \\
&\left.\cdot \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right). \qquad (111)
\end{aligned}
$$

Plugging expression (111) in (110), the third addend in expression (103) happens to be:

$$
\begin{aligned}
&-4\int \frac{\Psi_1'(x)\left(f^0\right)''(x)\Psi_1(x)\left(f^0\right)'(x)}{f^0(x)^4}\left[\hat{f}_g^1(x)-f^1(x)\right]dx\\
&-4\int \frac{\Psi_1'(x)\left(f^0\right)''(x)\Psi_1(x)f^1(x)}{f^0(x)^4}\left[\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right]dx\\
&-4\int \frac{\Psi_1'(x)\left(f^0\right)''(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^4}\left[\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right]dx\\
&-4\int \frac{\Psi_1'(x)\Psi_1(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^4}\left[\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right]dx\\
&-4\int \frac{\left(f^0\right)''(x)\Psi_1(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^4}\left[\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right]dx\\
&-12\int \frac{\Psi_1'(x)\left(f^0\right)''(x)\Psi_1(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^5}\left[\hat{f}_g^0(x)-f^0(x)\right]dx\\
&+O\left(\int\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)^2dx\right)\\
&+O\left(\int\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}'(x)\left(\hat{f}_g^0\right)''(x)\hat{\Psi}_{1,g}(x)\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x)\right.\right.\\
&\left.\left.-\Psi_1'(x)\left(f^0\right)''(x)\Psi_1(x)\left(f^0\right)'(x)f^1(x)\right)dx\right)\\
&+O\left(\int\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+O\left(\int\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+O\left(\int\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)dx\right)\\
&+O\left(\int\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+O\left(\int\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+O\left(\int\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)dx\right)\\
&+O\left(\int\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\right.\\
&\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+O\left(\int\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)dx\right)\\
&+O\left(\int\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+O\left(\int\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right.\\
&\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)dx\right)\\
&+O\left(\int\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right.
\end{aligned}
$$

$$
\begin{aligned}
&\cdot \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x) \right) \cdot \left(\hat{f}_g^1(x) - f^1(x) \right) dx \Big) \\
&+ O\left(\int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x) \right) \cdot \left(\hat{f}_g^1(x) - f^1(x) \right) dx \right) \\
&+ O\left(\int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x) \right) \cdot \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x) \right) dx \right) \\
&+ O\left(\int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x) \right) \cdot \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x) \right) \cdot \left(\hat{f}_g^1(x) - f^1(x) \right) dx \right) \\
&+ O\left(\int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x) \right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x) \right) \right) \\
&+ O\left(\int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x) \right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x) \right) \cdot \left(\hat{f}_g^1(x) - f^1(x) \right) dx \right) \\
&+ O\left(\int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x) \right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x) \right) \cdot \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x) \right) dx \right) \\
&+ O\Big(\int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x) \right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x) \right) \cdot \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x) \right) \\
&\cdot \left(\hat{f}_g^1(x) - f^1(x) \right) dx \Big) \\
&+ O\left(\int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x) \right) \cdot \left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x) \right) dx \right) \\
&+ O\left(\int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x) \right) \cdot \left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x) \right) \cdot \left(\hat{f}_g^1(x) - f^1(x) \right) dx \right) \\
&+ O\Big(\int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x) \right) \cdot \left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x) \right) \\
&\cdot \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x) \right) dx \Big) \\
&+ O\left(\int \left(\hat{f}_g^0(x) - f^0(x) \right)^2 + \left(\hat{f}_g^0(x) - f^0(x) \right) \cdot \left(\hat{f}_g^0(x)^2 - f^0(x)^2 \right) dx \right) \\
&+ O\Big(\int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x) \right) \cdot \left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x) \right) \\
&\cdot \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x) \right) \cdot \left(\hat{f}_g^1(x) - f^1(x) \right) dx \Big) \\
&+ O\left(\int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x) \right) \cdot \left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x) \right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x) \right) dx \right) \\
&+ O\Big(\int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x) \right) \cdot \left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x) \right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x) \right) \\
&\cdot \left(\hat{f}_g^1(x) - f^1(x) \right) dx \Big) + O\Big(\int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x) \right) \\
&\cdot \left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x) \right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x) \right) \cdot \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x) \right) dx \Big) \\
&+ O\Big(\int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x) \right) \cdot \left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x) \right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x) \right) \\
&\cdot \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x) \right) \cdot \left(\hat{f}_g^1(x) - f^1(x) \right) dx \Big). \qquad (112)
\end{aligned}
$$

The fourth addend in expression (103), using (93), (94), (95) and (96), happens to be:

$$
\begin{aligned}
&-8\int \frac{1}{f^0(x)}\left[\frac{\hat{\Psi}'_{1,g}(x)^2\left(\hat{f}^0_g\right)'(x)^2\hat{f}^1_g(x)}{\hat{f}^0_g(x)^3}-\frac{\Psi'_1(x)^2\left(f^0\right)'(x)^2 f^1(x)}{f^0(x)^3}\right]dx\\
&=-8\int \frac{1}{f^0(x)^4}\left[\hat{\Psi}'_{1,g}(x)^2\left(\hat{f}^0_g\right)'(x)^2\hat{f}^1_g(x)-\Psi'_1(x)^2\left(f^0\right)'(x)^2 f^1(x)\right]dx\\
&+8\int \frac{\Psi'_1(x)^2\left(f^0\right)'(x)^2 f^1(x)}{f^0(x)^7}\left[\hat{f}^0_g(x)^3-f^0(x)^3\right]dx\\
&+O\left(\int\left(\hat{f}^0_g(x)^3-f^0(x)^3\right)^2 dx\right)+O\left(\int\left(\hat{f}^0_g(x)^3-f^0(x)^3\right)\right.\\
&\left.\cdot\left(\hat{\Psi}'_{1,g}(x)^2\left(\hat{f}^0_g\right)'(x)^2\hat{f}^1_g(x)-\Psi'_1(x)^2\left(f^0\right)'(x)^2 f^1(x)\right)dx\right)\\
&=-8\int \frac{1}{f^0(x)^4}\left[\hat{\Psi}'_{1,g}(x)^2\left(\hat{f}^0_g\right)'(x)^2\hat{f}^1_g(x)-\Psi'_1(x)^2\left(f^0\right)'(x)^2 f^1(x)\right]dx\\
&+24\int \frac{\Psi'_1(x)^2\left(f^0\right)'(x)^2 f^1(x)}{f^0(x)^5}\left[\hat{f}^0_g(x)-f^0(x)\right]dx\\
&+O\left(\int\left(\hat{f}^0_g(x)^3-f^0(x)^3\right)^2 dx\right)+O\left(\int\left(\hat{f}^0_g(x)^3-f^0(x)^3\right)\right.\\
&\left.\cdot\left(\hat{\Psi}'_{1,g}(x)^2\left(\hat{f}^0_g\right)'(x)^2\hat{f}^1_g(x)-\Psi'_1(x)^2\left(f^0\right)'(x)^2 f^1(x)\right)dx\right)\\
&+O\left(\int\left(\hat{f}^0_g(x)-f^0(x)\right)^2 dx+\int\left(\hat{f}^0_g(x)^2-f^0(x)^2\right)dx\right)\\
&=-8\int \frac{\Psi'_1(x)^2}{f^0(x)^4}\left[\left(\hat{f}^0_g\right)'(x)^2\hat{f}^1_g(x)-\left(f^0\right)'(x)^2 f^1(x)\right]dx\\
&-8\int \frac{1}{f^0(x)^4}\left[\left(\hat{\Psi}'_{1,g}(x)^2-\Psi'_1(x)^2\right)\cdot\left(\hat{f}^0_g\right)'(x)^2\hat{f}^1_g(x)\right]dx\\
&+24\int \frac{\Psi'_1(x)^2\left(f^0\right)'(x)^2 f^1(x)}{f^0(x)^5}\left[\hat{f}^0_g(x)-f^0(x)\right]dx\\
&+O\left(\int\left(\hat{f}^0_g(x)^3-f^0(x)^3\right)^2\right)+O\left(\int\left(\hat{f}^0_g(x)^3-f^0(x)^3\right)\right.\\
&\left.\cdot\left(\hat{\Psi}'_{1,g}(x)^2\left(\hat{f}^0_g\right)'(x)^2\hat{f}^1_g(x)-\Psi'_1(x)^2\left(f^0\right)'(x)^2 f^1(x)\right)\right)\\
&+O\left(\int\left(\hat{f}^0_g(x)-f^0(x)\right)^2+\left(\hat{f}^0_g(x)^2-f^0(x)^2\right)\right)\\
&=-8\int \frac{\Psi'_1(x)^2\left(f^0\right)'(x)^2}{f^0(x)^4}\left[\hat{f}^1_g(x)-f^1(x)\right]dx\\
&-8\int \frac{\Psi'_1(x)^2}{f^0(x)^4}\left[\left(\left(\hat{f}^0_g\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\hat{f}^1_g(x)\right]dx\\
&-8\int \frac{\left(f^0\right)'(x)^2}{f^0(x)^4}\left[\left(\hat{\Psi}'_{1,g}(x)^2-\Psi'_1(x)^2\right)\cdot\hat{f}^1_g(x)\right]dx-8\int\frac{1}{f^0(x)^4}\\
&\left[\left(\hat{\Psi}'_{1,g}(x)^2-\Psi'_1(x)^2\right)\cdot\left(\left(\hat{f}^0_g\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\hat{f}^1_g(x)\right]dx\\
&+24\int \frac{\Psi'_1(x)^2\left(f^0\right)'(x)^2 f^1(x)}{f^0(x)^5}\left[\hat{f}^0_g(x)-f^0(x)\right]dx\\
&+O\left(\int\left(\hat{f}^0_g(x)^3-f^0(x)^3\right)^2 dx\right)+O\left(\int\left(\hat{f}^0_g(x)^3-f^0(x)^3\right)\right.
\end{aligned}
$$

$$
\begin{aligned}
&\cdot \left(\hat{\Psi}'_{1,g}(x)^2 \left(\hat{f}_g^0\right)'(x)^2 \hat{f}_g^1(x) - \Psi'_1(x)^2 \left(f^0\right)'(x)^2 f^1(x) \right) dx \Big) \\
&+ O\left(\int \left(\hat{f}_g^0(x) - f^0(x)\right)^2 dx + \int \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right) dx \right) \\
&= -8 \int \frac{\Psi'_1(x)^2 \left(f^0\right)'(x)^2}{f^0(x)^4} \left[\hat{f}_g^1(x) - f^1(x)\right] dx \\
&- 8 \int \frac{\Psi'_1(x)^2 f^1(x)}{f^0(x)^4} \left[\left(\hat{f}_g^0\right)'(x)^2 - \left(f^0\right)'(x)^2\right] dx \\
&- 8 \int \frac{\Psi'_1(x)^2}{f^0(x)^4} \left[\left(\left(\hat{f}_g^0\right)'(x)^2 - \left(f^0\right)'(x)^2\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right)\right] dx \\
&- 8 \int \frac{\left(f^0\right)'(x)^2 f^1(x)}{f^0(x)^4} \left[\hat{\Psi}'_{1,g}(x)^2 - \Psi'_1(x)^2\right] dx \\
&- 8 \int \frac{\left(f^0\right)'(x)^2}{f^0(x)^4} \left[\left(\hat{\Psi}'_{1,g}(x)^2 - \Psi'_1(x)^2\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right)\right] dx \\
&- 8 \int \frac{f^1(x)}{f^0(x)^4} \left[\left(\hat{\Psi}'_{1,g}(x)^2 - \Psi'_1(x)^2\right) \cdot \left(\left(\hat{f}_g^0\right)'(x)^2 - \left(f^0\right)'(x)^2\right)\right] dx \\
&- 8 \int \frac{1}{f^0(x)^4} \Big[\left(\hat{\Psi}'_{1,g}(x)^2 - \Psi'_1(x)^2\right) \cdot \left(\left(\hat{f}_g^0\right)'(x)^2 - \left(f^0\right)'(x)^2\right) \\
&\cdot \left(\hat{f}_g^1(x) - f^1(x)\right)\Big] dx + 24 \int \frac{\Psi'_1(x)^2 \left(f^0\right)'(x)^2 f^1(x)}{f^0(x)^5} \left[\hat{f}_g^0(x) - f^0(x)\right] dx \\
&+ O\left(\int \left(\hat{f}_g^0(x)^3 - f^0(x)^3\right)^2 dx \right) + O\Big(\int \left(\hat{f}_g^0(x)^3 - f^0(x)^3\right) \\
&\cdot \left(\hat{\Psi}'_{1,g}(x)^2 \left(\hat{f}_g^0\right)'(x)^2 \hat{f}_g^1(x) - \Psi'_1(x)^2 \left(f^0\right)'(x)^2 f^1(x) \right) dx \Big) \\
&+ O\left(\int \left(\hat{f}_g^0(x) - f^0(x)\right)^2 dx + \int \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right) dx \right) \\
&= -8 \int \frac{\Psi'_1(x)^2 \left(f^0\right)'(x)^2}{f^0(x)^4} \left[\hat{f}_g^1(x) - f^1(x)\right] dx \\
&- 16 \int \frac{\Psi'_1(x)^2 f^1(x) \left(f^0\right)'(x)}{f^0(x)^4} \left[\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right] dx \\
&- 16 \int \frac{\left(f^0\right)'(x)^2 f^1(x) \Psi'_1(x)}{f^0(x)^4} \left[\hat{\Psi}'_{1,g}(x) - \Psi'_1(x)\right] dx \\
&+ 24 \int \frac{\Psi'_1(x)^2 \left(f^0\right)'(x)^2 f^1(x)}{f^0(x)^5} \left[\hat{f}_g^0(x) - f^0(x)\right] dx
\end{aligned}
$$

$$
\begin{aligned}
&+ O\left(\int \left(\left(\hat{f}_g^0\right)' (x)^2 - \left(f^0\right)' (x)^2\right) \cdot \left(\hat{f}_g^1 (x) - f^1 (x)\right) dx\right) \\
&+ O\left(\int \left(\hat{\Psi}_{1,g}' (x)^2 - \Psi_1' (x)^2\right) \cdot \left(\hat{f}_g^1 (x) - f^1 (x)\right) dx\right) \\
&+ O\left(\int \left(\hat{\Psi}_{1,g}' (x)^2 - \Psi_1' (x)^2\right) \cdot \left(\left(\hat{f}_g^0\right)' (x)^2 - \left(f^0\right)' (x)^2\right) dx\right) \\
&+ O\left(\int \left(\hat{f}_g^0 (x)^3 - f^0 (x)^3\right)^2 dx\right) + O\left(\int \left(\hat{\Psi}_{1,g}' (x)^2 - \Psi_1' (x)^2\right)\right. \\
&\left.\cdot \left(\left(\hat{f}_g^0\right)' (x)^2 - \left(f^0\right)' (x)^2\right) \cdot \left(\hat{f}_g^1 (x) - f^1 (x)\right) dx\right) \\
&+ O\left(\int \left(\hat{\Psi}_{1,g}' (x) - \Psi_1' (x)\right)^2 dx\right) + O\left(\int \left(\hat{f}_g^0 (x)^3 - f^0 (x)^3\right)\right. \\
&\left.\cdot \left(\hat{\Psi}_{1,g}' (x)^2 \left(\hat{f}_g^0\right)' (x)^2 \hat{f}_g^1 (x) - \Psi_1' (x)^2 \left(f^0\right)' (x)^2 f^1 (x)\right) dx\right) \\
&+ O\left(\int \left(\hat{f}_g^0 (x) - f^0 (x)\right)^2 dx + \int \left(\hat{f}_g^0 (x)^2 - f^0 (x)^2\right) dx\right) \\
&+ O\left(\int \left(\left(\hat{f}_g^0\right)' (x) - \left(f^0\right)' (x)\right)^2 dx\right). \qquad (113)
\end{aligned}
$$

The fifth addend in expression (103) turns out to be, considering (93), (94), (95) and (96):

$$
\begin{aligned}
&-4\int \frac{1}{f^0(x)}\left[\frac{\left(\hat{f}_g^0\right)'(x)^2\hat{\Psi}_{1,g}(x)\hat{\Psi}_{1,g}''(x)\hat{f}_g^1(x)}{\hat{f}_g^0(x)^3}\right.\\
&\left.-\frac{\left(f^0\right)'(x)^2\Psi_1(x)\Psi_1''(x)f^1(x)}{f^0(x)^3}\right]dx\\
&=-4\int \frac{1}{f^0(x)^4}\left[\left(\hat{f}_g^0\right)'(x)^2\hat{\Psi}_{1,g}(x)\hat{\Psi}_{1,g}''(x)\hat{f}_g^1(x)\right.\\
&\left.-\left(f^0\right)'(x)^2\Psi_1(x)\Psi_1''(x)f^1(x)\right]dx\\
&+4\int \frac{\left(f^0\right)'(x)^2\Psi_1(x)\Psi_1''(x)f^1(x)}{f^0(x)^7}\left[\hat{f}_g^0(x)^3-f^0(x)^3\right]dx\\
&+O\left(\int\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)^2dx\right)+O\left(\int\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)\right.\\
&\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)^2\hat{\Psi}_{1,g}(x)\hat{\Psi}_{1,g}''(x)\hat{f}_g^1(x)-\left(f^0\right)'(x)^2\Psi_1(x)\Psi_1''(x)f^1(x)\right)dx\right)\\
&=-4\int \frac{1}{f^0(x)^4}\left[\left(\hat{f}_g^0\right)'(x)^2\hat{\Psi}_{1,g}(x)\hat{\Psi}_{1,g}''(x)\hat{f}_g^1(x)\right.\\
&\left.-\left(f^0\right)'(x)^2\Psi_1(x)\Psi_1''(x)f^1(x)\right]dx\\
&+12\int \frac{\left(f^0\right)'(x)^2\Psi_1(x)\Psi_1''(x)f^1(x)}{f^0(x)^5}\left[\hat{f}_g^0(x)-f^0(x)\right]dx\\
&+O\left(\int\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)^2dx\right)+O\left(\int\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)\right.\\
&\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)^2\hat{\Psi}_{1,g}(x)\hat{\Psi}_{1,g}''(x)\hat{f}_g^1(x)-\left(f^0\right)'(x)^2\Psi_1(x)\Psi_1''(x)f^1(x)\right)dx\right)\\
&+O\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2dx+\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)dx\right)\\
&=-4\int \frac{\left(f^0\right)'(x)^2}{f^0(x)^4}\left[\hat{\Psi}_{1,g}(x)\hat{\Psi}_{1,g}''(x)\hat{f}_g^1(x)-\Psi_1(x)\Psi_1''(x)f^1(x)\right]dx\\
&-4\int \frac{1}{f^0(x)^4}\left[\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\hat{\Psi}_{1,g}(x)\hat{\Psi}_{1,g}''(x)\hat{f}_g^1(x)\right]dx\\
&+12\int \frac{\left(f^0\right)'(x)^2\Psi_1(x)\Psi_1''(x)f^1(x)}{f^0(x)^5}\left[\hat{f}_g^0(x)-f^0(x)\right]dx\\
&+O\left(\int\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)^2dx\right)+O\left(\int\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)\right.\\
&\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)^2\hat{\Psi}_{1,g}(x)\hat{\Psi}_{1,g}''(x)\hat{f}_g^1(x)-\left(f^0\right)'(x)^2\Psi_1(x)\Psi_1''(x)f^1(x)\right)dx\right)\\
&+O\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2dx+\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)dx\right)
\end{aligned}
$$

$$= -4 \int \frac{\left(f^0\right)'(x)^2 \Psi_1(x)}{f^0(x)^4} \left[\hat{\Psi}''_{1,g}(x)\hat{f}^1_g(x) - \Psi''_1(x) f^1(x)\right] dx$$
$$-4 \int \frac{\left(f^0\right)'(x)^2}{f^0(x)^4} \left[\left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \hat{\Psi}''_{1,g}(x)\hat{f}^1_g(x)\right] dx$$
$$-4 \int \frac{\Psi_1(x)}{f^0(x)^4} \left[\left(\left(\hat{f}^0_g\right)'(x)^2 - \left(f^0\right)'(x)^2\right) \cdot \hat{\Psi}''_{1,g}(x)\hat{f}^1_g(x)\right] dx - 4 \int \frac{1}{f^0(x)^4}$$
$$\left[\left(\left(\hat{f}^0_g\right)'(x)^2 - \left(f^0\right)'(x)^2\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \hat{\Psi}''_{1,g}(x)\hat{f}^1_g(x)\right] dx$$
$$+12 \int \frac{\left(f^0\right)'(x)^2 \Psi_1(x)\Psi''_1(x) f^1(x)}{f^0(x)^5} \left[\hat{f}^0_g(x) - f^0(x)\right] dx$$
$$+ O\left(\int \left(\hat{f}^0_g(x)^3 - f^0(x)^3\right)^2 dx\right) + O\left(\int \left(\hat{f}^0_g(x)^3 - f^0(x)^3\right)\right.$$
$$\left. \cdot \left(\left(\hat{f}^0_g\right)'(x)^2 \hat{\Psi}_{1,g}(x)\hat{\Psi}''_{1,g}(x)\hat{f}^1_g(x) - \left(f^0\right)'(x)^2 \Psi_1(x)\Psi''_1(x) f^1(x)\right) dx\right)$$
$$+ O\left(\int \left(\hat{f}^0_g(x) - f^0(x)\right)^2 dx + \int \left(\hat{f}^0_g(x)^2 - f^0(x)^2\right) dx\right)$$
$$= -4 \int \frac{\left(f^0\right)'(x)^2 \Psi_1(x)\Psi''_1(x)}{f^0(x)^4} \left[\hat{f}^1_g(x) - f^1(x)\right] dx$$
$$-4 \int \frac{\left(f^0\right)'(x)^2 \Psi_1(x)}{f^0(x)^4} \left[\left(\hat{\Psi}''_{1,g}(x) - \Psi''_1(x)\right) \cdot \hat{f}^1_g(x)\right] dx$$
$$-4 \int \frac{\left(f^0\right)'(x)^2 \Psi''_1(x)}{f^0(x)^4} \left[\left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \hat{f}^1_g(x)\right] dx$$
$$-4 \int \frac{\left(f^0\right)'(x)^2}{f^0(x)^4} \left[\left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\hat{\Psi}''_{1,g}(x) - \Psi''_1(x)\right) \cdot \hat{f}^1_g(x)\right] dx$$
$$-4 \int \frac{\Psi_1(x)\Psi''_1(x)}{f^0(x)^4} \left[\left(\left(\hat{f}^0_g\right)'(x)^2 - \left(f^0\right)'(x)^2\right) \cdot \hat{f}^1_g(x)\right] dx$$
$$-4 \int \frac{\Psi_1(x)}{f^0(x)^4} \left[\left(\left(\hat{f}^0_g\right)'(x)^2 - \left(f^0\right)'(x)^2\right) \cdot \left(\hat{\Psi}''_{1,g}(x) - \Psi''_1(x)\right) \cdot \hat{f}^1_g(x)\right] dx$$
$$-4 \int \frac{\Psi''_1(x)}{f^0(x)^4} \left[\left(\left(\hat{f}^0_g\right)'(x)^2 - \left(f^0\right)'(x)^2\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \hat{f}^1_g(x)\right] dx$$
$$-4 \int \frac{1}{f^0(x)^4} \left[\left(\left(\hat{f}^0_g\right)'(x)^2 - \left(f^0\right)'(x)^2\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right)\right.$$
$$\left. \cdot \left(\hat{\Psi}''_{1,g}(x) - \Psi''_1(x)\right) \cdot \hat{f}^1_g(x)\right] dx$$
$$+12 \int \frac{\left(f^0\right)'(x)^2 \Psi_1(x)\Psi''_1(x) f^1(x)}{f^0(x)^5} \left[\hat{f}^0_g(x) - f^0(x)\right] dx$$
$$+ O\left(\int \left(\hat{f}^0_g(x)^3 - f^0(x)^3\right)^2 dx\right) + O\left(\int \left(\hat{f}^0_g(x)^3 - f^0(x)^3\right)\right.$$
$$\left. \cdot \left(\left(\hat{f}^0_g\right)'(x)^2 \hat{\Psi}_{1,g}(x)\hat{\Psi}''_{1,g}(x)\hat{f}^1_g(x) - \left(f^0\right)'(x)^2 \Psi_1(x)\Psi''_1(x) f^1(x)\right) dx\right)$$
$$+ O\left(\int \left(\hat{f}^0_g(x) - f^0(x)\right)^2 dx + \int \left(\hat{f}^0_g(x)^2 - f^0(x)^2\right)\right)$$

$$
\begin{aligned}
&= -4\int \frac{\left(f^0\right)'(x)^2\Psi_1(x)\Psi_1''(x)}{f^0(x)^4}\left[\hat{f}_g^1(x)-f^1(x)\right]dx\\
&-4\int \frac{\left(f^0\right)'(x)^2\Psi_1(x)f^1(x)}{f^0(x)^4}\left[\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right]dx\\
&-4\int \frac{\left(f^0\right)'(x)^2\Psi_1(x)}{f^0(x)^4}\left[\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx\\
&-4\int \frac{\left(f^0\right)'(x)^2\Psi_1''(x)f^1(x)}{f^0(x)^4}\left[\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right]dx\\
&-4\int \frac{\Psi_1(x)\Psi_1''(x)f^1(x)}{f^0(x)^4}\left[\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right]dx\\
&-4\int \frac{\left(f^0\right)'(x)^2\Psi_1''(x)}{f^0(x)^4}\left[\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx\\
&-4\int \frac{\left(f^0\right)'(x)^2f^1(x)}{f^0(x)^4}\left[\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\right]dx\\
&-4\int \frac{\left(f^0\right)'(x)^2}{f^0(x)^4}\left[\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\right.\\
&\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx\\
&-4\int \frac{\Psi_1(x)\Psi_1''(x)}{f^0(x)^4}\left[\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx\\
&-4\int \frac{\Psi_1(x)f^1(x)}{f^0(x)^4}\left[\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\right]dx\\
&-4\int \frac{\Psi_1(x)}{f^0(x)^4}\left[\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\right.\\
&\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx\\
&-4\int \frac{\Psi_1''(x)f^1(x)}{f^0(x)^4}\left[\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right]dx\\
&-4\int \frac{\Psi_1''(x)}{f^0(x)^4}\left[\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right.\\
&\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx-4\int \frac{f^1(x)}{f^0(x)^4}\left[\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\right.\\
&\left.\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\right]dx\\
&-4\int \frac{1}{f^0(x)^4}\left[\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right.\\
&\left.\cdot\left(\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx\\
&+12\int \frac{\left(f^0\right)'(x)^2\Psi_1(x)\Psi_1''(x)f^1(x)}{f^0(x)^5}\left[\hat{f}_g^0(x)-f^0(x)\right]dx\\
&+O\left(\int\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)^2dx\right)+O\left(\int\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)\right.\\
&\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)^2\hat{\Psi}_{1,g}(x)\hat{\Psi}_{1,g}''(x)\hat{f}_g^1(x)-\left(f^0\right)'(x)^2\Psi_1(x)\Psi_1''(x)f^1(x)\right)dx\right)\\
&+O\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2+\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)dx\right)
\end{aligned}
$$

$$
\begin{aligned}
&= -4\int \frac{\left(f^0\right)'(x)^2\Psi_1(x)\Psi_1''(x)}{f^0(x)^4}\left[\hat f_g^1(x)-f^1(x)\right]dx \\
&- 4\int \frac{\left(f^0\right)'(x)^2\Psi_1(x)f^1(x)}{f^0(x)^4}\left[\hat\Psi_{1,g}''(x)-\Psi_1''(x)\right]dx \\
&- 4\int \frac{\left(f^0\right)'(x)^2\Psi_1''(x)f^1(x)}{f^0(x)^4}\left[\hat\Psi_{1,g}(x)-\Psi_1(x)\right]dx \\
&- 8\int \frac{\Psi_1(x)\Psi_1''(x)f^1(x)\left(f^0\right)'(x)}{f^0(x)^4}\left[\left(\hat f_g^0\right)'(x)-\left(f^0\right)'(x)\right]dx \\
&+ 12\int \frac{\left(f^0\right)'(x)^2\Psi_1(x)\Psi_1''(x)f^1(x)}{f^0(x)^5}\left[\hat f_g^0(x)-f^0(x)\right]dx \\
&+ O\left(\int\left(\hat f_g^0(x)^3-f^0(x)^3\right)^2dx\right) \\
&+ O\left(\int\left(\hat\Psi_{1,g}''(x)-\Psi_1''(x)\right)\cdot\left(\hat f_g^1(x)-f^1(x)\right)dx\right) \\
&+ O\left(\int\left(\hat\Psi_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat f_g^1(x)-f^1(x)\right)dx\right) \\
&+ O\left(\int\left(\hat\Psi_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat\Psi_{1,g}''(x)-\Psi_1''(x)\right)dx\right) \\
&+ O\left(\int\left(\left(\hat f_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat\Psi_{1,g}(x)-\Psi_1(x)\right)dx\right) \\
&+ O\left(\int\left(\hat\Psi_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat\Psi_{1,g}''(x)-\Psi_1''(x)\right)\cdot\left(\hat f_g^1(x)-f^1(x)\right)dx\right) \\
&+ O\left(\int\left(\left(\hat f_g^0\right)'(x)-\left(f^0\right)'(x)\right)^2dx\right) \\
&+ O\left(\int\left(\left(\hat f_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat f_g^1(x)-f^1(x)\right)dx\right) \\
&+ O\left(\int\left(\left(\hat f_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\right. \\
&\left.\cdot\left(\hat\Psi_{1,g}''(x)-\Psi_1''(x)\right)dx\right)+O\left(\int\left(\left(\hat f_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\right) \\
&\cdot\left(\hat\Psi_{1,g}''(x)-\Psi_1''(x)\right)\cdot\left(\hat f_g^1(x)-f^1(x)\right)dx\Big)+O\left(\int\left(\left(\hat f_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\right. \\
&\left.\cdot\left(\hat\Psi_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat f_g^1(x)-f^1(x)\right)dx\right) \\
&+ O\left(\int\left(\left(\hat f_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat\Psi_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat\Psi_{1,g}''(x)-\Psi_1''(x)\right)dx\right) \\
&+ O\left(\int\left(\left(\hat f_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat\Psi_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat\Psi_{1,g}''(x)-\Psi_1''(x)\right)\right. \\
&\left.\cdot\left(\hat f_g^1(x)-f^1(x)\right)dx\right)+O\left(\int\left(\hat f_g^0(x)^3-f^0(x)^3\right)\right. \\
&\left.\cdot\left(\left(\hat f_g^0\right)'(x)^2\hat\Psi_{1,g}(x)\hat\Psi_{1,g}''(x)\hat f_g^1(x)-\left(f^0\right)'(x)^2\Psi_1(x)\Psi_1''(x)f^1(x)\right)dx\right) \\
&+ O\left(\int\left(\hat f_g^0(x)-f^0(x)\right)^2dx+\int\left(\hat f_g^0(x)^2-f^0(x)^2\right)dx\right). \qquad (114)
\end{aligned}
$$

Working out calculations with the sixth addend in expression (103), using (93), (94), (95) and (96), it leads to:

$$
\begin{aligned}
& -8 \int \frac{1}{f^0(x)} \left[\frac{\left(\hat{f}_g^0\right)'(x)^4 \hat{\Psi}_{1,g}^2(x) \hat{f}_g^1(x)}{\hat{f}_g^0(x)^5} - \frac{\left(f^0\right)'(x)^4 \Psi_1^2(x) f^1(x)}{f^0(x)^5} \right] dx \\
& = -8 \int \frac{1}{f^0(x)^6} \left[\left(\hat{f}_g^0\right)'(x)^4 \hat{\Psi}_{1,g}^2(x) \hat{f}_g^1(x) - \left(f^0\right)'(x)^4 \Psi_1^2(x) f^1(x) \right] dx \\
& + 8 \int \frac{\left(f^0\right)'(x)^4 \Psi_1^2(x) f^1(x)}{f^0(x)^{11}} \left[\hat{f}_g^0(x)^5 - f^0(x)^5 \right] dx \\
& + O\left(\left(\hat{f}_g^0(x)^5 - f^0(x)^5 \right)^2 dx \right) + O\left(\int \left(\hat{f}_g^0(x)^5 - f^0(x)^5 \right) \right. \\
& \left. \cdot \left(\left(\hat{f}_g^0\right)'(x)^4 \hat{\Psi}_{1,g}^2(x) \hat{f}_g^1(x) - \left(f^0\right)'(x)^4 \Psi_1^2(x) f^1(x) \right) dx \right) \\
& = -8 \int \frac{1}{f^0(x)^6} \left[\left(\hat{f}_g^0\right)'(x)^4 \hat{\Psi}_{1,g}^2(x) \hat{f}_g^1(x) - \left(f^0\right)'(x)^4 \Psi_1^2(x) f^1(x) \right] dx \\
& + 32 \int \frac{\left(f^0\right)'(x)^4 \Psi_1^2(x) f^1(x)}{f^0(x)^7} \left[\hat{f}_g^0(x) - f^0(x) \right] dx \\
& + O\left(\left(\hat{f}_g^0(x)^5 - f^0(x)^5 \right)^2 dx \right) + O\left(\int \left(\hat{f}_g^0(x)^5 - f^0(x)^5 \right) \right. \\
& \left. \cdot \left(\left(\hat{f}_g^0\right)'(x)^4 \hat{\Psi}_{1,g}^2(x) \hat{f}_g^1(x) - \left(f^0\right)'(x)^4 \Psi_1^2(x) f^1(x) \right) dx \right) \\
& + O\left(\int \left(\hat{f}_g^0(x) - f^0(x) \right) \cdot \left(\hat{f}_g^0(x)^4 - f^0(x)^4 \right) dx + \int \left(\hat{f}_g^0(x) - f^0(x) \right)^2 dx \right) \\
& + O\left(\int \left(\hat{f}_g^0(x) - f^0(x) \right) \cdot \left(\hat{f}_g^0(x)^3 - f^0(x)^3 \right) dx \right) \\
& = -8 \int \frac{\left(f^0\right)'(x)^4}{f^0(x)^6} \left[\hat{\Psi}_{1,g}^2(x) \hat{f}_g^1(x) - \Psi_1^2(x) f^1(x) \right] dx \\
& - 8 \int \frac{1}{f^0(x)^6} \left[\left(\left(\hat{f}_g^0\right)'(x)^4 - \left(f^0\right)'(x)^4 \right) \cdot \hat{\Psi}_{1,g}^2(x) \hat{f}_g^1(x) \right] dx \\
& + 32 \int \frac{\left(f^0\right)'(x)^4 \Psi_1^2(x) f^1(x)}{f^0(x)^7} \left[\hat{f}_g^0(x) - f^0(x) \right] dx \\
& + O\left(\left(\hat{f}_g^0(x)^5 - f^0(x)^5 \right)^2 dx \right) + O\left(\int \left(\hat{f}_g^0(x)^5 - f^0(x)^5 \right) \right. \\
& \left. \cdot \left(\left(\hat{f}_g^0\right)'(x)^4 \hat{\Psi}_{1,g}^2(x) \hat{f}_g^1(x) - \left(f^0\right)'(x)^4 \Psi_1^2(x) f^1(x) \right) dx \right) \\
& + O\left(\int \left(\hat{f}_g^0(x) - f^0(x) \right) \cdot \left(\hat{f}_g^0(x)^4 - f^0(x)^4 \right) dx + \int \left(\hat{f}_g^0(x) - f^0(x) \right)^2 dx \right) \\
& + O\left(\int \left(\hat{f}_g^0(x) - f^0(x) \right) \cdot \left(\hat{f}_g^0(x)^3 - f^0(x)^3 \right) dx \right)
\end{aligned}
$$

$$= -8\int \frac{\left(f^0\right)'(x)^4\Psi_1^2(x)}{f^0(x)^6}\left[\hat f_g^1(x) - f^1(x)\right]dx$$
$$-8\int \frac{\left(f^0\right)'(x)^4}{f^0(x)^6}\left[\left(\hat\Psi_{1,g}^2(x) - \Psi_1^2(x)\right)\cdot \hat f_g^1(x)\right]dx$$
$$-8\int \frac{\Psi_1^2(x)}{f^0(x)^6}\left[\left(\left(\hat f_g^0\right)'(x)^4 - \left(f^0\right)'(x)^4\right)\cdot \hat f_g^1(x)\right]dx$$
$$-8\int \frac{1}{f^0(x)^6}\left[\left(\left(\hat f_g^0\right)'(x)^4 - \left(f^0\right)'(x)^4\right)\cdot\left(\hat\Psi_{1,g}^2(x) - \Psi_1^2(x)\right)\cdot \hat f_g^1(x)\right]dx$$
$$+32\int \frac{\left(f^0\right)'(x)^4\Psi_1^2(x)f^1(x)}{f^0(x)^7}\left[\hat f_g^0(x) - f^0(x)\right]dx$$
$$+O\left(\left(\hat f_g^0(x)^5 - f^0(x)^5\right)^2 dx\right) + O\left(\int\left(\hat f_g^0(x)^5 - f^0(x)^5\right)\right.$$
$$\left.\cdot\left(\left(\hat f_g^0\right)'(x)^4\hat\Psi_{1,g}^2(x)\hat f_g^1(x) - \left(f^0\right)'(x)^4\Psi_1^2(x)f^1(x)\right)dx\right)$$
$$+O\left(\int\left(\hat f_g^0(x) - f^0(x)\right)\cdot\left(\hat f_g^0(x)^4 - f^0(x)^4\right)dx + \int\left(\hat f_g^0(x) - f^0(x)\right)^2 dx\right)$$
$$+O\left(\int\left(\hat f_g^0(x) - f^0(x)\right)\cdot\left(\hat f_g^0(x)^3 - f^0(x)^3\right)dx\right)$$

$$= -8\int \frac{\left(f^0\right)'(x)^4\Psi_1^2(x)}{f^0(x)^6}\left[\hat f_g^1(x) - f^1(x)\right]dx$$
$$-8\int \frac{\left(f^0\right)'(x)^4 f^1(x)}{f^0(x)^6}\left[\hat\Psi_{1,g}^2(x) - \Psi_1^2(x)\right]dx$$
$$-8\int \frac{\Psi_1^2(x)f^1(x)}{f^0(x)^6}\left[\left(\hat f_g^0\right)'(x)^4 - \left(f^0\right)'(x)^4\right]dx$$
$$+32\int \frac{\left(f^0\right)'(x)^4\Psi_1^2(x)f^1(x)}{f^0(x)^7}\left[\hat f_g^0(x) - f^0(x)\right]dx$$
$$-8\int \frac{\left(f^0\right)'(x)^4}{f^0(x)^6}\left[\left(\hat\Psi_{1,g}^2(x) - \Psi_1^2(x)\right)\cdot\left(\hat f_g^1(x) - f^1(x)\right)\right]dx$$
$$-8\int \frac{\Psi_1^2(x)}{f^0(x)^6}\left[\left(\left(\hat f_g^0\right)'(x)^4 - \left(f^0\right)'(x)^4\right)\cdot\left(\hat f_g^1(x) - f^1(x)\right)\right]dx$$
$$-8\int \frac{f^1(x)}{f^0(x)^6}\left[\left(\left(\hat f_g^0\right)'(x)^4 - \left(f^0\right)'(x)^4\right)\cdot\left(\hat\Psi_{1,g}^2(x) - \Psi_1^2(x)\right)\right]dx$$
$$-8\int \frac{1}{f^0(x)^6}\left[\left(\left(\hat f_g^0\right)'(x)^4 - \left(f^0\right)'(x)^4\right)\cdot\left(\hat\Psi_{1,g}^2(x) - \Psi_1^2(x)\right)\right.$$
$$\left.\cdot\left(\hat f_g^1(x) - f^1(x)\right)\right]dx + O\left(\int\left(\hat f_g^0(x)^5 - f^0(x)^5\right)\right.$$
$$\left.\cdot\left(\left(\hat f_g^0\right)'(x)^4\hat\Psi_{1,g}^2(x)\hat f_g^1(x) - \left(f^0\right)'(x)^4\Psi_1^2(x)f^1(x)\right)dx\right)$$
$$+O\left(\int\left(\hat f_g^0(x) - f^0(x)\right)\cdot\left(\hat f_g^0(x)^4 - f^0(x)^4\right)dx + \int\left(\hat f_g^0(x) - f^0(x)\right)^2 dx\right)$$
$$+O\left(\int\left(\hat f_g^0(x) - f^0(x)\right)\cdot\left(\hat f_g^0(x)^3 - f^0(x)^3\right)\right)dx$$
$$+\int O\left(\left(\hat f_g^0(x)^5 - f^0(x)^5\right)^2 dx\right)$$

$$
\begin{aligned}
&= -8 \int \frac{\left(f^0\right)'(x)^4 \Psi_1^2(x)}{f^0(x)^6} \left[\hat{f}_g^1(x) - f^1(x)\right] dx \\
&- 16 \int \frac{\left(f^0\right)'(x)^4 f^1(x) \Psi_1(x)}{f^0(x)^6} \left[\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right] dx \\
&- 32 \int \frac{\Psi_1^2(x) f^1(x)}{f^0(x)^3} \left[\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right] dx \\
&+ 32 \int \frac{\left(f^0\right)'(x)^4 \Psi_1^2(x) f^1(x)}{f^0(x)^7} \left[\hat{f}_g^0(x) - f^0(x)\right] dx + O\left(\int \left(\hat{f}_g^0(x) - f^0(x)\right)^2\right. \\
&+ \left(\hat{f}_g^0(x) - f^0(x)\right) \cdot \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right) + \left(\hat{f}_g^0(x) - f^0(x)\right) \\
&\left. \cdot \left(\hat{f}_g^0(x)^3 - f^0(x)^3\right) dx\right) + O\left(\int \left(\hat{\Psi}_{1,g}^2(x) - \Psi_1^2(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right) \\
&+ O\left(\int \left(\left(\hat{f}_g^0\right)'(x)^4 - \left(f^0\right)'(x)^4\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right) \\
&+ O\left(\int \left(\left(\hat{f}_g^0\right)'(x)^4 - \left(f^0\right)'(x)^4\right) \cdot \left(\hat{\Psi}_{1,g}^2(x) - \Psi_1^2(x)\right) dx\right) \\
&+ O\left(\int \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right)^2 dx\right) \\
&+ O\left(\int \left(\left(\hat{f}_g^0\right)'(x)^4 - \left(f^0\right)'(x)^4\right) \cdot \left(\hat{\Psi}_{1,g}^2(x) - \Psi_1^2(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right) \\
&+ O\left(\int \left(\hat{f}_g^0(x)^5 - f^0(x)^5\right)\right. \\
&\left. \cdot \left(\left(\hat{f}_g^0\right)'(x)^4 \hat{\Psi}_{1,g}^2(x) \hat{f}_g^1(x) - \left(f^0\right)'(x)^4 \Psi_1^2(x) f^1(x)\right) dx\right) \\
&+ O\left(\int \left(\hat{f}_g^0(x) - f^0(x)\right) \cdot \left(\hat{f}_g^0(x)^4 - f^0(x)^4\right) dx + \int \left(\hat{f}_g^0(x) - f^0(x)\right)^2 dx\right) \\
&+ O\left(\int \left(\hat{f}_g^0(x) - f^0(x)\right) \cdot \left(\hat{f}_g^0(x)^3 - f^0(x)^3\right) dx\right) \\
&+ O\left(\left(\hat{f}_g^0(x)^5 - f^0(x)^5\right)^2 dx\right).
\end{aligned} \tag{115}
$$

Carrying on with calculations with the seventh addend in expression (103) following similar steps as in the previous expressions, it turns out:

$$
\begin{aligned}
&4\int \frac{1}{f^0(x)}\left[\frac{\left(\hat f_g^0\right)'(x)^2\hat\Psi_{1,g}(x)^2\left(\hat f_g^0\right)''(x)\hat f_g^1(x)}{\hat f_g^0(x)^4}\right.\\
&\left.-\frac{\left(f^0\right)'(x)^2\Psi_1(x)^2\left(f^0\right)''(x)f^1(x)}{f^0(x)^4}\right]dx\\
&=4\int \frac{1}{f^0(x)^5}\left[\left(\hat f_g^0\right)'(x)^2\hat\Psi_{1,g}(x)^2\left(\hat f_g^0\right)''(x)\hat f_g^1(x)\right.\\
&\left.-\left(f^0\right)'(x)^2\Psi_1(x)^2\left(f^0\right)''(x)f^1(x)\right]dx\\
&-4\int \frac{\left(f^0\right)'(x)^2\Psi_1(x)^2\left(f^0\right)''(x)f^1(x)}{f^0(x)^9}\left[\hat f_g^0(x)^4-f^0(x)^4\right]dx\\
&+O\left(\int\left(\hat f_g^0(x)^4-f^0(x)^4\right)^2dx\right)+O\left(\int\left(\hat f_g^0(x)^4-f^0(x)^4\right)\right.\\
&\left.\cdot\left(\left(\hat f_g^0\right)'(x)^2\hat\Psi_{1,g}(x)^2\left(\hat f_g^0\right)''(x)\hat f_g^1(x)-\left(f^0\right)'(x)^2\Psi_1(x)^2\left(f^0\right)''(x)f^1(x)\right)dx\right)\\
&=4\int \frac{1}{f^0(x)^5}\left[\left(\hat f_g^0\right)'(x)^2\hat\Psi_{1,g}(x)^2\left(\hat f_g^0\right)''(x)\hat f_g^1(x)\right.\\
&\left.-\left(f^0\right)'(x)^2\Psi_1(x)^2\left(f^0\right)''(x)f^1(x)\right]dx\\
&-16\int \frac{\left(f^0\right)'(x)^2\Psi_1(x)^2\left(f^0\right)''(x)f^1(x)}{f^0(x)^6}\left[\hat f_g^0(x)-f^0(x)\right]dx\\
&+O\left(\int\left(\hat f_g^0(x)^4-f^0(x)^4\right)^2dx\right)+O\left(\int\left(\hat f_g^0(x)^4-f^0(x)^4\right)\right.\\
&\left.\cdot\left(\left(\hat f_g^0\right)'(x)^2\hat\Psi_{1,g}(x)^2\left(\hat f_g^0\right)''(x)\hat f_g^1(x)-\left(f^0\right)'(x)^2\Psi_1(x)^2\left(f^0\right)''(x)f^1(x)\right)dx\right)\\
&+O\left(\int\left(\hat f_g^0(x)-f^0(x)\right)^2dx+\int\left(\hat f_g^0(x)-f^0(x)\right)\cdot\left(\hat f_g^0(x)^2-f^0(x)^2\right)dx\right)\\
&+O\left(\int\left(\hat f_g^0(x)-f^0(x)\right)\cdot\left(\hat f_g^0(x)^3-f^0(x)^3\right)dx\right)\\
&=4\int\frac{\left(f^0\right)'(x)^2}{f^0(x)^5}\left[\hat\Psi_{1,g}(x)^2\left(\hat f_g^0\right)''(x)\hat f_g^1(x)-\Psi_1(x)^2\left(f^0\right)''(x)f^1(x)\right]dx
\end{aligned}
$$

$$+4\int \frac{1}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\hat{\Psi}_{1,g}(x)^2\left(\hat{f}_g^0\right)''(x)\hat{f}_g^1(x)\right]dx$$

$$-16\int \frac{\left(f^0\right)'(x)^2\Psi_1(x)^2\left(f^0\right)''(x)f^1(x)}{f^0(x)^6}\left[\hat{f}_g^0(x)-f^0(x)\right]dx$$

$$+O\left(\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)^2dx\right)+O\left(\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)\right.$$

$$\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)^2\hat{\Psi}_{1,g}(x)^2\left(\hat{f}_g^0\right)''(x)\hat{f}_g^1(x)-\left(f^0\right)'(x)^2\Psi_1(x)^2\left(f^0\right)''(x)f^1(x)\right)dx\right)$$

$$+O\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2dx+\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)dx\right.$$

$$\left.+\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)dx\right)$$

$$=4\int \frac{\left(f^0\right)'(x)^2\Psi_1(x)^2}{f^0(x)^5}\left[\left(\hat{f}_g^0\right)''(x)\hat{f}_g^1(x)-\left(f^0\right)''(x)f^1(x)\right]dx$$

$$+4\int \frac{\left(f^0\right)'(x)^2}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}(x)^2-\Psi_1(x)^2\right)\cdot\left(\hat{f}_g^0\right)''(x)\hat{f}_g^1(x)\right]dx$$

$$+4\int \frac{\Psi_1(x)^2}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{f}_g^0\right)''(x)\hat{f}_g^1(x)\right]dx$$

$$+4\int \frac{1}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)^2-\hat{\Psi}_{1,g}(x)^2\right)\right.$$

$$\left.\cdot\left(\hat{f}_g^0\right)''(x)\hat{f}_g^1(x)\right]dx-16\int \frac{\left(f^0\right)'(x)^2\Psi_1(x)^2\left(f^0\right)''(x)f^1(x)}{f^0(x)^6}$$

$$\left[\hat{f}_g^0(x)-f^0(x)\right]dx+O\left(\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)^2dx\right)$$

$$+O\left(\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)\right.$$

$$\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)^2\hat{\Psi}_{1,g}(x)^2\left(\hat{f}_g^0\right)''(x)\hat{f}_g^1(x)-\left(f^0\right)'(x)^2\Psi_1(x)^2\left(f^0\right)''(x)f^1(x)\right)dx\right)$$

$$+O\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2dx+\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)dx\right.$$

$$\left.+\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)dx\right)$$

$$= 4\int \frac{\left(f^0\right)'(x)^2\Psi_1(x)^2\left(f^0\right)''(x)}{f^0(x)^5}\left[\hat{f}_g^1(x) - f^1(x)\right]dx$$
$$+ 4\int \frac{\left(f^0\right)'(x)^2\Psi_1(x)^2}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right)\cdot \hat{f}_g^1(x)\right]dx$$
$$+ 4\int \frac{\left(f^0\right)'(x)^2\left(f^0\right)''(x)}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}(x)^2 - \Psi_1(x)^2\right)\cdot \hat{f}_g^1(x)\right]dx$$
$$+ 4\int \frac{\left(f^0\right)'(x)^2}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}(x)^2 - \Psi_1(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right)\cdot \hat{f}_g^1(x)\right]dx$$
$$+ 4\int \frac{\Psi_1(x)^2\left(f^0\right)''(x)}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)'(x)^2 - \left(f^0\right)'(x)^2\right)\cdot \hat{f}_g^1(x)\right]dx$$
$$+ 4\int \frac{\Psi_1(x)^2}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)'(x)^2 - \left(f^0\right)'(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right)\right.$$
$$\left.\cdot\hat{f}_g^1(x)\right]dx$$
$$+ 4\int \frac{\left(f^0\right)''(x)}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)'(x)^2 - \left(f^0\right)'(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)^2 - \hat{\Psi}_{1,g}(x)^2\right)\right.$$
$$\left.\cdot\hat{f}_g^1(x)\right]dx + 4\int \frac{1}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)'(x)^2 - \left(f^0\right)'(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)^2 - \hat{\Psi}_{1,g}(x)^2\right)\right.$$
$$\left.\cdot\left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right)\cdot\hat{f}_g^1(x)\right]dx - 16\int \frac{\left(f^0\right)'(x)^2\Psi_1(x)^2\left(f^0\right)''(x)f^1(x)}{f^0(x)^6}$$
$$\left[\hat{f}_g^0(x) - f^0(x)\right]dx + O\left(\int\left(\hat{f}_g^0(x)^4 - f^0(x)^4\right)^2dx\right)$$
$$+ O\left(\int\left(\hat{f}_g^0(x)^4 - f^0(x)^4\right)\right.$$
$$\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)^2\hat{\Psi}_{1,g}(x)^2\left(\hat{f}_g^0\right)''(x)\hat{f}_g^1(x) - \left(f^0\right)'(x)^2\Psi_1(x)^2\left(f^0\right)''(x)f^1(x)\right)dx\right)$$
$$+ O\left(\int\left(\hat{f}_g^0(x) - f^0(x)\right)^2dx + \int\left(\hat{f}_g^0(x) - f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^2 - f^0(x)^2\right)dx\right.$$
$$\left.+\int\left(\hat{f}_g^0(x) - f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^3 - f^0(x)^3\right)dx\right)$$
$$= 4\int \frac{\left(f^0\right)'(x)^2\Psi_1(x)^2\left(f^0\right)''(x)}{f^0(x)^5}\left[\hat{f}_g^1(x) - f^1(x)\right]dx$$
$$+ 4\int \frac{\left(f^0\right)'(x)^2\Psi_1(x)^2f^1(x)}{f^0(x)^5}\left[\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right]dx$$
$$+ 4\int \frac{\left(f^0\right)'(x)^2\left(f^0\right)''(x)f^1(x)}{f^0(x)^5}\left[\hat{\Psi}_{1,g}(x)^2 - \Psi_1(x)^2\right]dx$$

$$+4\int \frac{\Psi_1(x)^2\left(f^0\right)''(x)f^1(x)}{f^0(x)^5}\left[\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right]dx$$

$$+4\int \frac{\left(f^0\right)'(x)^2\Psi_1(x)^2}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx$$

$$+4\int \frac{\left(f^0\right)'(x)^2\left(f^0\right)''(x)}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}(x)^2-\Psi_1(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx$$

$$+4\int \frac{\left(f^0\right)'(x)^2f^1(x)}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}(x)^2-\Psi_1(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\right]dx$$

$$+4\int \frac{\left(f^0\right)'(x)^2}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}(x)^2-\Psi_1(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\right.$$

$$\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx$$

$$+4\int \frac{\Psi_1(x)^2\left(f^0\right)''(x)}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx$$

$$+4\int \frac{\Psi_1(x)^2f^1(x)}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\right]dx$$

$$+4\int \frac{\Psi_1(x)^2}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\right.$$

$$\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx$$

$$+4\int \frac{\left(f^0\right)''(x)f^1(x)}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)^2-\hat{\Psi}_{1,g}(x)^2\right)\right]dx$$

$$+4\int \frac{\left(f^0\right)''(x)}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)^2-\hat{\Psi}_{1,g}(x)^2\right)\right.$$

$$\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx+4\int \frac{f^1(x)}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\right.$$

$$\left.\cdot\left(\hat{\Psi}_{1,g}(x)^2-\hat{\Psi}_{1,g}(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\right]dx$$

$$+4\int \frac{1}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)^2-\hat{\Psi}_{1,g}(x)^2\right)\right.$$

$$\left.\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx$$

$$-16\int \frac{\left(f^0\right)'(x)^2\Psi_1(x)^2\left(f^0\right)''(x)f^1(x)}{f^0(x)^6}\left[\hat{f}_g^0(x)-f^0(x)\right]dx$$

$$+O\left(\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)^2dx\right)+O\left(\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)\right.$$

$$\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)^2\hat{\Psi}_{1,g}(x)^2\left(\hat{f}_g^0\right)''(x)\hat{f}_g^1(x)-\left(f^0\right)'(x)^2\Psi_1(x)^2\left(f^0\right)''(x)f^1(x)\right)dx\right)$$

$$+O\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2dx+\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)dx\right.$$

$$\left.+\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)dx\right)$$

$$=4\int \frac{\left(f^0\right)'(x)^2\Psi_1(x)^2\left(f^0\right)''(x)}{f^0(x)^5}\left[\hat{f}_g^1(x)-f^1(x)\right]dx$$

$$
\begin{aligned}
&+4\int \frac{\left(f^0\right)'(x)^2\Psi_1(x)^2 f^1(x)}{f^0(x)^5}\left[\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right]dx\\
&+8\int \frac{\left(f^0\right)'(x)^2\left(f^0\right)''(x)f^1(x)\Psi_1(x)}{f^0(x)^5}\left[\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right]dx\\
&+8\int \frac{\Psi_1(x)^2\left(f^0\right)''(x)f^1(x)\left(f^0\right)'(x)}{f^0(x)^5}\left[\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right]dx\\
&-16\int \frac{\left(f^0\right)'(x)^2\Psi_1(x)^2\left(f^0\right)''(x)f^1(x)}{f^0(x)^6}\left[\hat{f}_g^0(x)-f^0(x)\right]dx\\
&+O\left(\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)^2dx\right)\\
&+O\left(\int\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+O\left(\int\left(\hat{\Psi}_{1,g}(x)^2-\Psi_1(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+O\left(\int\left(\hat{\Psi}_{1,g}(x)^2-\Psi_1(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)dx\right)\\
&+O\left(\int\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)^2-\hat{\Psi}_{1,g}(x)^2\right)dx\right)\\
&+O\left(\int\left(\hat{\Psi}_{1,g}(x)^2-\Psi_1(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\right.\\
&\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+O\left(\int\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+O\left(\int\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)^2dx\right)+O\left(\int\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\right.
\end{aligned}
$$

$$
\begin{aligned}
&\cdot \left(\left(\hat{f}_g^0\right)'' (x) - \left(f^0\right)'' (x) \right) dx \Bigg) + O\left(\int \left(\left(\hat{f}_g^0\right)' (x) - \left(f^0\right)' (x) \right)^2 dx \right) \\
&+ O\Bigg(\int \left(\left(\hat{f}_g^0\right)' (x)^2 - \left(f^0\right)' (x)^2 \right) \cdot \left(\left(\hat{f}_g^0\right)'' (x) - \left(f^0\right)'' (x) \right) \\
&\cdot \left(\hat{f}_g^1 (x) - f^1 (x) \right) dx \Bigg) + O\Bigg(\int \left(\left(\hat{f}_g^0\right)' (x)^2 - \left(f^0\right)' (x)^2 \right) \\
&\cdot \left(\hat{\Psi}_{1,g} (x)^2 - \hat{\Psi}_{1,g} (x)^2 \right) \cdot \left(\hat{f}_g^1 (x) - f^1 (x) \right) dx \Bigg) \\
&+ O\Bigg(\int \left(\left(\hat{f}_g^0\right)' (x)^2 - \left(f^0\right)' (x)^2 \right) \cdot \left(\hat{\Psi}_{1,g} (x)^2 - \hat{\Psi}_{1,g} (x)^2 \right) \\
&\cdot \left(\left(\hat{f}_g^0\right)'' (x) - \left(f^0\right)'' (x) \right) dx \Bigg) + O\Bigg(\int \left(\left(\hat{f}_g^0\right)' (x)^2 - \left(f^0\right)' (x)^2 \right) \\
&\cdot \left(\hat{\Psi}_{1,g} (x)^2 - \hat{\Psi}_{1,g} (x)^2 \right) \cdot \left(\left(\hat{f}_g^0\right)'' (x) - \left(f^0\right)'' (x) \right) \cdot \left(\hat{f}_g^1 (x) - f^1 (x) \right) dx \Bigg) \\
&+ O\Bigg(\int \left(\hat{f}_g^0 (x)^4 - f^0 (x)^4 \right) \\
&\cdot \left(\left(\hat{f}_g^0\right)' (x)^2 \hat{\Psi}_{1,g} (x)^2 \left(\hat{f}_g^0\right)'' (x) \hat{f}_g^1 (x) - \left(f^0\right)' (x)^2 \Psi_1 (x)^2 \left(f^0\right)'' (x) f^1 (x) \right) dx \Bigg) \\
&+ O\Bigg(\int \left(\hat{f}_g^0 (x) - f^0 (x) \right)^2 + \left(\hat{f}_g^0 (x) - f^0 (x) \right) \cdot \left(\hat{f}_g^0 (x)^2 - f^0 (x)^2 \right) dx \\
&+ \int \left(\hat{f}_g^0 (x) - f^0 (x) \right) \cdot \left(\hat{f}_g^0 (x)^3 - f^0 (x)^3 \right) dx \Bigg). \qquad (116)
\end{aligned}
$$

Finally, following analogous steps as in previous expressions, the eigthth addend in expression (103) happens to be:

$$8\int \frac{1}{f^0(x)}\left[\frac{\left(\hat{f}_g^0\right)'(x)^3\hat{\Psi}_{1,g}(x)\hat{\Psi}_{1,g}'(x)\hat{f}_g^1(x)}{\hat{f}_g^0(x)^4}-\frac{\left(f^0\right)'(x)^3\Psi_1(x)\Psi_1'(x)f^1(x)}{f^0(x)^4}\right]dx$$

$$=8\int \frac{1}{f^0(x)^5}\left[\left(\hat{f}_g^0\right)'(x)^3\hat{\Psi}_{1,g}(x)\hat{\Psi}_{1,g}'(x)\hat{f}_g^1(x)-\left(f^0\right)'(x)^3\Psi_1(x)\Psi_1'(x)f^1(x)\right]dx$$

$$-8\int \frac{\left(f^0\right)'(x)^3\Psi_1(x)\Psi_1'(x)f^1(x)}{f^0(x)^9}\left[\hat{f}_g^0(x)^4-f^0(x)^4\right]dx$$

$$+O\left(\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)^2dx\right)+O\left(\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)\right.$$

$$\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)^3\hat{\Psi}_{1,g}(x)\hat{\Psi}_{1,g}'(x)\hat{f}_g^1(x)-\left(f^0\right)'(x)^3\Psi_1(x)\Psi_1'(x)f^1(x)\right)dx\right)$$

$$=8\int \frac{1}{f^0(x)^5}\left[\left(\hat{f}_g^0\right)'(x)^3\hat{\Psi}_{1,g}(x)\hat{\Psi}_{1,g}'(x)\hat{f}_g^1(x)-\left(f^0\right)'(x)^3\Psi_1(x)\Psi_1'(x)f^1(x)\right]dx$$

$$-32\int \frac{\left(f^0\right)'(x)^3\Psi_1(x)\Psi_1'(x)f^1(x)}{f^0(x)^6}\left[\hat{f}_g^0(x)-f^0(x)\right]dx$$

$$+O\left(\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)^2dx\right)+O\left(\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)\right.$$

$$\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)^3\hat{\Psi}_{1,g}(x)\hat{\Psi}_{1,g}'(x)\hat{f}_g^1(x)-\left(f^0\right)'(x)^3\Psi_1(x)\Psi_1'(x)f^1(x)\right)dx\right)$$

$$+O\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2dx+\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)dx\right.$$

$$\left.+\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)dx\right)$$

$$=8\int \frac{\left(f^0\right)'(x)^3}{f^0(x)^5}\left[\hat{\Psi}_{1,g}(x)\hat{\Psi}_{1,g}'(x)\hat{f}_g^1(x)-\Psi_1(x)\Psi_1'(x)f^1(x)\right]dx$$

$$+8\int \frac{1}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\hat{\Psi}_{1,g}(x)\hat{\Psi}_{1,g}'(x)\hat{f}_g^1(x)\right]dx$$

$$-32\int \frac{\left(f^0\right)'(x)^3\Psi_1(x)\Psi_1'(x)f^1(x)}{f^0(x)^6}\left[\hat{f}_g^0(x)-f^0(x)\right]dx$$

$$+O\left(\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)^2dx\right)+O\left(\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)\right.$$

$$\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)^3\hat{\Psi}_{1,g}(x)\hat{\Psi}_{1,g}'(x)\hat{f}_g^1(x)-\left(f^0\right)'(x)^3\Psi_1(x)\Psi_1'(x)f^1(x)\right)dx\right)$$

$$+O\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2dx+\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)dx\right.$$

$$\left.+\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)dx\right)$$

$$=8\int \frac{\left(f^0\right)'(x)^3\Psi_1(x)}{f^0(x)^5}\left[\hat{\Psi}_{1,g}'(x)\hat{f}_g^1(x)-\Psi_1'(x)f^1(x)\right]dx$$

$$+8\int \frac{\left(f^0\right)'(x)^3}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\hat{\Psi}_{1,g}'(x)\hat{f}_g^1(x)\right]dx$$

$$+8\int \frac{\Psi_1(x)}{f^0(x)^5}\left[\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\hat{\Psi}_{1,g}'(x)\hat{f}_g^1(x)\right]dx+8\int\frac{1}{f^0(x)^5}$$

$$\left[\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\hat{\Psi}_{1,g}'(x)\hat{f}_g^1(x)\right]dx$$

$$- 32 \int \frac{\left(f^0\right)' (x)^3 \Psi_1(x) \Psi_1'(x) f^1(x)}{f^0(x)^6} \left[\hat{f}_g^0(x) - f^0(x)\right] dx$$
$$+ O\left(\int \left(\hat{f}_g^0(x)^4 - f^0(x)^4\right)^2 dx\right) + O\left(\int \left(\hat{f}_g^0(x)^4 - f^0(x)^4\right)\right.$$
$$\left. \cdot \left(\left(\hat{f}_g^0\right)' (x)^3 \hat{\Psi}_{1,g}(x) \hat{\Psi}_{1,g}'(x) \hat{f}_g^1(x) - \left(f^0\right)' (x)^3 \Psi_1(x) \Psi_1'(x) f^1(x)\right) dx\right)$$
$$+ O\left(\int \left(\hat{f}_g^0(x) - f^0(x)\right)^2 dx + \int \left(\hat{f}_g^0(x) - f^0(x)\right) \cdot \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right) dx\right.$$
$$\left. + \int \left(\hat{f}_g^0(x) - f^0(x)\right) \cdot \left(\hat{f}_g^0(x)^3 - f^0(x)^3\right) dx\right)$$
$$= 8 \int \frac{\left(f^0\right)' (x)^3 \Psi_1(x) \Psi_1'(x)}{f^0(x)^5} \left[\hat{f}_g^1(x) - f^1(x)\right] dx$$
$$+ 8 \int \frac{\left(f^0\right)' (x)^3 \Psi_1(x)}{f^0(x)^5} \left[\left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right) \cdot \hat{f}_g^1(x)\right] dx$$
$$+ 8 \int \frac{\left(f^0\right)' (x)^3 \Psi_1'(x)}{f^0(x)^5} \left[\left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \hat{f}_g^1(x)\right] dx$$
$$+ 8 \int \frac{\left(f^0\right)' (x)^3}{f^0(x)^5} \left[\left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right) \cdot \hat{f}_g^1(x)\right] dx$$

$$+ 8 \int \frac{\Psi_1(x) \Psi_1'(x)}{f^0(x)^5} \left[\left(\left(\hat{f}_g^0\right)' (x)^3 - \left(f^0\right)' (x)^3\right) \cdot \hat{f}_g^1(x)\right] dx$$
$$+ 8 \int \frac{\Psi_1(x)}{f^0(x)^5} \left[\left(\left(\hat{f}_g^0\right)' (x)^3 - \left(f^0\right)' (x)^3\right) \cdot \left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right) \cdot \hat{f}_g^1(x)\right] dx$$
$$+ 8 \int \frac{\Psi_1'(x)}{f^0(x)^5} \left[\left(\left(\hat{f}_g^0\right)' (x)^3 - \left(f^0\right)' (x)^3\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \hat{f}_g^1(x)\right] dx$$
$$+ 8 \int \frac{1}{f^0(x)^5} \left[\left(\left(\hat{f}_g^0\right)' (x)^3 - \left(f^0\right)' (x)^3\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right)\right.$$
$$\left. \cdot \left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right) \cdot \hat{f}_g^1(x)\right] dx - 32 \int \frac{\left(f^0\right)' (x)^3 \Psi_1(x) \Psi_1'(x) f^1(x)}{f^0(x)^6}$$
$$\left[\hat{f}_g^0(x) - f^0(x)\right] dx + O\left(\int \left(\hat{f}_g^0(x)^4 - f^0(x)^4\right)^2 dx\right)$$
$$+ O\left(\int \left(\hat{f}_g^0(x)^4 - f^0(x)^4\right)\right.$$

$$
\begin{aligned}
&\cdot\left(\left(\hat{f}_g^0\right)^{'}(x)^3\hat{\Psi}_{1,g}(x)\hat{\Psi}_{1,g}'(x)\hat{f}_g^1(x)-\left(f^0\right)^{'}(x)^3\Psi_1(x)\Psi_1'(x)f^1(x)\right)dx\Bigg)\\
&+O\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2dx+\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)dx\right.\\
&\left.+\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)dx\right)\\
&=8\int\frac{\left(f^0\right)^{'}(x)^3\Psi_1(x)\Psi_1'(x)}{f^0(x)^5}\left[\hat{f}_g^1(x)-f^1(x)\right]dx\\
&+8\int\frac{\left(f^0\right)^{'}(x)^3\Psi_1(x)f^1(x)}{f^0(x)^5}\left[\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right]dx\\
&+8\int\frac{\left(f^0\right)^{'}(x)^3\Psi_1'(x)f^1(x)}{f^0(x)^5}\left[\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right]dx\\
&+8\int\frac{\Psi_1(x)\Psi_1'(x)f^1(x)}{f^0(x)^5}\left[\left(\hat{f}_g^0\right)^{'}(x)^3-\left(f^0\right)^{'}(x)^3\right]dx\\
&+8\int\frac{\left(f^0\right)^{'}(x)^3\Psi_1(x)}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx
\end{aligned}
$$

$$
\begin{aligned}
&+8\int\frac{\left(f^0\right)^{'}(x)^3\Psi_1'(x)}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx\\
&+8\int\frac{\left(f^0\right)^{'}(x)^3f^1(x)}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\right]dx
\end{aligned}
$$

$$+8\int \frac{\left(f^0\right)'(x)^3}{f^0(x)^5}\left[\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\right.$$
$$\left.\cdot\left(\hat{f}^1_g(x)-f^1(x)\right)\right]dx$$
$$+8\int \frac{\Psi_1(x)\Psi'_1(x)}{f^0(x)^5}\left[\left(\left(\hat{f}^0_g\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\left(\hat{f}^1_g(x)-f^1(x)\right)\right]dx$$
$$+8\int \frac{\Psi_1(x)f^1(x)}{f^0(x)^5}\left[\left(\left(\hat{f}^0_g\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\right]dx$$
$$+8\int \frac{\Psi_1(x)}{f^0(x)^5}\left[\left(\left(\hat{f}^0_g\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\right.$$
$$\left.\cdot\left(\hat{f}^1_g(x)-f^1(x)\right)\right]dx+8\int \frac{\Psi'_1(x)f^1(x)}{f^0(x)^5}\left[\left(\left(\hat{f}^0_g\right)'(x)^3-\left(f^0\right)'(x)^3\right)\right.$$
$$\left.\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right]dx$$
$$+8\int \frac{\Psi'_1(x)}{f^0(x)^5}\left[\left(\left(\hat{f}^0_g\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right.$$
$$\left.\cdot\left(\hat{f}^1_g(x)-f^1(x)\right)\right]dx+8\int \frac{f^1(x)}{f^0(x)^5}\left[\left(\left(\hat{f}^0_g\right)'(x)^3-\left(f^0\right)'(x)^3\right)\right.$$
$$\left.\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\right]dx$$
$$+8\int \frac{1}{f^0(x)^5}\left[\left(\left(\hat{f}^0_g\right)'(x)^3-\left(f^0\right)'(x)^3\right)\right.$$
$$\left.\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\hat{f}^1_g(x)-f^1(x)\right)\right]dx$$
$$-32\int \frac{\left(f^0\right)'(x)^3\Psi_1(x)\Psi'_1(x)f^1(x)}{f^0(x)^6}\left[\hat{f}^0_g(x)-f^0(x)\right]dx$$
$$+O\left(\int\left(\hat{f}^0_g(x)^4-f^0(x)^4\right)^2dx\right)+O\left(\int\left(\hat{f}^0_g(x)^4-f^0(x)^4\right)\right.$$
$$\left.\cdot\left(\left(\hat{f}^0_g\right)'(x)^3\hat{\Psi}_{1,g}(x)\hat{\Psi}'_{1,g}(x)\hat{f}^1_g(x)-\left(f^0\right)'(x)^3\Psi_1(x)\Psi'_1(x)f^1(x)\right)dx\right)$$

$$+O\left(\int\left(\hat{f}^0_g(x)-f^0(x)\right)^2dx+\int\left(\hat{f}^0_g(x)-f^0(x)\right)\cdot\left(\hat{f}^0_g(x)^2-f^0(x)^2\right)dx\right.$$
$$\left.+\left(\hat{f}^0_g(x)-f^0(x)\right)\cdot\left(\hat{f}^0_g(x)^3-f^0(x)^3\right)dx\right)$$

$$
\begin{aligned}
&= 8 \int \frac{\left(f^0\right)'(x)^3 \Psi_1(x) \Psi_1'(x)}{f^0(x)^5} \left[\hat{f}_g^1(x) - f^1(x)\right] dx \\
&+ 8 \int \frac{\left(f^0\right)'(x)^3 \Psi_1(x) f^1(x)}{f^0(x)^5} \left[\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right] dx \\
&+ 8 \int \frac{\left(f^0\right)'(x)^3 \Psi_1'(x) f^1(x)}{f^0(x)^5} \left[\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right] dx \\
&+ 24 \int \frac{\Psi_1(x) \Psi_1'(x) f^1(x) \left(f^0\right)'(x)^3}{f^0(x)^5} \left[\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right] dx \\
&- 32 \int \frac{\left(f^0\right)'(x)^3 \Psi_1(x) \Psi_1'(x) f^1(x)}{f^0(x)^6} \left[\hat{f}_g^0(x) - f^0(x)\right] dx \\
&+ O\left(\int \left(\hat{f}_g^0(x)^4 - f^0(x)^4\right)^2 dx\right) \\
&+ O\left(\int \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right)^2 + \left(\hat{f}_g^0\right)'(x)^2 - \left(f^0\right)'(x)^2\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ O\left(\int \left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right) \\
&+ O\left(\int \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right) \\
&+ O\left(\int \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right) dx\right) \\
&+ O\left(\int \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right) \\
&+ O\left(\int \left(\left(\hat{f}_g^0\right)'(x)^3 - \left(f^0\right)'(x)^3\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right) \\
&+ O\left(\int \left(\left(\hat{f}_g^0\right)'(x)^3 - \left(f^0\right)'(x)^3\right) \cdot \left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right) dx\right) \\
&+ O\left(\int \left(\left(\hat{f}_g^0\right)'(x)^3 - \left(f^0\right)'(x)^3\right) \cdot \left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right) \\
&+ O\left(\int \left(\left(\hat{f}_g^0\right)'(x)^3 - \left(f^0\right)'(x)^3\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) dx\right) \\
&+ O\left(\int \left(\left(\hat{f}_g^0\right)'(x)^3 - \left(f^0\right)'(x)^3\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right) \\
&+ O\left(\int \left(\left(\hat{f}_g^0\right)'(x)^3 - \left(f^0\right)'(x)^3\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right) dx\right) \\
&+ O\left(\int \left(\left(\hat{f}_g^0\right)'(x)^3 - \left(f^0\right)'(x)^3\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right)\right. \\
&\left. \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right) + O\left(\int \left(\hat{f}_g^0(x)^4 - f^0(x)^4\right)\right.
\end{aligned}
$$

$$\cdot\left(\left(\hat{f}_g^0\right)'(x)^3\hat{\Psi}_{1,g}(x)\hat{\Psi}_{1,g}'(x)\hat{f}_g^1(x)-\left(f^0\right)'(x)^3\Psi_1(x)\Psi_1'(x)f^1(x)\right)dx\Bigg)$$
$$+O\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2dx+\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)dx\right.$$
$$\left.+\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)dx\right). \quad (117)$$

Collecting terms (106)-(117) and plugging them into (103), we obtain an expression for term B_2 (102), given by:

$$\begin{aligned}
B_2 := {} & 4\int\frac{\Psi_1'(x)\Psi_1''(x)\left(f^0\right)'(x)}{f^0(x)^3}\left[\hat{f}_g^1(x)-f^1(x)\right]dx\\
&+4\int\frac{\Psi_1'(x)\Psi_1''(x)f^1(x)}{f^0(x)^3}\left[\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right]dx\\
&+4\int\frac{\Psi_1'(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^3}\left[\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right]dx\\
&+4\int\frac{\Psi_1''(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^3}\left[\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right]dx\\
&-4\int\frac{m'(x)m''(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^2}\left[\hat{f}_g^0(x)-f^0(x)\right]dx\\
&-8\int\frac{\Psi_1'(x)\Psi_1''(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^4}\left[\hat{f}_g^0(x)-f^0(x)\right]dx\\
&+8\int\frac{\Psi_1'(x)\left(f^0\right)'(x)^3\Psi_1(x)}{f^0(x)^5}\left[\hat{f}_g^1(x)-f^1(x)\right]dx\\
&+8\int\frac{\Psi_1'(x)\left(f^0\right)'(x)^3f^1(x)}{f^0(x)^5}\left[\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right]dx\\
&+24\int\frac{\Psi_1'(x)\Psi_1(x)f^1(x)\left(f^0\right)'(x)^2}{f^0(x)^5}\left[\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right]dx\\
&+8\int\frac{\left(f^0\right)'(x)^3\Psi_1(x)f^1(x)}{f^0(x)^5}\left[\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right]dx\\
&-4\int\frac{\Psi_1'(x)\left(f^0\right)''(x)\Psi_1(x)\left(f^0\right)'(x)}{f^0(x)^4}\left[\hat{f}_g^1(x)-f^1(x)\right]dx\\
&-32\int\frac{\Psi_1'(x)\left(f^0\right)'(x)^3\Psi_1(x)f^1(x)}{f^0(x)^5}\left[\hat{f}_g^0(x)-f^0(x)\right]dx
\end{aligned}$$

$$
\begin{aligned}
&-4\int \frac{\Psi_1'(x)\left(f^0\right)''(x)\Psi_1(x)f^1(x)}{f^0(x)^4}\left[\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right]dx\\
&-4\int \frac{\Psi_1'(x)\left(f^0\right)''(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^4}\left[\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right]dx\\
&-4\int \frac{\Psi_1'(x)\Psi_1(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^4}\left[\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right]dx\\
&-4\int \frac{\left(f^0\right)''(x)\Psi_1(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^4}\left[\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right]dx\\
&-12\int \frac{\Psi_1'(x)\left(f^0\right)''(x)\Psi_1(x)\left(f^0\right)'(x)f^1(x)}{f^0(x)^5}\left[\hat{f}_g^0(x)-f^0(x)\right]dx\\
&-8\int \frac{\Psi_1'(x)^2\left(f^0\right)'(x)^2}{f^0(x)^4}\left[\hat{f}_g^1(x)-f^1(x)\right]dx\\
&-16\int \frac{\Psi_1'(x)^2f^1(x)\left(f^0\right)'(x)}{f^0(x)^4}\left[\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right]dx\\
&-16\int \frac{\left(f^0\right)'(x)^2f^1(x)\Psi_1'(x)}{f^0(x)^4}\left[\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right]dx\\
&+24\int \frac{\Psi_1'(x)^2\left(f^0\right)'(x)^2f^1(x)}{f^0(x)^5}\left[\hat{f}_g^0(x)-f^0(x)\right]dx\\
&-4\int \frac{\left(f^0\right)'(x)^2\Psi_1(x)\Psi_1''(x)}{f^0(x)^4}\left[\hat{f}_g^1(x)-f^1(x)\right]dx\\
&-4\int \frac{\left(f^0\right)'(x)^2\Psi_1(x)f^1(x)}{f^0(x)^4}\left[\hat{\Psi}_{1,g}''(x)-\Psi_1''(x)\right]dx\\
&-4\int \frac{\left(f^0\right)'(x)^2\Psi_1''(x)f^1(x)}{f^0(x)^4}\left[\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right]dx\\
&-8\int \frac{\Psi_1(x)\Psi_1''(x)f^1(x)\left(f^0\right)'(x)}{f^0(x)^4}\left[\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right]dx\\
&+12\int \frac{\left(f^0\right)'(x)^2\Psi_1(x)\Psi_1''(x)f^1(x)}{f^0(x)^5}\left[\hat{f}_g^0(x)-f^0(x)\right]dx\\
&-8\int \frac{\left(f^0\right)'(x)^4\Psi_1^2(x)}{f^0(x)^6}\left[\hat{f}_g^1(x)-f^1(x)\right]dx\\
&-16\int \frac{\left(f^0\right)'(x)^4f^1(x)\Psi_1(x)}{f^0(x)^6}\left[\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right]dx\\
&-32\int \frac{\Psi_1^2(x)f^1(x)}{f^0(x)^3}\left[\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right]dx
\end{aligned}
$$

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\begin{aligned}
&+ 32 \int \frac{\left(f^0\right)' (x)^4 \Psi_1^2(x) f^1(x)}{f^0(x)^7} \left[\hat{f}_g^0(x) - f^0(x)\right] dx \\
&+ 4 \int \frac{\left(f^0\right)' (x)^2 \Psi_1(x)^2 \left(f^0\right)'' (x)}{f^0(x)^5} \left[\hat{f}_g^1(x) - f^1(x)\right] dx \\
&+ 4 \int \frac{\left(f^0\right)' (x)^2 \Psi_1(x)^2 f^1(x)}{f^0(x)^5} \left[\left(\hat{f}_g^0\right)'' (x) - \left(f^0\right)'' (x)\right] dx \\
&+ 8 \int \frac{\left(f^0\right)' (x)^2 \left(f^0\right)'' (x) f^1(x) \Psi_1(x)}{f^0(x)^5} \left[\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right] dx \\
&+ 8 \int \frac{\Psi_1(x)^2 \left(f^0\right)'' (x) f^1(x) \left(f^0\right)' (x)}{f^0(x)^5} \left[\left(\hat{f}_g^0\right)' (x) - \left(f^0\right)' (x)\right] dx \\
&- 16 \int \frac{\left(f^0\right)' (x)^2 \Psi_1(x)^2 \left(f^0\right)'' (x) f^1(x)}{f^0(x)^6} \left[\hat{f}_g^0(x) - f^0(x)\right] dx \\
&+ 8 \int \frac{\left(f^0\right)' (x)^3 \Psi_1(x) \Psi_1'(x)}{f^0(x)^5} \left[\hat{f}_g^1(x) - f^1(x)\right] dx \\
&+ 8 \int \frac{\left(f^0\right)' (x)^3 \Psi_1(x) f^1(x)}{f^0(x)^5} \left[\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right] dx \\
&+ 8 \int \frac{\left(f^0\right)' (x)^3 \Psi_1'(x) f^1(x)}{f^0(x)^5} \left[\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right] dx \\
&+ 24 \int \frac{\Psi_1(x) \Psi_1'(x) f^1(x) \left(f^0\right)' (x)^3}{f^0(x)^5} \left[\left(\hat{f}_g^0\right)' (x) - \left(f^0\right)' (x)\right] dx \\
&- 32 \int \frac{\left(f^0\right)' (x)^3 \Psi_1(x) \Psi_1'(x) f^1(x)}{f^0(x)^6} \left[\hat{f}_g^0(x) - f^0(x)\right] dx \\
&+ O\left(\int \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right)^2 dx\right) + O\left(\left(\hat{f}_g^0(x)^4 - f^0(x)^4\right)^2 dx\right) \\
&+ O\left(\int \left(\hat{f}_g^0(x)^3 - f^0(x)^3\right)^2 dx\right) + O\left(\int \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right)\right. \\
&\cdot \left.\left(\hat{\Psi}_{1,g}'(x) \hat{\Psi}_{1,g}''(x) \left(\hat{f}_g^0\right)' (x) \hat{f}_g^1(x) - \Psi_1'(x) \Psi_1''(x) \left(f^0\right)' (x) f^1(x)\right) dx\right) \\
&+ O\left(\int \left(\left(\hat{f}_g^0\right)' (x) - \left(f^0\right)' (x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right) \\
&+ O\left(\int \left(\hat{\Psi}_{1,g}''(x) - \Psi_1''(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right)
\end{aligned}
$$

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\begin{aligned}
&+ O\left(\int \left(\hat{\Psi}''_{1,g}(x) - \Psi''_1(x)\right) \cdot \left(\left(\hat{f}^0_g\right)'(x) - \left(f^0\right)'(x)\right) dx\right) \\
&+ O\left(\int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x)\right) \cdot \left(\hat{f}^1_g(x) - f^1(x)\right) dx\right) \\
&+ O\left(\int \left(\hat{\Psi}''_{1,g}(x) - \Psi''_1(x)\right) \cdot \left(\left(\hat{f}^0_g\right)'(x) - \left(f^0\right)'(x)\right) \right.\\
&\left.\cdot \left(\hat{f}^1_g(x) - f^1(x)\right) dx\right) + O\left(\int \left(\hat{f}^0_g(x) - f^0(x)\right)^2 dx\right) \\
&+ O\left(\int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x)\right) \cdot \left(\left(\hat{f}^0_g\right)'(x) - \left(f^0\right)'(x)\right) dx\right) \\
&+ O\left(\int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x)\right) \cdot \left(\hat{\Psi}''_{1,g}(x) - \Psi''_1(x)\right) dx\right) \\
&+ O\left(\int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x)\right) \cdot \left(\left(\hat{f}^0_g\right)'(x) - \left(f^0\right)'(x)\right) \cdot \left(\hat{f}^1_g(x) - f^1(x)\right) dx\right) \\
&+ O\left(\int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x)\right) \cdot \left(\hat{\Psi}''_{1,g}(x) - \Psi''_1(x)\right) \cdot \left(\hat{f}^1_g(x) - f^1(x)\right) dx\right) \\
&+ O\left(\int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x)\right) \cdot \left(\hat{\Psi}''_{1,g}(x) - \Psi''_1(x)\right) \cdot \left(\left(\hat{f}^0_g\right)'(x) - \left(f^0\right)'(x)\right) dx\right) \\
&+ O\left(\int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x)\right) \cdot \left(\hat{\Psi}''_{1,g}(x) - \Psi''_1(x)\right) \cdot \left(\left(\hat{f}^0_g\right)'(x) - \left(f^0\right)'(x)\right) \right.\\
&\left.\cdot \left(\hat{f}^1_g(x) - f^1(x)\right) dx\right) + O\left(\int \left(\hat{f}^0_g(x)^4 - f^0(x)^4\right) \right.\\
&\left.\cdot \left(\hat{\Psi}'_{1,g}(x)\left(\hat{f}^0_g\right)'(x)^3 \hat{\Psi}_{1,g}(x)\hat{f}^1_g(x) - \Psi'_1(x)\left(f^0\right)'(x)^3 \Psi_1(x) f^1(x)\right) dx\right) \\
&+ O\left(\int \left(\hat{f}^0_g(x) - f^0(x)\right) \cdot \left(\hat{f}^0_g(x)^4 - f^0(x)^4\right) dx + \int \left(\hat{f}^0_g(x) - f^0(x)\right)^2 dx\right) \\
&+ O\left(\int \left(\left(\hat{f}^0_g\right)'(x) - \left(f^0\right)'(x)\right)^2 dx\right) \\
&+ O\left(\int \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\hat{f}^1_g(x) - f^1(x)\right) dx\right) \\
&+ O\left(\int \left(\left(\hat{f}^0_g\right)'(x)^3 - \left(f^0\right)'(x)^3\right) \cdot \left(\hat{f}^1_g(x) - f^1(x)\right) dx\right) \\
&+ O\left(\int \left(\left(\hat{f}^0_g\right)'(x)^3 - \left(f^0\right)'(x)^3\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) dx\right) \\
&+ O\left(\int \left(\hat{f}^0_g\right)'(x)^2 - \left(f^0\right)'(x)^2 dx\right) \\
&+ O\left(\int \left(\left(\hat{f}^0_g\right)'(x)^3 - \left(f^0\right)'(x)^3\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\hat{f}^1_g(x) - f^1(x)\right) dx\right) \\
&+ O\left(\int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x)\right) \cdot \left(\hat{f}^1_g(x) - f^1(x)\right) dx\right)
\end{aligned}
$$

$$+ O\left(\int \left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) dx\right)$$
$$+ O\left(\int \left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right)$$
$$+ O\left(\int \left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right) \cdot \left(\left(\hat{f}_g^0\right)'(x)^3 - \left(f^0\right)'(x)^3\right) dx\right)$$
$$+ O\left(\int \left(\hat{f}_g^0(x) - f^0(x)\right) \cdot \left(\hat{f}_g^0(x)^3 - f^0(x)^3\right) dx\right)$$
$$+ O\left(\int \left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right) \cdot \left(\left(\hat{f}_g^0\right)'(x)^3 - \left(f^0\right)'(x)^3\right)\right.$$
$$\left.\cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right) + O\left(\int \left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right)\right.$$
$$\left.\cdot \left(\left(\hat{f}_g^0\right)'(x)^3 - \left(f^0\right)'(x)^3\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) dx\right)$$
$$+ O\left(\int \left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right) \cdot \left(\left(\hat{f}_g^0\right)'(x)^3 - \left(f^0\right)'(x)^3\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right)\right.$$
$$\left.\cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right) + O\left(\int \left(\hat{f}_g^0(x)^3 - f^0(x)^3\right) \cdot \left(\hat{\Psi}_{1,g}'(x) \left(\hat{f}_g^0\right)''(x)\right.\right.$$
$$\left.\left.\hat{\Psi}_{1,g}(x) \left(\hat{f}_g^0\right)'(x) \hat{f}_g^1(x) - \Psi_1'(x) \left(f^0\right)''(x) \Psi_1(x) \left(f^0\right)'(x) f^1(x)\right) dx\right)$$
$$+ O\left(\int \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right)$$
$$+ O\left(\int \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right)$$
$$+ O\left(\int \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right) dx\right)$$
$$+ O\left(\int \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right)$$
$$+ O\left(\int \left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right)$$
$$+ O\left(\int \left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right) \cdot \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right) dx\right)$$
$$+ O\left(\int \left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right) \cdot \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right)\right.$$
$$\left.\cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right)$$
$$+ O\left(\int \left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) dx\right)$$
$$+ O\left(\int \left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right)$$

$$
\begin{aligned}
&+ O\left(\int \left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right)\right.\\
&\left.\cdot \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right) dx\right) + O\left(\int \left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right)\right.\\
&\left.\cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right)\\
&+ O\left(\int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right)\\
&+ O\left(\int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x)\right) \cdot \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right) dx\right)\\
&+ O\left(\int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x)\right) \cdot \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right)\\
&+ O\left(\int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x)\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) dx\right)\\
&+ O\left(\int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x)\right) \cdot \left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right) dx\right)\\
&+ O\left(\int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x)\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right)
\end{aligned}
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\begin{aligned}
&+O\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)dx\right)\\
&+O\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\right.\\
&\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)+O\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\right.\\
&\left.\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)dx\right)\\
&+O\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\right.\\
&\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+O\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)dx\right)\\
&+O\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right.\\
&\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)+O\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\right.\\
&\left.\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)dx\right)\\
&+O\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)\cdot\left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\right.\\
&\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+O\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2+\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)dx\right)\\
&+O\left(\int\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+O\left(\int\left(\hat{\Psi}_{1,g}'(x)^2-\Psi_1'(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+O\left(\int\left(\hat{\Psi}_{1,g}'(x)^2-\Psi_1'(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)dx\right)\\
&+O\left(\int\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)^2dx\right)+O\left(\int\left(\hat{\Psi}_{1,g}'(x)^2-\Psi_1'(x)^2\right)\right.\\
&\left.\cdot\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\right)\\
&+O\left(\int\left(\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right)^2dx\right)+O\left(\int\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)\right.\\
&\left.\cdot\left(\hat{\Psi}_{1,g}'(x)^2\left(\hat{f}_g^0\right)'(x)^2\hat{f}_g^1(x)-\Psi_1'(x)^2\left(f^0\right)'(x)^2f^1(x)\right)dx\right)\\
&+O\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2+\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)dx\right)\\
&+O\left(\int\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)^2dx\right)+O\left(\int\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)^2dx\right)
\end{aligned}
$$

$$+ O\left(\int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x)\right) \cdot \left(\left(\hat{f}^0_g\right)''(x) - \left(f^0\right)''(x)\right) \cdot \left(\hat{f}^1_g(x) - f^1(x)\right) dx\right)$$
$$+ O\left(\int \left(\hat{\Psi}''_{1,g}(x) - \Psi''_1(x)\right) \cdot \left(\hat{f}^1_g(x) - f^1(x)\right) dx\right)$$
$$+ O\left(\int \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\hat{f}^1_g(x) - f^1(x)\right) dx\right)$$
$$+ O\left(\int \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\hat{\Psi}''_{1,g}(x) - \Psi''_1(x)\right) dx\right)$$
$$+ O\left(\int \left(\left(\hat{f}^0_g\right)'(x)^2 - \left(f^0\right)'(x)^2\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) dx\right)$$
$$+ O\left(\int \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\hat{\Psi}''_{1,g}(x) - \Psi''_1(x)\right) \cdot \left(\hat{f}^1_g(x) - f^1(x)\right) dx\right)$$
$$+ O\left(\int \left(\left(\hat{f}^0_g\right)'(x) - \left(f^0\right)'(x)\right)^2 dx\right)$$
$$+ O\left(\int \left(\left(\hat{f}^0_g\right)'(x)^2 - \left(f^0\right)'(x)^2\right) \cdot \left(\hat{\Psi}''_{1,g}(x) - \Psi''_1(x)\right) dx\right)$$
$$+ O\left(\int \left(\left(\hat{f}^0_g\right)'(x)^2 - \left(f^0\right)'(x)^2\right) \cdot \left(\hat{f}^1_g(x) - f^1(x)\right) dx\right)$$
$$+ O\left(\int \left(\hat{f}^0_g(x) - f^0(x)\right)^2 dx + \int \left(\hat{f}^0_g(x)^2 - f^0(x)^2\right) dx\right)$$
$$+ O\left(\int \left(\left(\hat{f}^0_g\right)'(x)^2 - \left(f^0\right)'(x)^2\right) \cdot \left(\hat{\Psi}''_{1,g}(x) - \Psi''_1(x)\right) \cdot \left(\hat{f}^1_g(x) - f^1(x)\right) dx\right)$$
$$+ O\left(\int \left(\left(\hat{f}^0_g\right)'(x)^2 - \left(f^0\right)'(x)^2\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\hat{f}^1_g(x) - f^1(x)\right) dx\right)$$
$$+ O\left(\int \left(\left(\hat{f}^0_g\right)'(x)^2 - \left(f^0\right)'(x)^2\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right)\right.$$
$$\left. \cdot \left(\hat{\Psi}''_{1,g}(x) - \Psi''_1(x)\right) dx\right)$$
$$+ O\left(\int \left(\left(\hat{f}^0_g\right)'(x)^2 - \left(f^0\right)'(x)^2\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\hat{\Psi}''_{1,g}(x) - \Psi''_1(x)\right)\right.$$

$$\cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right) + O\left(\int \left(\hat{f}_g^0(x)^3 - f^0(x)^3\right)\right.$$

$$\left. \cdot \left(\left(\hat{f}_g^0\right)'(x)^2 \hat{\Psi}_{1,g}(x) \hat{\Psi}''_{1,g}(x) \hat{f}_g^1(x) - \left(f^0\right)'(x)^2 \Psi_1(x) \Psi''_1(x) f^1(x)\right) dx\right)$$

$$+ O\left(\int \left(\hat{f}_g^0(x) - f^0(x)\right) \cdot \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right) dx\right.$$

$$\left. + \int \left(\hat{f}_g^0(x) - f^0(x)\right) \cdot \left(\hat{f}_g^0(x)^3 - f^0(x)^3\right) dx\right)$$

$$+ O\left(\int \left(\hat{\Psi}_{1,g}^2(x) - \Psi_1^2(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right)$$

$$+ O\left(\int \left(\left(\hat{f}_g^0\right)'(x)^4 - \left(f^0\right)'(x)^4\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right)$$

$$+ O\left(\int \left(\left(\hat{f}_g^0\right)'(x)^4 - \left(f^0\right)'(x)^4\right) \cdot \left(\hat{\Psi}_{1,g}^2(x) - \Psi_1^2(x)\right) dx\right)$$

$$+ O\left(\int \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right)^2 dx\right)$$

$$+ O\left(\int \left(\left(\hat{f}_g^0\right)'(x)^4 - \left(f^0\right)'(x)^4\right) \cdot \left(\hat{\Psi}_{1,g}^2(x) - \Psi_1^2(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right)$$

$$+ O\left(\int \left(\hat{f}_g^0(x)^5 - f^0(x)^5\right)\right.$$

$$\left. \cdot \left(\left(\hat{f}_g^0\right)'(x)^4 \hat{\Psi}_{1,g}^2(x) \hat{f}_g^1(x) - \left(f^0\right)'(x)^4 \Psi_1^2(x) f^1(x)\right) dx\right)$$

$$+ O\left(\int \left(\hat{f}_g^0(x) - f^0(x)\right) \cdot \left(\hat{f}_g^0(x)^4 - f^0(x)^4\right) + \left(\hat{f}_g^0(x) - f^0(x)\right)^2 dx\right)$$

$$
\begin{aligned}
&+ O\left(\int \left(\hat{f}_g^0(x) - f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^3 - f^0(x)^3\right)\,dx\right) + O\left(\left(\hat{f}_g^0(x)^5 - f^0(x)^5\right)^2\right)\\
&+ O\left(\int \left(\hat{f}_g^0(x)^4 - f^0(x)^4\right)^2 dx\right)\\
&+ O\left(\int \left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right)\cdot\left(\hat{f}_g^1(x) - f^1(x)\right)dx\right)\\
&+ O\left(\int \left(\hat{\Psi}_{1,g}(x)^2 - \Psi_1(x)^2\right)\cdot\left(\hat{f}_g^1(x) - f^1(x)\right)dx\right)\\
&+ O\left(\int \left(\hat{\Psi}_{1,g}(x)^2 - \Psi_1(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right)dx\right)\\
&+ O\left(\int \left(\left(\hat{f}_g^0\right)'(x)^2 - \left(f^0\right)'(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)^2 - \hat{\Psi}_{1,g}(x)^2\right)dx\right)\\
&+ O\left(\int \left(\hat{\Psi}_{1,g}(x)^2 - \Psi_1(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right)\cdot\left(\hat{f}_g^1(x) - f^1(x)\right)dx\right)\\
&+ O\left(\int \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right)^2 dx\right)\\
&+ O\left(\int \left(\left(\hat{f}_g^0\right)'(x)^2 - \left(f^0\right)'(x)^2\right)\cdot\left(\hat{f}_g^1(x) - f^1(x)\right)dx\right)\\
&+ O\left(\int \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right)^2 dx\right)\\
&+ O\left(\int \left(\left(\hat{f}_g^0\right)'(x)^2 - \left(f^0\right)'(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right)dx\right)\\
&+ O\left(\int \left(\left(\hat{f}_g^0\right)'(x)^2 - \left(f^0\right)'(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right)\right.\\
&\left.\cdot\left(\hat{f}_g^1(x) - f^1(x)\right)dx\right) + O\left(\int \left(\left(\hat{f}_g^0\right)'(x)^2 - \left(f^0\right)'(x)^2\right)\right.\\
&\left.\cdot\left(\hat{\Psi}_{1,g}(x)^2 - \hat{\Psi}_{1,g}(x)^2\right)\cdot\left(\hat{f}_g^1(x) - f^1(x)\right)dx\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ O\left(\int \left(\left(\hat{f}_g^0\right)'(x)^2 - \left(f^0\right)'(x)^2\right) \cdot \left(\hat{\Psi}_{1,g}(x)^2 - \hat{\Psi}_{1,g}(x)^2\right)\right.\\
&\left.\cdot \left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right) dx\right) + O\left(\int \left(\left(\hat{f}_g^0\right)'(x)^2 - \left(f^0\right)'(x)^2\right)\right.\\
&\left.\cdot \left(\hat{\Psi}_{1,g}(x)^2 - \hat{\Psi}_{1,g}(x)^2\right) \cdot \left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right)\\
&+ O\left(\int \left(\hat{f}_g^0(x)^4 - f^0(x)^4\right)\right.\\
&\left.\cdot \left(\left(\hat{f}_g^0\right)'(x)^2 \hat{\Psi}_{1,g}(x)^2 \left(\hat{f}_g^0\right)''(x) \hat{f}_g^1(x) - \left(f^0\right)'(x)^2 \Psi_1(x)^2 \left(f^0\right)''(x) f^1(x)\right) dx\right)\\
&+ O\left(\int \left(\hat{f}_g^0(x) - f^0(x)\right)^2 dx + \int \left(\hat{f}_g^0(x) - f^0(x)\right) \cdot \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right) dx\right.\\
&\left.+ \int \left(\hat{f}_g^0(x) - f^0(x)\right) \cdot \left(\hat{f}_g^0(x)^3 - f^0(x)^3\right) dx\right)\\
&+ O\left(\int \left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right)\\
&+ O\left(\int \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right)^2 dx + \int \left(\hat{f}_g^0\right)'(x)^2 - \left(f^0\right)'(x)^2 dx\right)\\
&+ O\left(\int \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right)\\
&+ O\left(\int \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right) dx\right)\\
&+ O\left(\int \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right)\\
&+ O\left(\int \left(\hat{f}_g^0(x)^4 - f^0(x)^4\right)^2 dx\right) + O\left(\int \left(\left(\hat{f}_g^0\right)'(x)^3 - \left(f^0\right)'(x)^3\right)\right.\\
&\left.\cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx + \int \left(\left(\hat{f}_g^0\right)'(x)^3 - \left(f^0\right)'(x)^3\right) \cdot \left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right) dx\right)\\
&+ O\left(\int \left(\left(\hat{f}_g^0\right)'(x)^3 - \left(f^0\right)'(x)^3\right) \cdot \left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right)\\
&+ O\left(\int \left(\left(\hat{f}_g^0\right)'(x)^3 - \left(f^0\right)'(x)^3\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) dx\right)\\
&+ O\left(\int \left(\hat{f}_g^0(x) - f^0(x)\right)^2 dx\right)\\
&+ O\left(\int \left(\left(\hat{f}_g^0\right)'(x)^3 - \left(f^0\right)'(x)^3\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right)\\
&+ O\left(\int \left(\left(\hat{f}_g^0\right)'(x)^3 - \left(f^0\right)'(x)^3\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right)\right.\\
&\left.\cdot \left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right) dx\right)\\
&+ O\left(\int \left(\left(\hat{f}_g^0\right)'(x)^3 - \left(f^0\right)'(x)^3\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right)\right.\\
&\left.\cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx\right) + O\left(\int \left(\hat{f}_g^0(x)^4 - f^0(x)^4\right)\right.\\
&\left.\cdot \left(\left(\hat{f}_g^0\right)'(x)^3 \hat{\Psi}_{1,g}(x) \hat{\Psi}_{1,g}'(x) \hat{f}_g^1(x) - \left(f^0\right)'(x)^3 \Psi_1(x) \Psi_1'(x) f^1(x)\right) dx\right)\\
&+ O\left(\int \left(\hat{f}_g^0(x) - f^0(x)\right)\right.\\
&\left.\cdot \left(\hat{m}_g'(x) \hat{m}_g''(x) \left(\hat{f}_g^0\right)'(x) \hat{f}_g^1(x) - m'(x) m''(x) \left(f^0\right)'(x) f^1(x)\right) dx\right). \qquad (118)
\end{aligned}
$$

Finally, it remains to work out further calculations with term B_3. Considering expression (93), it turns out:

$$
\begin{aligned}
B_3 := 4 \int & \left[\frac{\hat{m}_g'(x)^2 \left(\hat{f}_g^0\right)'(x)^2 \hat{f}_g^1(x)}{\hat{f}_g^0(x)^2} - \frac{m'(x)^2 \left(f^0\right)'(x)^2 f^1(x)}{f^0(x)^2} \right] dx \\
&= 4 \int \frac{1}{f^0(x)^2} \left[\hat{m}_g'(x)^2 \left(\hat{f}_g^0\right)'(x)^2 \hat{f}_g^1(x) - m'(x)^2 \left(f^0\right)'(x)^2 f^1(x) \right] dx \\
&- 4 \int \frac{m'(x)^2 \left(f^0\right)'(x)^2 f^1(x)}{f^0(x)^4} \left[\hat{f}_g^0(x)^2 - f^0(x)^2 \right] dx \\
&+ O\left(\left(\hat{f}_g^0(x)^2 - f^0(x)^2 \right)^2 dx \right) + O\left(\int \left(\hat{f}_g^0(x)^2 - f^0(x)^2 \right) \right. \\
&\left. \cdot \left(\hat{m}_g'(x)^2 \left(\hat{f}_g^0\right)'(x)^2 \hat{f}_g^1(x) - m'(x)^2 \left(f^0\right)'(x)^2 f^1(x) \right) dx \right) \\
&= 4 \int \frac{1}{f^0(x)^2} \left[\hat{m}_g'(x)^2 \left(\hat{f}_g^0\right)'(x)^2 \hat{f}_g^1(x) - m'(x)^2 \left(f^0\right)'(x)^2 f^1(x) \right] dx \\
&- 8 \int \frac{m'(x)^2 \left(f^0\right)'(x)^2 f^1(x)}{f^0(x)^3} \left[\hat{f}_g^0(x) - f^0(x) \right] dx \\
&+ O\left(\left(\hat{f}_g^0(x)^2 - f^0(x)^2 \right)^2 dx \right) + O\left(\int \left(\hat{f}_g^0(x) - f^0(x) \right)^2 dx \right) \\
&+ O\left(\int \left(\hat{f}_g^0(x)^2 - f^0(x)^2 \right) \left(\hat{m}_g'(x)^2 \left(\hat{f}_g^0\right)'(x)^2 \hat{f}_g^1(x) - m'(x)^2 \left(f^0\right)'(x)^2 f^1(x) \right) dx \right) \\
&= 4 \int \frac{1}{f^0(x)^2} \left[\hat{m}_g'(x)^2 \left(\left(\hat{f}_g^0\right)'(x)^2 \hat{f}_g^1(x) + \left(f^0\right)'(x)^2 f^1(x) - \left(f^0\right)'(x)^2 f^1(x) \right) \right. \\
&\left. - m'(x)^2 \left(f^0\right)'(x)^2 f^1(x) \right] dx - 8 \int \frac{m'(x)^2 \left(f^0\right)'(x)^2 f^1(x)}{f^0(x)^3} \left[\hat{f}_g^0(x) - f^0(x) \right] dx \\
&+ O\left(\left(\hat{f}_g^0(x)^2 - f^0(x)^2 \right)^2 dx \right) + O\left(\int \left(\hat{f}_g^0(x) - f^0(x) \right)^2 dx \right) \\
&+ O\left(\int \left(\hat{f}_g^0(x)^2 - f^0(x)^2 \right) \left(\hat{m}_g'(x)^2 \left(\hat{f}_g^0\right)'(x)^2 \hat{f}_g^1(x) - m'(x)^2 \left(f^0\right)'(x)^2 f^1(x) \right) dx \right) \\
&= 4 \int \frac{\left(f^0\right)'(x)^2 f^1(x)}{f^0(x)^2} \left[\hat{m}_g'(x)^2 - m'(x)^2 \right] dx \\
&+ 4 \int \frac{1}{f^0(x)^2} \left[\hat{m}_g'(x)^2 \cdot \left(\left(\hat{f}_g^0\right)'(x)^2 \hat{f}_g^1(x) - \left(f^0\right)'(x)^2 f^1(x) \right) \right] dx \\
&- 8 \int \frac{m'(x)^2 \left(f^0\right)'(x)^2 f^1(x)}{f^0(x)^3} \left[\hat{f}_g^0(x) - f^0(x) \right] dx \\
&+ O\left(\left(\hat{f}_g^0(x)^2 - f^0(x)^2 \right)^2 dx \right) + O\left(\int \left(\hat{f}_g^0(x) - f^0(x) \right)^2 dx \right) \\
&+ O\left(\int \left(\hat{f}_g^0(x)^2 - f^0(x)^2 \right) \left(\hat{m}_g'(x)^2 \left(\hat{f}_g^0\right)'(x)^2 \hat{f}_g^1(x) - m'(x)^2 \left(f^0\right)'(x)^2 f^1(x) \right) dx \right)
\end{aligned}
$$

$$
\begin{aligned}
&= 8 \int \frac{\left(f^0\right)^{'}(x)^2 f^1(x) m^{'}(x)}{f^0(x)^2} \left[\hat{m}_g^{'}(x) - m^{'}(x)\right] dx \\
&+ 4 \int \frac{m^{'}(x)^2}{f^0(x)^2} \left[\left(\hat{f}_g^0\right)^{'}(x)^2 \hat{f}_g^1(x) - \left(f^0\right)^{'}(x)^2 f^1(x)\right] dx \\
&+ 4 \int \frac{1}{f^0(x)^2} \left[\left(\hat{m}_g^{'}(x)^2 - m^{'}(x)^2\right) \cdot \left(\left(\hat{f}_g^0\right)^{'}(x)^2 \hat{f}_g^1(x) - \left(f^0\right)^{'}(x)^2 f^1(x)\right)\right] dx \\
&- 8 \int \frac{m^{'}(x)^2 \left(f^0\right)^{'}(x)^2 f^1(x)}{f^0(x)^3} \left[\hat{f}_g^0(x) - f^0(x)\right] dx \\
&+ O\left(\left(\hat{f}_g^0(x)^2 - f^0(x)^2\right)^2 dx\right) + O\left(\int \left(\hat{f}_g^0(x) - f^0(x)\right)^2 dx\right) \\
&+ O\left(\int \left(\hat{m}_g^{'}(x) - m^{'}(x)\right)^2 dx\right) + O\left(\int \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right)\right. \\
&\left.\cdot \left(\hat{m}_g^{'}(x)^2 \left(\hat{f}_g^0\right)^{'}(x)^2 \hat{f}_g^1(x) - m^{'}(x)^2 \left(f^0\right)^{'}(x)^2 f^1(x)\right) dx\right) \\
&= 8 \int \frac{\left(f^0\right)^{'}(x)^2 f^1(x) m^{'}(x)}{f^0(x)^2} \left[\frac{\hat{\Psi}_{1,g}^{'}(x)}{\hat{f}_g^0(x)} - \frac{\Psi_1^{'}(x)}{f^0(x)}\right] dx \\
&- 8 \int \frac{\left(f^0\right)^{'}(x)^2 f^1(x) m^{'}(x)}{f^0(x)^2} \left[\frac{\left(\hat{f}_g^0\right)^{'}(x) \hat{\Psi}_{1,g}(x)}{\hat{f}_g^0(x)^2} - \frac{\left(f^0\right)^{'}(x) \Psi_1(x)}{f^0(x)^2}\right] dx
\end{aligned}
$$

$$+4\int \frac{m'(x)^2 \left(f^0\right)'(x)^2}{f^0(x)^2}\left[\hat{f}_g^1(x)-f^1(x)\right]dx$$

$$+4\int \frac{m'(x)^2}{f^0(x)^2}\left[\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot \hat{f}_g^1(x)\right]dx$$

$$+4\int \frac{\left(f^0\right)'(x)^2}{f^0(x)^2}\left[\left(\hat{m}_g'(x)^2-m'(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx+4\int \frac{1}{f^0(x)^2}$$

$$\left[\left(\hat{m}_g'(x)^2-m'(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\hat{f}_g^1(x)\right]dx$$

$$-8\int \frac{m'(x)^2\left(f^0\right)'(x)^2 f^1(x)}{f^0(x)^3}\left[\hat{f}_g^0(x)-f^0(x)\right]dx$$

$$+O\left(\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)^2 dx\right)+O\left(\int\left(\hat{f}_g^0(x)-f^0(x)\right)^2 dx\right)$$

$$+O\left(\int\left(\hat{m}_g'(x)-m'(x)\right)^2 dx\right)+O\left(\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)\right.$$

$$\left.\cdot\left(\hat{m}_g'(x)^2\left(\hat{f}_g^0\right)'(x)^2\hat{f}_g^1(x)-m'(x)^2\left(f^0\right)'(x)^2 f^1(x)\right)dx\right)$$

$$=8\int \frac{\left(f^0\right)'(x)^2 f^1(x)m'(x)}{f^0(x)^3}\left[\hat{\Psi}_{1,g}'(x)-\Psi_1'(x)\right]dx$$

$$-8\int \frac{\left(f^0\right)'(x)^2 f^1(x)m'(x)\Psi_1'(x)}{f^0(x)^4}\left[\hat{f}_g^0(x)-f^0(x)\right]dx$$

$$-8\int \frac{\left(f^0\right)'(x)^2 f^1(x)m'(x)}{f^0(x)^4}\left[\left(\hat{f}_g^0\right)'(x)\hat{\Psi}_{1,g}(x)-\left(f^0\right)'(x)\Psi_1(x)\right]dx$$

$$-8\int \frac{\left(f^0\right)'(x)^2 f^1(x)m'(x)\left(f^0\right)'(x)\Psi_1(x)}{f^0(x)^6}\left[\hat{f}_g^0(x)^2-f^0(x)^2\right]dx$$

$$+4\int \frac{m'(x)^2\left(f^0\right)'(x)^2}{f^0(x)^2}\left[\hat{f}_g^1(x)-f^1(x)\right]dx$$

$$+4\int \frac{m'(x)^2 f^1(x)}{f^0(x)^2}\left[\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right]dx$$

$$+4\int \frac{m'(x)^2}{f^0(x)^2}\left[\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx$$

$$+4\int \frac{\left(f^0\right)'(x)^2}{f^0(x)^2}\left[\left(\hat{m}_g'(x)^2-m'(x)^2\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx$$

$$+4\int \frac{f^1(x)}{f^0(x)^2}\left[\left(\hat{m}_g'(x)^2-m'(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\right]dx$$

$$+4\int \frac{1}{f^0(x)^2}\left[\left(\hat{m}_g'(x)^2-m'(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)\right.$$

$$\left.\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)\right]dx-8\int \frac{m'(x)^2\left(f^0\right)'(x)^2 f^1(x)}{f^0(x)^3}\left[\hat{f}_g^0(x)-f^0(x)\right]dx$$

$$+ O\left(\left(\hat{f}_g^0(x)^2 - f^0(x)^2\right)^2 dx\right) + O\left(\int \left(\hat{f}_g^0(x) - f^0(x)\right)^2 dx\right)$$
$$+ O\left(\int \left(\hat{m}_g'(x) - m'(x)\right)^2 dx\right) + O\left(\int \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right)\right.$$
$$\left.\cdot\left(\hat{m}_g'(x)^2 \left(\hat{f}_g^0\right)'(x)^2 \hat{f}_g^1(x) - m'(x)^2 \left(f^0\right)'(x)^2 f^1(x)\right) dx\right)$$
$$+ O\left(\int \left(\hat{f}_g^0(x) - f^0(x)\right)\cdot\left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right) dx\right)$$
$$+ O\left(\int \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)\hat{\Psi}_{1,g}(x) - \left(f^0\right)'(x)\Psi_1(x)\right) dx\right)$$
$$= 8\int \frac{\left(f^0\right)'(x)^2 f^1(x) m'(x)}{f^0(x)^3}\left[\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right] dx$$
$$- 8\int \frac{\left(f^0\right)'(x)^2 f^1(x) m'(x)\Psi_1'(x)}{f^0(x)^4}\left[\hat{f}_g^0(x) - f^0(x)\right] dx$$
$$- 8\int \frac{\left(f^0\right)'(x)^2 f^1(x) m'(x)\left(f^0\right)'(x)}{f^0(x)^4}\left[\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right] dx$$
$$- 16\int \frac{\left(f^0\right)'(x)^2 f^1(x) m'(x)\left(f^0\right)'(x)\Psi_1(x)}{f^0(x)^4}\left[\hat{f}_g^0(x) - f^0(x)\right] dx$$
$$+ 4\int \frac{m'(x)^2\left(f^0\right)'(x)^2}{f^0(x)^2}\left[\hat{f}_g^1(x) - f^1(x)\right] dx$$
$$+ 8\int \frac{m'(x)^2 f^1(x)\left(f^0\right)'(x)}{f^0(x)^2}\left[\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right] dx$$
$$- 8\int \frac{\left(f^0\right)'(x)^2 f^1(x) m'(x)}{f^0(x)^4}\left[\left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right)\cdot\hat{\Psi}_{1,g}(x)\right] dx$$
$$+ 4\int \frac{m'(x)^2}{f^0(x)^2}\left[\left(\left(\hat{f}_g^0\right)'(x)^2 - \left(f^0\right)'(x)^2\right)\cdot\left(\hat{f}_g^1(x) - f^1(x)\right)\right] dx$$
$$+ 4\int \frac{\left(f^0\right)'(x)^2}{f^0(x)^2}\left[\left(\hat{m}_g'(x)^2 - m'(x)^2\right)\cdot\left(\hat{f}_g^1(x) - f^1(x)\right)\right] dx$$
$$+ 4\int \frac{f^1(x)}{f^0(x)^2}\left[\left(\hat{m}_g'(x)^2 - m'(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)^2 - \left(f^0\right)'(x)^2\right)\right] dx$$
$$+ 4\int \frac{1}{f^0(x)^2}\left[\left(\hat{m}_g'(x)^2 - m'(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)^2 - \left(f^0\right)'(x)^2\right)\right.$$
$$\left.\cdot\left(\hat{f}_g^1(x) - f^1(x)\right)\right] dx - 8\int \frac{m'(x)^2\left(f^0\right)'(x)^2 f^1(x)}{f^0(x)^3}\left[\hat{f}_g^0(x) - f^0(x)\right] dx$$
$$+ O\left(\int \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right)^2 dx\right) + O\left(\int \left(\hat{f}_g^0(x) - f^0(x)\right)^2 dx\right)$$
$$+ O\left(\int \left(\hat{m}_g'(x) - m'(x)\right)^2 dx\right) + O\left(\int \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right)\right.$$
$$\left.\cdot\left(\hat{m}_g'(x)^2 \left(\hat{f}_g^0\right)'(x)^2 \hat{f}_g^1(x) - m'(x)^2 \left(f^0\right)'(x)^2 f^1(x)\right) dx\right)$$
$$+ O\left(\int \left(\hat{f}_g^0(x) - f^0(x)\right)\cdot\left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right) dx\right)$$
$$+ O\left(\int \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right)^2 dx\right)$$
$$+ O\left(\int \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)\hat{\Psi}_{1,g}(x) - \left(f^0\right)'(x)\Psi_1(x)\right) dx\right)$$

$$
\begin{aligned}
&= 8 \int \frac{\left(f^0\right)'(x)^2 f^1(x) m'(x)}{f^0(x)^3} \left[\hat{\Psi}'_{1,g}(x) - \Psi'_1(x)\right] dx \\
&- 8 \int \frac{\left(f^0\right)'(x)^2 f^1(x) m'(x) \Psi'_1(x)}{f^0(x)^4} \left[\hat{f}^0_g(x) - f^0(x)\right] dx \\
&- 8 \int \frac{\left(f^0\right)'(x)^2 f^1(x) m'(x) \left(f^0\right)'(x)}{f^0(x)^4} \left[\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right] dx \\
&- 8 \int \frac{m'(x)^2 \left(f^0\right)'(x)^2 f^1(x)}{f^0(x)^3} \left[\hat{f}^0_g(x) - f^0(x)\right] dx \\
&+ 4 \int \frac{m'(x)^2 \left(f^0\right)'(x)^2}{f^0(x)^2} \left[\hat{f}^1_g(x) - f^1(x)\right] dx \\
&+ 8 \int \frac{m'(x)^2 f^1(x) \left(f^0\right)'(x)}{f^0(x)^2} \left[\left(\hat{f}^0_g\right)'(x) - \left(f^0\right)'(x)\right] dx \\
&- 16 \int \frac{\left(f^0\right)'(x)^2 f^1(x) m'(x) \left(f^0\right)'(x) \Psi_1(x)}{f^0(x)^4} \left[\hat{f}^0_g(x) - f^0(x)\right] dx \\
&- 8 \int \frac{\left(f^0\right)'(x)^2 f^1(x) m'(x) \Psi_1(x)}{f^0(x)^4} \left[\left(\hat{f}^0_g\right)'(x) - \left(f^0\right)'(x)\right] dx \\
&- 8 \int \frac{\left(f^0\right)'(x)^2 f^1(x) m'(x)}{f^0(x)^4} \left[\left(\left(\hat{f}^0_g\right)'(x) - \left(f^0\right)'(x)\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right)\right] dx \\
&+ 4 \int \frac{m'(x)^2}{f^0(x)^2} \left[\left(\left(\hat{f}^0_g\right)'(x)^2 - \left(f^0\right)'(x)^2\right) \cdot \left(\hat{f}^1_g(x) - f^1(x)\right)\right] dx \\
&+ 4 \int \frac{\left(f^0\right)'(x)^2}{f^0(x)^2} \left[\left(\hat{m}'_g(x)^2 - m'(x)^2\right) \cdot \left(\hat{f}^1_g(x) - f^1(x)\right)\right] dx \\
&+ 4 \int \frac{f^1(x)}{f^0(x)^2} \left[\left(\hat{m}'_g(x)^2 - m'(x)^2\right) \cdot \left(\left(\hat{f}^0_g\right)'(x)^2 - \left(f^0\right)'(x)^2\right)\right] dx \\
&+ 4 \int \frac{1}{f^0(x)^2} \left[\left(\hat{m}'_g(x)^2 - m'(x)^2\right) \cdot \left(\left(\hat{f}^0_g\right)'(x)^2 - \left(f^0\right)'(x)^2\right)\right. \\
&\left. \cdot \left(\hat{f}^1_g(x) - f^1(x)\right)\right] dx + O\left(\int \left(\hat{f}^0_g(x)^2 - f^0(x)^2\right)^2 dx\right) \\
&+ O\left(\int \left(\hat{f}^0_g(x) - f^0(x)\right)^2 dx\right) + O\left(\int \left(\left(\hat{f}^0_g\right)'(x) - \left(f^0\right)'(x)\right)^2 dx\right) \\
&+ O\left(\int \left(\hat{m}'_g(x) - m'(x)\right)^2 dx\right) + O\left(\int \left(\hat{f}^0_g(x)^2 - f^0(x)^2\right)\right. \\
&\left. \cdot \left(\hat{m}'_g(x)^2 \left(\hat{f}^0_g\right)'(x)^2 \hat{f}^1_g(x) - m'(x)^2 \left(f^0\right)'(x)^2 f^1(x)\right) dx\right) \\
&+ O\left(\int \left(\hat{f}^0_g(x) - f^0(x)\right) \cdot \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x)\right) dx\right) \\
&+ O\left(\int \left(\hat{f}^0_g(x)^2 - f^0(x)^2\right) \cdot \left(\left(\hat{f}^0_g\right)'(x) \hat{\Psi}_{1,g}(x) - \left(f^0\right)'(x) \Psi_1(x)\right) dx\right)
\end{aligned}
$$

$$
\begin{aligned}
&= 8 \int \frac{\left(f^0\right)'(x)^2 f^1(x) m'(x)}{f^0(x)^3} \left[\hat{\Psi}'_{1,g}(x) - \Psi'_1(x)\right] dx \\
&- 8 \int \frac{\left(f^0\right)'(x)^2 f^1(x) m'(x) \Psi'_1(x)}{f^0(x)^4} \left[\hat{f}^0_g(x) - f^0(x)\right] dx \\
&- 8 \int \frac{\left(f^0\right)'(x)^2 f^1(x) m'(x) \left(f^0\right)'(x)}{f^0(x)^4} \left[\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right] dx \\
&- 8 \int \frac{m'(x)^2 \left(f^0\right)'(x)^2 f^1(x)}{f^0(x)^3} \left[\hat{f}^0_g(x) - f^0(x)\right] dx \\
&+ 4 \int \frac{m'(x)^2 \left(f^0\right)'(x)^2}{f^0(x)^2} \left[\hat{f}^1_g(x) - f^1(x)\right] dx \\
&+ 8 \int \frac{m'(x)^2 f^1(x) \left(f^0\right)'(x)}{f^0(x)^2} \left[\left(\hat{f}^0_g\right)'(x) - \left(f^0\right)'(x)\right] dx \\
&- 16 \int \frac{\left(f^0\right)'(x)^2 f^1(x) m'(x) \left(f^0\right)'(x) \Psi_1(x)}{f^0(x)^4} \left[\hat{f}^0_g(x) - f^0(x)\right] dx \\
&- 8 \int \frac{\left(f^0\right)'(x)^2 f^1(x) m'(x) \Psi_1(x)}{f^0(x)^4} \left[\left(\hat{f}^0_g\right)'(x) - \left(f^0\right)'(x)\right] dx \\
&+ O\left(\int \left(\left(\hat{f}^0_g\right)'(x) - \left(f^0\right)'(x)\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) dx\right) \\
&+ O\left(\int \left(\left(\hat{f}^0_g\right)'(x)^2 - \left(f^0\right)'(x)^2\right) \cdot \left(\hat{f}^1_g(x) - f^1(x)\right) dx\right) \\
&+ O\left(\int \left(\hat{f}^0_g(x) - f^0(x)\right) \cdot \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x)\right) dx\right) \\
&+ O\left(\int \left(\hat{m}'_g(x)^2 - m'(x)^2\right) \cdot \left(\hat{f}^1_g(x) - f^1(x)\right) dx\right) \\
&+ O\left(\int \left(\hat{m}'_g(x)^2 - m'(x)^2\right) \cdot \left(\left(\hat{f}^0_g\right)'(x)^2 - \left(f^0\right)'(x)^2\right) dx\right) \\
&+ O\left(\int \left(\hat{m}'_g(x) - m'(x)\right)^2 dx\right) + O\left(\int \left(\hat{f}^0_g(x)^2 - f^0(x)^2\right)\right. \\
&\cdot \left.\left(\hat{m}'_g(x)^2 \left(\hat{f}^0_g\right)'(x)^2 \hat{f}^1_g(x) - m'(x)^2 \left(f^0\right)'(x)^2 f^1(x)\right) dx\right) \\
&+ O\left(\int \left(\hat{f}^0_g(x)^2 - f^0(x)^2\right) \cdot \left(\left(\hat{f}^0_g\right)'(x) \hat{\Psi}_{1,g}(x) - \left(f^0\right)'(x) \Psi_1(x)\right) dx\right) \\
&+ O\left(\left(\hat{f}^0_g(x)^2 - f^0(x)^2\right)^2\right) + O\left(\int \left(\hat{f}^0_g(x) - f^0(x)\right)^2 dx\right) \\
&+ O\left(\int \left(\left(\hat{f}^0_g\right)'(x) - \left(f^0\right)'(x)\right)^2 dx\right) + O\left(\int \left(\hat{m}'_g(x)^2 - m'(x)^2\right)\right. \\
&\cdot \left.\left(\left(\hat{f}^0_g\right)'(x)^2 - \left(f^0\right)'(x)^2\right) \cdot \left(\hat{f}^1_g(x) - f^1(x)\right) dx\right). \qquad (119)
\end{aligned}
$$

Bringing together terms (101), (118) and (119), and afterwards plugging them into (98), Lemma 4 is proven.

Furthermore, term r_{1,n_0} in Lemma 4 is given by:

$$
\begin{aligned}
r_{1,n_0} = & \int \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right)^2 dx + \int \left(\hat{f}_g^0(x)^4 - f^0(x)^4\right)^2 dx \\
& + \int \left(\hat{f}_g^0(x)^3 - f^0(x)^3\right)^2 dx + \int \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right) \\
& \cdot \left(\hat{\Psi}'_{1,g}(x)\hat{\Psi}''_{1,g}(x)\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x) - \Psi'_1(x)\Psi''_1(x)\left(f^0\right)'(x)f^1(x)\right) dx \\
& + \int \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx \\
& + \int \left(\hat{\Psi}''_{1,g}(x) - \Psi''_1(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx \\
& + \int \left(\hat{\Psi}''_{1,g}(x) - \Psi''_1(x)\right) \cdot \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right) dx \\
& + \int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx \\
& + \int \left(\hat{\Psi}''_{1,g}(x) - \Psi''_1(x)\right) \cdot \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx \\
& + \int \left(\hat{f}_g^0(x) - f^0(x)\right)^2 dx \\
& + \int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x)\right) \cdot \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right) dx \\
& + \int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x)\right) \cdot \left(\hat{\Psi}''_{1,g}(x) - \Psi''_1(x)\right) dx \\
& + \int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x)\right) \cdot \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx \\
& + \int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x)\right) \cdot \left(\hat{\Psi}''_{1,g}(x) - \Psi''_1(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx \\
& + \int \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right) dx \\
& + \int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x)\right) \cdot \left(\hat{\Psi}''_{1,g}(x) - \Psi''_1(x)\right) \cdot \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right) dx \\
& + \int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x)\right) \cdot \left(\hat{\Psi}''_{1,g}(x) - \Psi''_1(x)\right) \cdot \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right) \\
& \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx + \int \left(\hat{f}_g^0(x)^4 - f^0(x)^4\right) \\
& \cdot \left(\hat{\Psi}'_{1,g}(x)\left(\hat{f}_g^0\right)'(x)^3\hat{\Psi}_{1,g}(x)\hat{f}_g^1(x) - \Psi'_1(x)\left(f^0\right)'(x)^3\Psi_1(x)f^1(x)\right) dx \\
& + \int \left(\hat{f}_g^0(x) - f^0(x)\right) \cdot \left(\hat{f}_g^0(x)^4 - f^0(x)^4\right) + \left(\hat{f}_g^0(x) - f^0(x)\right)^2 dx \\
& + \int \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right)^2 dx + \int \left(\left(\hat{f}_g^0\right)'(x)^2 - \left(f^0\right)'(x)^2\right) dx \\
& + \int \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx \\
& + \int \left(\left(\hat{f}_g^0\right)'(x)^3 - \left(f^0\right)'(x)^3\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx
\end{aligned}
$$

$$+ \int \left(\left(\hat{f}_g^0 \right)' (x)^3 - \left(f^0 \right)' (x)^3 \right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x) \right) dx$$
$$+ \int \left(\left(\hat{f}_g^0 \right)' (x)^3 - \left(f^0 \right)' (x)^3 \right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x) \right) \cdot \left(\hat{f}_g^1(x) - f^1(x) \right) dx$$
$$+ \int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x) \right) \cdot \left(\hat{f}_g^1(x) - f^1(x) \right) dx$$
$$+ \int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x) \right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x) \right) dx$$
$$+ \int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x) \right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x) \right) \cdot \left(\hat{f}_g^1(x) - f^1(x) \right) dx$$
$$+ \int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x) \right) \cdot \left(\left(\hat{f}_g^0 \right)' (x)^3 - \left(f^0 \right)' (x)^3 \right) dx$$
$$+ \int \left(\hat{f}_g^0(x) - f^0(x) \right) \cdot \left(\hat{f}_g^0(x)^3 - f^0(x)^3 \right) dx$$
$$+ \int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x) \right) \cdot \left(\left(\hat{f}_g^0 \right)' (x)^3 - \left(f^0 \right)' (x)^3 \right) \cdot \left(\hat{f}_g^1(x) - f^1(x) \right) dx$$
$$+ \int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x) \right) \cdot \left(\left(\hat{f}_g^0 \right)' (x)^3 - \left(f^0 \right)' (x)^3 \right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x) \right) dx$$
$$+ \int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x) \right) \cdot \left(\left(\hat{f}_g^0 \right)' (x)^3 - \left(f^0 \right)' (x)^3 \right)$$
$$\cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x) \right) \cdot \left(\hat{f}_g^1(x) - f^1(x) \right) dx + \int \left(\hat{f}_g^0(x)^3 - f^0(x)^3 \right)$$
$$\cdot \left(\hat{\Psi}'_{1,g}(x) \left(\hat{f}_g^0 \right)'' (x) \hat{\Psi}_{1,g}(x) \left(\hat{f}_g^0 \right)' (x) \hat{f}_g^1(x) \right.$$
$$\left. - \Psi'_1(x) \left(f^0 \right)'' (x) \Psi_1(x) \left(f^0 \right)' (x) f^1(x) \right) dx$$
$$+ \int \left(\left(\hat{f}_g^0 \right)' (x) - \left(f^0 \right)' (x) \right) \cdot \left(\hat{f}_g^1(x) - f^1(x) \right) dx$$

$$
\begin{aligned}
&+\int \left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\\
&+\int \left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)dx\\
&+\int \left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\\
&+\int \left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\\
&+\int \left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)dx\\
&+\int \left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\\
&+\int \left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)dx\\
&+\int \left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\\
&+\int \left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)dx\\
&+\int \left(\left(\hat{f}_g^0\right)''(x)-\left(f^0\right)''(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\\
&\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx+\int \left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\\
&+\int \left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)dx\\
&+\int \left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx\\
&+\int \left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)dx+\int \left(\hat{f}_g^0(x)^3-f^0(x)^3\right)^2 dx\\
&+\int \left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)dx\\
&+\int \left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)
\end{aligned}
$$

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\begin{aligned}
&\cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx + \int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x)\right) \cdot \left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right) dx \\
&+ \int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x)\right) \cdot \left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx \\
&+ \int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x)\right) \cdot \left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right) \cdot \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right) dx \\
&+ \int \left(\hat{f}_g^0(x) - f^0(x)\right)^2 dx + \int \left(\hat{f}_g^0(x) - f^0(x)\right) \cdot \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right) dx \\
&+ \int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x)\right) \cdot \left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right) \cdot \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right) \\
&\cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx + \int \left(\hat{f}_g^0(x)^3 - f^0(x)^3\right)^2 dx \\
&+ \int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x)\right) \cdot \left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) dx \\
&+ \int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x)\right) \cdot \left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \\
&\cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx + \int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x)\right) \cdot \left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right) \\
&\cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right) dx \\
&+ \int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x)\right) \cdot \left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \\
&\cdot \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx \\
&+ \int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x)\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx \\
&+ \int \left(\left(\hat{f}_g^0\right)'(x)^2 - \left(f^0\right)'(x)^2\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx
\end{aligned}
$$

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\begin{aligned}
&+ \int \left(\hat{\Psi}'_{1,g}(x)^2 - \Psi'_1(x)^2 \right) \cdot \left(\hat{f}^1_g(x) - f^1(x) \right) dx \\
&+ \int \left(\hat{\Psi}'_{1,g}(x)^2 - \Psi'_1(x)^2 \right) \cdot \left(\left(\hat{f}^0_g \right)'(x)^2 - \left(f^0 \right)'(x)^2 \right) dx \\
&+ \int \left(\hat{\Psi}'_{1,g}(x)^2 - \Psi'_1(x)^2 \right) \cdot \left(\left(\hat{f}^0_g \right)'(x)^2 - \left(f^0 \right)'(x)^2 \right) \cdot \left(\hat{f}^1_g(x) - f^1(x) \right) dx \\
&+ \int \left(\hat{\Psi}'_{1,g}(x) - \Psi'_1(x) \right)^2 dx + \int \left(\hat{f}^0_g(x)^3 - f^0(x)^3 \right) \\
&\cdot \left(\hat{\Psi}'_{1,g}(x)^2 \left(\hat{f}^0_g \right)'(x)^2 \hat{f}^1_g(x) - \Psi'_1(x)^2 \left(f^0 \right)'(x)^2 f^1(x) \right) dx \\
&+ \int \left(\hat{f}^0_g(x) - f^0(x) \right)^2 dx + \int \left(\hat{f}^0_g(x)^2 - f^0(x)^2 \right) dx \\
&+ \int \left(\left(\hat{f}^0_g \right)'(x) - \left(f^0 \right)'(x) \right)^2 dx \\
&+ \int \left(\hat{\Psi}''_{1,g}(x) - \Psi''_1(x) \right) \cdot \left(\hat{f}^1_g(x) - f^1(x) \right) dx + \int \left(\left(\hat{f}^0_g \right)'(x) - \left(f^0 \right)'(x) \right)^2 dx \\
&+ \int \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x) \right) \cdot \left(\hat{f}^1_g(x) - f^1(x) \right) dx \\
&+ \int \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x) \right) \cdot \left(\hat{\Psi}''_{1,g}(x) - \Psi''_1(x) \right) dx \\
&+ \int \left(\left(\hat{f}^0_g \right)'(x)^2 - \left(f^0 \right)'(x)^2 \right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x) \right) dx \\
&+ \int \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x) \right) \cdot \left(\hat{\Psi}''_{1,g}(x) - \Psi''_1(x) \right) \cdot \left(\hat{f}^1_g(x) - f^1(x) \right) dx \\
&+ \int \left(\left(\hat{f}^0_g \right)'(x)^2 - \left(f^0 \right)'(x)^2 \right) \cdot \left(\hat{f}^1_g(x) - f^1(x) \right) dx \\
&+ \int \left(\left(\hat{f}^0_g \right)'(x)^2 - \left(f^0 \right)'(x)^2 \right) \cdot \left(\hat{\Psi}''_{1,g}(x) - \Psi''_1(x) \right) dx \\
&+ \int \left(\left(\hat{f}^0_g \right)'(x)^2 - \left(f^0 \right)'(x)^2 \right) \cdot \left(\hat{\Psi}''_{1,g}(x) - \Psi''_1(x) \right) \cdot \left(\hat{f}^1_g(x) - f^1(x) \right) dx \\
&+ \int \left(\left(\hat{f}^0_g \right)'(x)^2 - \left(f^0 \right)'(x)^2 \right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x) \right) \cdot \left(\hat{f}^1_g(x) - f^1(x) \right) dx
\end{aligned}
$$

$$
\begin{aligned}
&+ \int \left(\left(\hat{f}_g^0\right)'(x)^2 - \left(f^0\right)'(x)^2 \right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x) \right) \cdot \left(\hat{\Psi}''_{1,g}(x) - \Psi''_1(x) \right) dx \\
&+ \int \left(\left(\hat{f}_g^0\right)'(x)^2 - \left(f^0\right)'(x)^2 \right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x) \right) \cdot \left(\hat{\Psi}''_{1,g}(x) - \Psi''_1(x) \right) \\
&\cdot \left(\hat{f}_g^1(x) - f^1(x) \right) dx + \int \left(\hat{f}_g^0(x)^3 - f^0(x)^3 \right) + \int \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x) \right)^2 dx \\
&\cdot \left(\left(\hat{f}_g^0\right)'(x)^2 \hat{\Psi}_{1,g}(x) \hat{\Psi}''_{1,g}(x) \hat{f}_g^1(x) - \left(f^0\right)'(x)^2 \Psi_1(x) \Psi''_1(x) f^1(x) \right) dx \\
&+ \int \left(\hat{f}_g^0(x) - f^0(x) \right)^2 dx + \int \left(\hat{f}_g^0(x)^2 - f^0(x)^2 \right) dx \\
&+ \int \left(\hat{f}_g^0(x) - f^0(x) \right)^2 dx + \int \left(\hat{f}_g^0(x) - f^0(x) \right) \cdot \left(\hat{f}_g^0(x)^2 - f^0(x)^2 \right) dx \\
&+ \int \left(\hat{f}_g^0(x) - f^0(x) \right) \cdot \left(\hat{f}_g^0(x)^3 - f^0(x)^3 \right) dx \\
&+ \int \left(\hat{\Psi}^2_{1,g}(x) - \Psi_1^2(x) \right) \cdot \left(\hat{f}_g^1(x) - f^1(x) \right) dx \\
&+ \int \left(\left(\hat{f}_g^0\right)'(x)^4 - \left(f^0\right)'(x)^4 \right) \cdot \left(\hat{f}_g^1(x) - f^1(x) \right) dx \\
&+ \int \left(\left(\hat{f}_g^0\right)'(x)^4 - \left(f^0\right)'(x)^4 \right) \cdot \left(\hat{\Psi}^2_{1,g}(x) - \Psi_1^2(x) \right) dx \\
&+ \int \left(\left(\hat{f}_g^0\right)'(x)^4 - \left(f^0\right)'(x)^4 \right) \cdot \left(\hat{\Psi}^2_{1,g}(x) - \Psi_1^2(x) \right) \cdot \left(\hat{f}_g^1(x) - f^1(x) \right) dx \\
&+ \int \left(\hat{f}_g^0(x)^5 - f^0(x)^5 \right) \cdot \left(\left(\hat{f}_g^0\right)'(x)^4 \hat{\Psi}^2_{1,g}(x) \hat{f}_g^1(x) - \left(f^0\right)'(x)^4 \Psi_1^2(x) f^1(x) \right) dx \\
&+ \int \left(\hat{f}_g^0(x) - f^0(x) \right) \cdot \left(\hat{f}_g^0(x)^4 - f^0(x)^4 \right) + \int \left(\hat{f}_g^0(x) - f^0(x) \right)^2 dx \\
&+ \int \left(\hat{f}_g^0(x) - f^0(x) \right) \cdot \left(\hat{f}_g^0(x)^3 - f^0(x)^3 \right) dx + \int \left(\hat{f}_g^0(x)^5 - f^0(x)^5 \right)^2 dx \\
&+ \int \left(\hat{f}_g^0(x)^4 - f^0(x)^4 \right)^2 dx + \int \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x) \right)^2 dx \\
&+ \int \left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x) \right) \cdot \left(\hat{f}_g^1(x) - f^1(x) \right) dx \\
&+ \int \left(\hat{\Psi}_{1,g}(x)^2 - \Psi_1(x)^2 \right) \cdot \left(\hat{f}_g^1(x) - f^1(x) \right) dx
\end{aligned}
$$

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\begin{aligned}
&+\int \left(\hat{\Psi}_{1,g}(x)^2 - \Psi_1(x)^2\right) \cdot \left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right) dx \\
&+\int \left(\left(\hat{f}_g^0\right)'(x)^2 - \left(f^0\right)'(x)^2\right) \cdot \left(\hat{\Psi}_{1,g}(x)^2 - \hat{\Psi}_{1,g}(x)^2\right) dx \\
&+\int \left(\hat{\Psi}_{1,g}(x)^2 - \Psi_1(x)^2\right) \cdot \left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx \\
&+\int \left(\left(\hat{f}_g^0\right)'(x)^2 - \left(f^0\right)'(x)^2\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx \\
&+\int \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right)^2 dx + \int \left(\hat{f}_g^0(x)^3 - f^0(x)^3\right)^2 dx \\
&+\int \left(\left(\hat{f}_g^0\right)'(x)^2 - \left(f^0\right)'(x)^2\right) \cdot \left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right) dx \\
&+\int \left(\left(\hat{f}_g^0\right)'(x)^2 - \left(f^0\right)'(x)^2\right) \cdot \left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx \\
&+\int \left(\left(\hat{f}_g^0\right)'(x)^2 - \left(f^0\right)'(x)^2\right) \cdot \left(\hat{\Psi}_{1,g}(x)^2 - \hat{\Psi}_{1,g}(x)^2\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx \\
&+\int \left(\left(\hat{f}_g^0\right)'(x)^2 - \left(f^0\right)'(x)^2\right) \cdot \left(\hat{\Psi}_{1,g}(x)^2 - \hat{\Psi}_{1,g}(x)^2\right) \\
&\cdot \left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right) dx + \int \left(\left(\hat{f}_g^0\right)'(x)^2 - \left(f^0\right)'(x)^2\right) \\
&\cdot \left(\hat{\Psi}_{1,g}(x)^2 - \hat{\Psi}_{1,g}(x)^2\right) \cdot \left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx \\
&+\int \left(\hat{f}_g^0(x)^4 - f^0(x)^4\right) dx + \int \left(\hat{f}_g^0(x)^4 - f^0(x)^4\right)^2 dx \\
&\cdot \left(\left(\hat{f}_g^0\right)'(x)^2 \hat{\Psi}_{1,g}(x)^2 \left(\hat{f}_g^0\right)''(x) \hat{f}_g^1(x) - \left(f^0\right)'(x)^2 \Psi_1(x)^2 \left(f^0\right)''(x) f^1(x)\right) dx \\
&+\int \left(\hat{f}_g^0(x) - f^0(x)\right) \cdot \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right) dx + \int \left(\hat{f}_g^0(x) - f^0(x)\right)^2 dx \\
&+\int \left(\hat{f}_g^0(x) - f^0(x)\right) \cdot \left(\hat{f}_g^0(x)^3 - f^0(x)^3\right) dx
\end{aligned}
$$

$$+\int\left(\left(\hat{f}_g^0\right)'(x)-\left(f^0\right)'(x)\right)^2dx+\int\left(\left(\hat{f}_g^0\right)'(x)^2-\left(f^0\right)'(x)^2\right)dx$$
$$+\int\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx$$
$$+\int\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx$$
$$+\int\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx$$
$$+\int\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)dx$$
$$+\int\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx$$
$$+\int\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)dx$$
$$+\int\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx$$
$$+\int\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)dx$$
$$+\int\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx$$
$$+\int\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)dx$$
$$+\int\left(\left(\hat{f}_g^0\right)'(x)^3-\left(f^0\right)'(x)^3\right)\cdot\left(\hat{\Psi}_{1,g}(x)-\Psi_1(x)\right)\cdot\left(\hat{\Psi}'_{1,g}(x)-\Psi'_1(x)\right)$$
$$\cdot\left(\hat{f}_g^1(x)-f^1(x)\right)dx+\int\left(\hat{f}_g^0(x)^4-f^0(x)^4\right)$$
$$\cdot\left(\left(\hat{f}_g^0\right)'(x)^3\hat{\Psi}_{1,g}(x)\hat{\Psi}'_{1,g}(x)\hat{f}_g^1(x)-\left(f^0\right)'(x)^3\Psi_1(x)\Psi'_1(x)f^1(x)\right)dx$$
$$+\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)dx+\int\left(\hat{m}'_g(x)-m'(x)\right)^2dx$$
$$+\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)dx+\int\left(\hat{f}_g^0(x)-f^0(x)\right)$$
$$\cdot\left(\hat{m}'_g(x)\hat{m}''_g(x)\left(\hat{f}_g^0\right)'(x)\hat{f}_g^1(x)-m'(x)m''(x)\left(f^0\right)'(x)f^1(x)\right)dx$$
$$+\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{\Psi}''_{1,g}(x)-\Psi''_1(x)\right)dx+\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)^2dx$$
$$+\int\left(\hat{f}_g^0(x)^2-f^0(x)^2\right)\cdot\left(\hat{\Psi}_{1,g}(x)\left(\hat{f}_g^0\right)''(x)-\Psi_1(x)\left(f^0\right)''(x)\right)dx$$
$$+\int\left(\hat{f}_g^0(x)-f^0(x)\right)\cdot\left(\hat{f}_g^0(x)^3-f^0(x)^3\right)dx$$

$$
\begin{aligned}
&+\int \left(\hat{f}_g^0(x) - f^0(x)\right) \cdot \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right) dx \\
&+\int \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right) \cdot \left(\hat{\Psi}_{1,g}'(x) \left(\hat{f}_g^0\right)'(x) - \Psi_1'(x) \left(f^0\right)'(x)\right) dx \\
&+\int \left(\hat{f}_g^0(x)^3 - f^0(x)^3\right) \cdot \left(\hat{\Psi}_{1,g}(x) \left(\hat{f}_g^0\right)'(x)^2 - \Psi_1(x) \left(f^0\right)'(x)^2\right) dx \\
&+\int \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\left(\hat{f}_g^0\right)''(x) - \left(f^0\right)''(x)\right) dx \\
&+\int \left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right) \cdot \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right) dx \\
&+\int \left(\hat{f}_g^0(x) - f^0(x)\right) \cdot \left(\hat{\Psi}_{1,g}'(x) - \Psi_1'(x)\right) dx \\
&+\int \left(\hat{m}_g''(x)^2 - m''(x)^2\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx + \int \left(\hat{m}_g''(x) - m''(x)\right)^2 dx \\
&+\int \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) \cdot \left(\left(\hat{f}_g^0\right)'(x)^2 - \left(f^0\right)'(x)^2\right) dx \\
&+\int \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right)^2 dx \\
&+\int \left(\left(\hat{f}_g^0\right)'(x) - \left(f^0\right)'(x)\right) \cdot \left(\hat{\Psi}_{1,g}(x) - \Psi_1(x)\right) dx \\
&+\int \left(\left(\hat{f}_g^0\right)'(x)^2 - \left(f^0\right)'(x)^2\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx \\
&+\int \left(\hat{m}_g'(x)^2 - m'(x)^2\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx \\
&+\int \left(\hat{m}_g'(x)^2 - m'(x)^2\right) \cdot \left(\left(\hat{f}_g^0\right)'(x)^2 - \left(f^0\right)'(x)^2\right) dx \\
&+\int \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right) \cdot \left(\hat{m}_g'(x)^2 \left(\hat{f}_g^0\right)'(x)^2 \hat{f}_g^1(x) - m'(x)^2 \left(f^0\right)'(x)^2 f^1(x)\right) dx \\
&+\int \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right) \cdot \left(\left(\hat{f}_g^0\right)'(x) \hat{\Psi}_{1,g}(x) - \left(f^0\right)'(x) \Psi_1(x)\right) dx \\
&+\int \left(\hat{m}_g'(x)^2 - m'(x)^2\right) \cdot \left(\left(\hat{f}_g^0\right)'(x)^2 - \left(f^0\right)'(x)^2\right) \cdot \left(\hat{f}_g^1(x) - f^1(x)\right) dx \\
&+\int \left(\hat{f}_g^0(x)^2 - f^0(x)^2\right)^2 dx + \int \left(\hat{f}_g^0(x) - f^0(x)\right)^2 dx. \qquad (120)
\end{aligned}
$$

Remark 3. *Consider two random variables η, ξ. Thanks to Cauchy-Schwartz inequality, we have:*

$$\mathbb{E}\left[(\eta + \xi)^2\right] \leq 2\left(\mathbb{E}\left[\eta^2\right] + \mathbb{E}\left[\xi^2\right]\right). \qquad (121)$$

Similarly, given an additional random variable ζ, it follows that:

$$\mathbb{E}\left[(\eta + \xi + \zeta)^2\right] \leq 3\left(\mathbb{E}\left[\eta^2\right] + \mathbb{E}\left[\xi^2\right] + \mathbb{E}\left[\zeta^2\right]\right). \qquad (122)$$

Proof of Remark 3 Using that $2\mathbb{E}[\eta]\,\mathbb{E}[\xi] \leq \mathbb{E}[\eta]^2 + \mathbb{E}[\xi]^2$, since $0 \leq (\mathbb{E}[\eta] - \mathbb{E}[\xi])^2 = \mathbb{E}[\eta]^2 + \mathbb{E}[\xi]^2 - 2\mathbb{E}[\eta]\,\mathbb{E}[\xi]$.

Then,

$$
\begin{aligned}
0 &\leq \mathbb{E}\left[(\eta+\xi)^2\right] = \left|\mathbb{E}\left[\eta^2\right] + \mathbb{E}\left[\xi^2\right] + 2\mathbb{E}\left[\eta\xi\right]\right| \\
&\leq \mathbb{E}\left[\eta^2\right] + \mathbb{E}\left[\xi^2\right] + \mathbb{E}\left[\eta^2\right] + \mathbb{E}\left[\xi^2\right] \\
&= 2\left(\mathbb{E}\left[\eta^2\right] + \mathbb{E}\left[\xi^2\right]\right).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
0 &\leq \mathbb{E}\left[(\eta+\xi+\zeta)^2\right] = \mathbb{E}\left[\eta^2\right] + \mathbb{E}\left[\xi^2\right] + \mathbb{E}\left[\zeta^2\right] \\
&+ 2\mathbb{E}\left[\eta\xi\right] + 2\mathbb{E}\left[\eta\zeta\right] + 2\mathbb{E}\left[\zeta\xi\right] \\
&\leq \mathbb{E}\left[\eta^2\right] + \mathbb{E}\left[\xi^2\right] + \mathbb{E}\left[\zeta^2\right] + \left(\mathbb{E}\left[\eta^2\right] + \mathbb{E}\left[\xi^2\right]\right) \\
&+ \left(\mathbb{E}\left[\eta^2\right] + \mathbb{E}\left[\zeta^2\right]\right) + \left(\mathbb{E}\left[\xi^2\right] + \mathbb{E}\left[\zeta^2\right]\right) \\
&= 3\left(\mathbb{E}\left[\eta^2\right] + \mathbb{E}\left[\xi^2\right]\right).
\end{aligned}
$$

4 Consider a pilot bandwidth $g > 0$ of exact order $n_0^{-1/2}$, then:

$$
\begin{aligned}
MISE^{a*}(h) &= \frac{R(K)}{n_0\, h}\hat{A}_g + \frac{h^4}{4}\mu_2(K)^2\hat{B}_g + O_P\left(h^6\, n_1^{-1}\left(n_0^{7/2} + n_0^4 + n_0^{9/2}\right)\right) \\
&+ O_P\left(h^{-1}\, n_0^{-1}\, n_1^{-1}\right) + O_P\left(h\, n_1^{-1}\left(1 + n_0^{1/2} + n_0\right)\right).
\end{aligned} \tag{123}
$$

Proof From now on, $C^{[s]}_{v,\ell,r}$, $\Psi^{(r)}_{s,\ell}$ will be denoted just by C and $\Psi^{(r)}_\ell$, respectively, for the sake of brevity, assuming $s = 0$. We begin by analyzing the expectation of expression (32) in the paper,

$$
\begin{aligned}
\mathbb{E}\left[\hat{\Psi}^{(r)}_\ell(x)\right] &= \frac{1}{n_0\, g^{r+1}}\sum_{i=1}^{n_0}\mathbb{E}\left[K^{(r)}\left(\frac{x-X_i^0}{g}\right)Y_i^{0\ell}\right] \\
&= \frac{1}{g^{r+1}}\mathbb{E}\left[\mathbb{E}\left[K^{(r)}\left(\frac{x-X_1^0}{g}\right)Y_1^{0\ell}\middle| X_1^0\right]\right] \\
&= \frac{1}{g^{r+1}}\mathbb{E}\left[K^{(r)}\left(\frac{x-X_1^0}{g}\right)\mathbb{E}\left[Y_1^{0\ell}\middle| X_1^0\right]\right] \\
&= \frac{1}{g^{r+1}}\mathbb{E}\left[K^{(r)}\left(\frac{x-X_1^0}{g}\right)m_\ell(X_1^0)\right] \\
&= \int K_g^{(r)}\,(x-y)\, m_\ell(y) f^0(y)\, dy = K_g^{(r)} * \phi_\ell(x),
\end{aligned} \tag{124}
$$

where $\phi_\ell(u) = m_\ell(u) f^0(u)$. Considering now expression (124), we can compute further calculations with expression

(32) in the paper. We denote by $\hat{\mu}_{i,g}(x) := K_g^{(r)}\left(x - X_i^0\right) Y_i^{0\ell} - K_g^{(r)} * \phi_\ell(x)$ and $\xi_i := \int \nu(x)\hat{\mu}_{i,g}(x)$. Then,

$$\begin{aligned}
C &= \int \nu(x)\left(\frac{1}{n_0\, g^{r+1}}\sum_{i=1}^{n_0} K^{(r)}\left(\frac{x - X_i^0}{g}\right) Y_i^{0\ell} - K_g^{(r)} * \phi_\ell(x) + K_g^{(r)} * \phi_\ell(x) - \Psi_\ell^{(r)}\right) dx \\
&= \int \nu(x)\left(\frac{1}{n_0}\sum_{i=1}^{n_0}\hat{\mu}_{i,g}(x) + K_g^{(r)} * \phi_\ell(x) - \Psi_\ell^{(r)}\right) dx \\
&= \frac{1}{n_0}\sum_{i=1}^{n_0}\int \nu(x)\hat{\mu}_{i,g}(x)\, dx + \int \nu(x)\left[K_g^{(r)} * \phi_\ell(x) - \Psi_\ell^{(r)}\right] dx \\
&= \frac{1}{n_0}\sum_{i=1}^{n_0}\xi_i + \int \nu(x)\left[K_g^{(r)} * \phi_\ell(x) - \Psi_\ell^{(r)}\right] dx. \qquad (125)
\end{aligned}$$

It follows that the MSE of $\hat{\Psi}_\ell^{(r)}(x)$ turns out to be, considering expression (125) and taking into account the fact that $\mathbb{E}\left[\xi_1\right] = 0$:

$$\begin{aligned}
\mathbb{E}\left[C^2\right] &= Var\left[C\right] + \mathbb{E}\left[C\right]^2 = \frac{Var\left[\xi_1\right]}{n_0} + \mathbb{E}\left[C\right]^2 = \frac{\mathbb{E}\left[\xi_1^2\right]}{n_0} + \mathbb{E}\left[C\right]^2 \\
&= \frac{\mathbb{E}\left[\xi_1^2\right]}{n_0} + \left(\int \nu(x)\left(K_g^{(r)} * \phi_\ell(x) - \Psi_\ell^{(r)}(x)\right) dx\right)^2. \qquad (126)
\end{aligned}$$

From now on, assume that f^0 and its derivatives tend to zero as $x \to \infty$ and m_ℓ and its derivatives are bounded as $x \to \infty$, where x is the point of evaluation. Focusing now on the second term of expression (126) leads to:

$$\begin{aligned}
\int \nu(x) K_g^{(r)} * \phi_\ell(x)\, dx &= \int \nu(x)\int K_g^{(r)}(x - y)\phi_\ell(y)\, dy\, dx \\
&\overset{(1)}{=} \int \nu(x)\int K_g(x - y)\phi_\ell^{(r)}(y)\, dy\, dx \\
&= \int \phi_\ell^{(r)}(y)\int \nu(x)\frac{1}{g}K\left(\frac{x - y}{g}\right) dy\, dx \\
&= \int \phi_\ell^{(r)}(y)\int \nu(y + g\,u)K(u)\, du\, dy \\
&= \int \nu(x)\phi_\ell^{(r)}(x)\, dx + \frac{g^2}{2}\mu_2(K)\int \nu''(x)\phi_\ell^{(r)}(x)\, dx \\
&\quad + O\left(g^4\right). \qquad (127)
\end{aligned}$$

It is straightforward to proof equality (1) just by applying integration by parts to:

$$\begin{aligned}
I &:= \int K_g^{(r)}(x - y)\phi_\ell(y)\, dy \\
&= \left[-K_g^{(r-1)}(x - y)\phi_\ell(y)\right]_{y=-\infty}^{y=+\infty} + \int_{-\infty}^{\infty} K_g^{(r-1)}(x - y)\phi_\ell'(y)\, dy \\
&= \int_{-\infty}^{\infty} K_g^{(r-1)}(x - y)\phi_\ell'(y)\, dy = \ldots = \int_{-\infty}^{\infty} K_g(x - y)\phi_\ell^{(r)}(y)\, dy.
\end{aligned}$$

Combining the second term in (126) and expression (26) in the paper leads to conclude that:

$$\begin{aligned}\left(\int v(x)\left(K_g^{(r)}*\phi_\ell(x)-\Psi_\ell^{(r)}(x)\right)dx\right)^2 &= \left(\frac{g^2}{2}\mu_2(K)\int v''(x)\phi_\ell^{(r)}(x)\,dx\right.\\ &\left.+O\left(g^4\right)\right)^2\\ &=\frac{g^4}{4}\mu_2(K)^2\left(\int v''(x)\phi_\ell^{(r)}(x)\,dx\right)^2\\ &+O\left(g^6\right).\end{aligned}\tag{128}$$

It remains to carry on with further computations with the first term of expression (126):

$$\begin{aligned}\frac{\mathbb{E}\left[\xi_1^2\right]}{n_0} &= \frac{1}{n_0}\mathbb{E}\left[\int v(x)\hat{\mu}_{1,g}(x)\,dx\int v(y)\hat{\mu}_{1,g}(y)\,dy\right]\\ &=\frac{1}{n_0}\int\int v(x)v(y)\mathbb{E}\left[\hat{\mu}_{1,g}(x)\hat{\mu}_{1,g}(y)\right]dx\,dy\\ &=\frac{1}{n_0}\int\int v(x)v(y)\mathbb{E}\left[\left(K_g^{(r)}\left(x-X_1^0\right)Y_1^{0\ell}-K_g^{(r)}*\phi_\ell(x)\right)\right.\\ &\left.\cdot\left(K_g^{(r)}\left(y-X_1^0\right)Y_1^{0\ell}-K_g^{(r)}*\phi_\ell(y)\right)\right]dx\,dy\\ &=\frac{1}{n_0}\int\int v(x)v(y)\mathbb{E}\left[K_g^{(r)}\left(x-X_1^0\right)K_g^{(r)}\left(y-X_1^0\right)\left(Y_1^{0\ell}\right)^2\right]dx\,dy\\ &-\frac{1}{n_0}\left(\int v(x)K_g^{(r)}*\phi_\ell(x)\,dx\right)^2.\end{aligned}\tag{129}$$

The second term in (129) has already been analyzed in (127), turning out:

$$\begin{aligned}\frac{1}{n_0}\left(\int v(x)K_g^{(r)}*\phi_\ell(x)\,dx\right)^2 &= \frac{1}{n_0}\left(\int v(x)\phi_\ell^{(r)}(x)\,dx\right)^2\\ &+\frac{g^2}{n_0}\mu_2(K)\int v''(x)\phi_\ell^{(r)}(x)\,dx\\ &\int v(x)\phi_\ell^{(r)}(x)\,dx+O\left(\frac{g^4}{n_0}\right).\end{aligned}\tag{130}$$

Finally, it remains to work out further computations with:

$$
\begin{aligned}
&\frac{1}{n_0}\int\int v(x)v(y)\mathbb{E}\left[K_g^{(r)}\left(x-X_1^0\right)K_g^{(r)}\left(y-X_1^0\right)\left(Y_1^{0\ell}\right)^2\right]dx\,dy\\
&=\frac{1}{n_0}\int\int v(x)v(y)\mathbb{E}\left[\mathbb{E}\left[K_g^{(r)}\left(x-X_1^0\right)K_g^{(r)}\left(y-X_1^0\right)\left(Y_1^{0\ell}\right)^2\middle|X_1^0\right]\right]dx\,dy\\
&=\frac{1}{n_0}\int\int v(x)v(y)\mathbb{E}\left[K_g^{(r)}\left(x-X_1^0\right)K_g^{(r)}\left(y-X_1^0\right)\mathbb{E}\left[\left(Y_1^{0\ell}\right)^2\middle|X_1^0\right]\right]dx\,dy\\
&=\frac{1}{n_0}\int\int v(x)v(y)\mathbb{E}\left[K_g^{(r)}\left(x-X_1^0\right)K_g^{(r)}\left(y-X_1^0\right)m_{2\ell}(X_1^0)\right]dx\,dy\\
&=\frac{1}{n_0}\int\int v(x)v(y)\int K_g^{(r)}(x-z)\,K_g^{(r)}(y-z)\,m_{2\ell}(z)f^0(z)\,dz\,dx\,dy\\
&=\frac{1}{n_0\,g^{2r+2}}\int\int v(x)v(y)\int K^{(r)}\left(\frac{x-z}{g}\right)K^{(r)}\left(\frac{y-z}{g}\right)m_{2\ell}(z)f^0(z)\,dz\,dx\,dy\\
&=\frac{1}{n_0\,g^{2r}}\int m_{2\ell}(z)f^0(z)\int\int v(z+g\,u)v(z+g\,v)K^{(r)}(u)\,K^{(r)}(v)\,du\,dv\,dz\\
&=\frac{2\,g^2}{r!\,(r+2)!\,n_0}\int u^rK^{(r)}(u)\,du\int u^{r+2}K^{(r)}(u)\,du\int v^{(r)}(z)v^{(r+2)}(z)m_{2\ell}(z)f^0(z)\,dz\\
&+\frac{1}{(r!)^2\,n_0}\left(\int u^rK^{(r)}(u)\,du\right)^2\int v^{(r)}(z)^2m_{2\ell}(z)f^0(z)\,dz+O\left(\frac{g^4}{n_0}\right). \qquad (131)
\end{aligned}
$$

Assume K is a symmetric function. Therefore, K' and K'' are antisymmetric and symmetric functions, respectively. Thus, it is easy to proof by integration by parts that:

- If $r=0$,

$$\int K(u)\,du=1,\int uK(u)\,du=0,\int u^2K(u)\,du=\mu_2(K).$$

- If $r=1$,

$$
\begin{aligned}
&\int K'(u)\,du=0,\int uK'(u)\,du=-1,\int u^2K'(u)\,du=0,\\
&\int u^3K'(u)\,du=-3\,\mu_2(K).
\end{aligned}
$$

- If $r=2$,

$$
\begin{aligned}
&\int K''(u)\,du=0,\int uK''(u)\,du=0,\int u^2K''(u)\,du=2,\\
&\int u^3K''(u)\,du=0,\int u^4K''(u)\,du=12\,\mu_2(K).
\end{aligned}
$$

In general, $\int u^rK^{(r)}(u)\,du=(-1)^r\,r!$ and $\int u^{r+2}K^{(r)}(u)\,du=\frac{1}{2!}\,(-1)^r\,(r+2)!\mu_2(K)$.

Hence, expression (131) results in:

$$\frac{1}{n_0}\int \nu^{(r)}(z)^2 m_{2\ell}(z) f^0(z)\,dz + \frac{g^2}{n_0}\mu_2(K)\int \nu^{(r)}(z)\nu^{(r+2)}(z) m_{2\ell}(z) f^0(z)\,dz + O\left(\frac{g^4}{n_0}\right). \tag{132}$$

Using expression (121), our aim is to work out further computations for:

$$\begin{aligned}
&\mathbb{E}\left\{\left[\left(\hat{A}_g - A\right) + \frac{A}{4B}\left(\hat{B}_g - B\right)\right]^2\right\} \\
&= \mathbb{E}\left[\left(\hat{A}_g - A\right)^2\right] + \frac{A^2}{16B^2}\mathbb{E}\left[\left(\hat{B}_g - B\right)^2\right] + \frac{A}{2B}\mathbb{E}\left[\left(\hat{A}_g - A\right)\cdot\left(\hat{B}_g - B\right)\right] \\
&\leq 2\mathbb{E}\left[\left(\hat{A}_g - A\right)^2\right] + \frac{A^2}{8B^2}\mathbb{E}\left[\left(\hat{B}_g - B\right)^2\right].
\end{aligned} \tag{133}$$

Finally, considering expression (122), collecting terms (128), (130) and (131), and afterwards plugging them into (126), it turns out that, $\forall \ell \in \{0,1,2\}, \forall r \in \{0,1,2\}$:

$$\begin{aligned}
\mathbb{E}\left[C^2\right] &= \frac{g^4}{4}\mu_2(K)^2\left(\int \nu''(x)\phi_\ell^{(r)}(x)\,dx\right)^2 \\
&- \frac{g^2}{n_0}\mu_2(K)\int \nu''(x)\phi_\ell^{(r)}(x)\,dx \int \nu(x)\phi_\ell^{(r)}(x)\,dx - \frac{1}{n_0}\left(\int \nu(x)\phi_\ell^{(r)}(x)\,dx\right)^2 \\
&+ \frac{2g^2}{r!\,(r+2)!\,n_0}\int u^r K^{(r)}(u)\,du \int u^{r+2}K^{(r)}(u)\,du \\
&\int \nu^{(r)}(x)\nu^{(r+2)}(x) m_{2\ell}(x) f^0(x)\,dx \\
&+ \frac{1}{(r!)^2\,n_0}\left(\int u^r K^{(r)}(u)\,du\right)^2 \int \nu^{(r)}(x)^2 m_{2\ell}(x) f^0(x)\,dx \\
&+ O\left(g^6\right) + O\left(\frac{g^4}{n_0}\right).
\end{aligned} \tag{134}$$

In other words, considering expression (132) instead of (131):

$$\begin{aligned}
\mathbb{E}\left[C^2\right] &= \frac{g^4}{4}\left(\mu_2(K)\int \nu''(x)\phi_\ell^{(r)}(x)\,dx\right)^2 \\
&+ \frac{1}{n_0}\left[\int \nu^{(r)}(x)^2 m_{2\ell}(x) f^0(x)\,dx - \left(\int \nu(x)\phi_\ell^{(r)}(x)\,dx\right)^2\right] \\
&+ \frac{g^2}{n_0}\mu_2(K)\left[\int \nu^{(r)}(x)\nu^{(r+2)}(x) m_{2\ell}(x) f^0(x)\,dx\right. \\
&\left.- \int \nu''(x)\phi_\ell^{(r)}(x)\,dx \int \nu(x)\phi_\ell^{(r)}(x)\,dx\right] \\
&+ O\left(g^6\right) + O\left(\frac{g^4}{n_0}\right), \forall \ell \in \{0,1,2\}, \forall r \in \{0,1,2\}.
\end{aligned} \tag{135}$$

Similarly, $\forall \ell \in \{0,1,2\}, \forall r \in \{0,1,2\}$,

$$\mathbb{E}\left[C^2\right] = g^4\, C_1 + \frac{g^2}{n_0} C_2 + \frac{1}{n_0} C_3 + O\left(g^6\right) + O\left(\frac{g^4}{n_0}\right), \tag{136}$$

where

$$C_1 := \left(\frac{\mu_2(K)}{2}\int v''(x)\phi_\ell^{(r)}(x)\,dx\right)^2, \tag{137}$$

$$C_2 := \mu_2(K)\left[\int v^{(r)}(x)v^{(r+2)}(x)m_{2\ell}(x)f^0(x)\,dx - \int v''(x)\phi_\ell^{(r)}(x)\,dx \int v(x)\phi_\ell^{(r)}(x)\,dx\right], \tag{138}$$

$$C_3 := \int v^{(r)}(x)^2 m_{2\ell}(x)f^0(x)\,dx - \left(\int v(x)\phi_\ell^{(r)}(x)\,dx\right)^2. \tag{139}$$

It is clear that C_1 in (137) is a non negative quantity. Let's focus now on C_2 in (138). Assume that f^0 and its derivatives tend to zero as $x \to \infty$; m_ℓ and v and their derivatives are bounded as $x \to \infty$, where x is the point of evaluation. Thus, by applying integration by parts to the second term in (138), it turns out:

$$\begin{aligned}\int v''(x)\phi_\ell^{(r)}(x)\,dx &= \left[v''(x)\phi_\ell^{(r-1)}(x)\right]_{-\infty}^{+\infty} - \int v'''(x)\phi_\ell^{(r-1)}(x)\,dx \\ &= -\int v'''(x)\phi_\ell^{(r-1)}(x)\,dx \\ &= -\left[v'''(x)\phi_\ell^{(r-2)}(x)\right]_{-\infty}^{+\infty} + \int v^{(4)}(x)\phi_\ell^{(r-2)}(x)\,dx \\ &= \int v^{(4)}(x)\phi_\ell^{(r-2)}(x)\,dx \\ &= \cdots = (-1)^r \int v^{(r+2)}(x)\phi_\ell(x)\,dx. \end{aligned} \tag{140}$$

Similarly,

$$\int v(x)\phi_\ell^{(r)}(x)\,dx = (-1)^r \int v^{(r)}(x)\phi_\ell(x)\,dx. \tag{141}$$

Hence, expression (138) can be rewritten as:

$$\begin{aligned} C_2 &:= \mu_2(K)\left[\int v^{(r)}(x)v^{(r+2)}(x)m_{2\ell}(x)f^0(x)\,dx \right.\\ &\left. -(-1)^{2r}\int v^{(r+2)}(x)\phi_\ell(x)\,dx \int v^{(r)}(x)\phi_\ell(x)\,dx\right] \\ &= \mu_2(K)\left[\int v^{(r)}(x)v^{(r+2)}(x)m_{2\ell}(x)f^0(x)\,dx \right.\\ &\left. -\int v^{(r+2)}(x)m_\ell(x)f^0(x)\,dx \int v^{(r)}(x)m_\ell(x)f^0(x)\,dx\right] \\ &= Cov\left[v^{(r)}(X^0)Y^{0\ell}, v^{(r+2)}(X^0)Y^{0\ell}\right]. \end{aligned} \tag{142}$$

Considering now expressions (137) and (140), it follows that:

$$C_1 := \left(\frac{\mu_2(K)}{2}\int v^{(r+2)}(x) m_\ell(x) f^0(x)\,dx\right)^2 = \left(\frac{\mu_2(K)}{2}\mathbb{E}\left[v^{(r+2)}(X^0)Y^{0\ell}\right]\right)^2.$$

Finally, combining expressions (139) and (141) leads to:

$$\begin{aligned} C_3 &:= \int v^{(r)}(x)^2 m_{2\ell}(x) f^0(x)\,dx - \left(\int v^{(r)}(x) m_\ell(x) f^0(x)\,dx\right)^2 \\ &= Var\left[v^{(r)}\left(X^0\right)Y^{0\ell}\right], \end{aligned} \tag{143}$$

turning out that C_3 (139) is, as well, greater than 0.

As a consequence of expressions (137), (142) and (143), the optimal g will be determined by the positive or negative sign accompanying term C_2 in (142). In particular, denoting by $\vartheta(g) := \mathbb{E}\left[C^2\right]$ and considering expressions (89), (97) and (122), expression (133) could be rewritten as:

$$\begin{aligned} &\sum_{i=1}^{k_0} a_{0,i}^2\vartheta_i(g) + \frac{A^2}{16B^2}\left(\sum_{i=1}^{k_1} a_{1,i}^2\vartheta_i(g) + \sum_{i=1}^{k_2} a_{2,i}^2\vartheta_i(g) + \sum_{i=1}^{k_3} a_{3,i}^2\vartheta_i(g)\right) \\ &= \left[\sum_{i=1}^{k_0} a_{0,i}^2 + \frac{A^2}{16B^2}\left(\sum_{i=1}^{k_1} a_{1,i}^2 + \sum_{i=1}^{k_2} a_{2,i}^2 + \sum_{i=1}^{k_3} a_{3,i}^2\right)\right]\cdot\vartheta_i(g) \\ &= \left[\sum_{i=1}^{k_0} a_{0,i}^2 + \frac{A^2}{16B^2}\left(\sum_{i=1}^{k_1} a_{1,i}^2 + \sum_{i=1}^{k_2} a_{2,i}^2 + \sum_{i=1}^{k_3} a_{3,i}^2\right)\right]\cdot\left(g^4\,C_{1,i} + \frac{g^2}{n_0}C_{2,i} + \frac{1}{n_0}C_{3,i}\right) \\ &= g^4\,C_1^0 + \frac{g^2}{n_0}\,C_2^0 + \frac{1}{n_0}\,C_3^0, \end{aligned}$$

where

$$\begin{aligned} C_1^0 &:= \left[\sum_{i=1}^{k_0} a_{0,i}^2 + \frac{A^2}{16B^2}\left(\sum_{i=1}^{k_1} a_{1,i}^2 + \sum_{i=1}^{k_2} a_{2,i}^2 + \sum_{i=1}^{k_3} a_{3,i}^2\right)\right] C_{1,i} \geq 0 \\ C_3^0 &:= \left[\sum_{i=1}^{k_0} a_{0,i}^2 + \frac{A^2}{16B^2}\left(\sum_{i=1}^{k_1} a_{1,i}^2 + \sum_{i=1}^{k_2} a_{2,i}^2 + \sum_{i=1}^{k_3} a_{3,i}^2\right)\right] C_{3,i} \geq 0, \text{ and} \\ C_2^0 &:= \left[\sum_{i=1}^{k_0} a_{0,i}^2 + \frac{A^2}{16B^2}\left(\sum_{i=1}^{k_1} a_{1,i}^2 + \sum_{i=1}^{k_2} a_{2,i}^2 + \sum_{i=1}^{k_3} a_{3,i}^2\right)\right] C_{2,i} \\ &= \left[\sum_{i=1}^{k_0} a_{0,i}^2 + \frac{A^2}{16B^2}\left(\sum_{i=1}^{k_1} a_{1,i}^2 + \sum_{i=1}^{k_2} a_{2,i}^2 + \sum_{i=1}^{k_3} a_{3,i}^2\right)\right] \\ &\cdot Cov\left[v_i^{(r)}(X^0)Y^{0\ell}, v_i^{(r+2)}(X^0)Y^{0\ell}\right]. \end{aligned}$$

If $C_2^0 \geq 0$, then the optimal pilot bandwidth g of the upper bound for $\hat{A}_g - A$ and $\hat{B}_g - B$ (expressions (89) and (120), respectively) is as close as 0 as possible. On the other hand, if $C_2^0 < 0$, then the optimal pilot bandwidth g of the upper bound for $\hat{A}_g - A$ and $\hat{B}_g - B$ (expressions (89) and (120), respectively) happens to be

$$\begin{aligned} g_{OPT} &\approx \arg\min_{g>0} \left[\sum_{i=1}^{k_0} a_{0,i}^2 + \frac{A^2}{16B^2} \left(\sum_{i=1}^{k_1} a_{1,i}^2 + \sum_{i=1}^{k_2} a_{2,i}^2 + \sum_{i=1}^{k_3} a_{3,i}^2 \right) \right] \cdot \vartheta_i(g) \\ &= \left(-\frac{C_2^0}{2\,C_1^0} \right)^{1/2} \cdot n_0^{-1/2}. \end{aligned} \tag{144}$$

Moreover, it is straightforward that:

$$\left[\sum_{i=1}^{k_0} a_{0,i}^2 + \frac{A^2}{16B^2} \left(\sum_{i=1}^{k_1} a_{1,i}^2 + \sum_{i=1}^{k_2} a_{2,i}^2 + \sum_{i=1}^{k_3} a_{3,i}^2 \right) \right] \vartheta_i(g_{OPT}) = n_0^{-1}\, C_3^0 - n_0^{-2}\, \frac{\left(C_2^0\right)^2}{4\,C_1^0}.$$

Combining expressions (66) and (144), Corollary 2 holds.

3 | ADDITIONAL MATERIAL FOR THE APPLICATION IN SECTION 4

In the following, Table 4 collects the sample sizes n_0 and n_1 from the real data application, while Table 5 contains the bandwidths for each sector, $h_{BOOT,Z}$.

	Studies					
CNAE	1	2	3	4	5	6
	$n_{0,z}$ = 474	$n_{0,z}$ = 415	$n_{0,z}$ = 321	$n_{0,z}$ = 92	$n_{0,z}$ = 67	$n_{0,z}$ = 202
B	$n_{1,z}$ = 13	$n_{1,z}$ = 14	$n_{1,z}$ = 39	$n_{1,z}$ = 23	$n_{1,z}$ = 28	$n_{1,z}$ = 97
	$n_{0,z}$ = 6592	$n_{0,z}$ = 9276	$n_{0,z}$ = 7479	$n_{0,z}$ = 5096	$n_{0,z}$ = 2134	$n_{0,z}$ = 3504
C	$n_{1,z}$ = 1712	$n_{1,z}$ = 2778	$n_{1,z}$ = 2222	$n_{1,z}$ = 1227	$n_{1,z}$ = 882	$n_{1,z}$ = 2213
	$n_{0,z}$ = 43	$n_{0,z}$ = 67	$n_{0,z}$ = 219	$n_{0,z}$ = 348	$n_{0,z}$ = 360	$n_{0,z}$ = 419
D	$n_{1,z}$ = 5	$n_{1,z}$ = 14	$n_{1,z}$ = 48	$n_{1,z}$ = 45	$n_{1,z}$ = 73	$n_{1,z}$ = 157
	$n_{0,z}$ = 1491	$n_{0,z}$ = 1636	$n_{0,z}$ = 609	$n_{0,z}$ = 350	$n_{0,z}$ = 186	$n_{0,z}$ = 354
E	$n_{1,z}$ = 145	$n_{1,z}$ = 174	$n_{1,z}$ = 165	$n_{1,z}$ = 109	$n_{1,z}$ = 91	$n_{1,z}$ = 230
	$n_{0,z}$ = 3413	$n_{0,z}$ = 3148	$n_{0,z}$ = 1672	$n_{0,z}$ = 815	$n_{0,z}$ = 677	$n_{0,z}$ = 858
F	$n_{1,z}$ = 88	$n_{1,z}$ = 115	$n_{1,z}$ = 338	$n_{1,z}$ = 189	$n_{1,z}$ = 267	$n_{1,z}$ = 424
	$n_{0,z}$ = 1044	$n_{0,z}$ = 2402	$n_{0,z}$ = 1996	$n_{0,z}$ = 635	$n_{0,z}$ = 346	$n_{0,z}$ = 913
G	$n_{1,z}$ = 618	$n_{1,z}$ = 1907	$n_{1,z}$ = 1623	$n_{1,z}$ = 357	$n_{1,z}$ = 327	$n_{1,z}$ = 716
	$n_{0,z}$ = 1156	$n_{0,z}$ = 2402	$n_{0,z}$ = 1754	$n_{0,z}$ = 629	$n_{0,z}$ = 414	$n_{0,z}$ = 597
H	$n_{1,z}$ = 171	$n_{1,z}$ = 913	$n_{1,z}$ = 640	$n_{1,z}$ = 196	$n_{1,z}$ = 210	$n_{1,z}$ = 374
	$n_{0,z}$ = 472	$n_{0,z}$ = 653	$n_{0,z}$ = 455	$n_{0,z}$ = 76	$n_{0,z}$ = 115	$n_{0,z}$ = 106
I	$n_{1,z}$ = 526	$n_{1,z}$ = 601	$n_{1,z}$ = 372	$n_{1,z}$ = 63	$n_{1,z}$ = 157	$n_{1,z}$ = 117
	$n_{0,z}$ = 107	$n_{0,z}$ = 306	$n_{0,z}$ = 1385	$n_{0,z}$ = 950	$n_{0,z}$ = 1269	$n_{0,z}$ = 2540
J	$n_{1,z}$ = 61	$n_{1,z}$ = 169	$n_{1,z}$ = 850	$n_{1,z}$ = 321	$n_{1,z}$ = 522	$n_{1,z}$ = 1783
	$n_{0,z}$ = 78	$n_{0,z}$ = 278	$n_{0,z}$ = 948	$n_{0,z}$ = 227	$n_{0,z}$ = 708	$n_{0,z}$ = 1934
K	$n_{1,z}$ = 115	$n_{1,z}$ = 227	$n_{1,z}$ = 916	$n_{1,z}$ = 352	$n_{1,z}$ = 786	$n_{1,z}$ = 1815
	$n_{0,z}$ = 85	$n_{0,z}$ = 85	$n_{0,z}$ = 259	$n_{0,z}$ = 36	$n_{0,z}$ = 65	$n_{0,z}$ = 181
L	$n_{1,z}$ = 66	$n_{1,z}$ = 73	$n_{1,z}$ = 225	$n_{1,z}$ = 53	$n_{1,z}$ = 88	$n_{1,z}$ = 199
	$n_{0,z}$ = 243	$n_{0,z}$ = 403	$n_{0,z}$ = 1022	$n_{0,z}$ = 791	$n_{0,z}$ = 1077	$n_{0,z}$ = 3123
M	$n_{1,z}$ = 172	$n_{1,z}$ = 411	$n_{1,z}$ = 1273	$n_{1,z}$ = 606	$n_{1,z}$ = 913	$n_{1,z}$ = 2971
	$n_{0,z}$ = 1278	$n_{0,z}$ = 2857	$n_{0,z}$ = 1627	$n_{0,z}$ = 412	$n_{0,z}$ = 349	$n_{0,z}$ = 614
N	$n_{1,z}$ = 842	$n_{1,z}$ = 1390	$n_{1,z}$ = 1118	$n_{1,z}$ = 333	$n_{1,z}$ = 593	$n_{1,z}$ = 670
	$n_{0,z}$ = 384	$n_{0,z}$ = 1339	$n_{0,z}$ = 1223	$n_{0,z}$ = 373	$n_{0,z}$ = 457	$n_{0,z}$ = 721
O	$n_{1,z}$ = 206	$n_{1,z}$ = 882	$n_{1,z}$ = 1030	$n_{1,z}$ = 310	$n_{1,z}$ = 812	$n_{1,z}$ = 1019
	$n_{0,z}$ = 41	$n_{0,z}$ = 84	$n_{0,z}$ = 175	$n_{0,z}$ = 86	$n_{0,z}$ = 322	$n_{0,z}$ = 985
P	$n_{1,z}$ = 73	$n_{1,z}$ = 120	$n_{1,z}$ = 281	$n_{1,z}$ = 183	$n_{1,z}$ = 1001	$n_{1,z}$ = 1239
	$n_{0,z}$ = 460	$n_{0,z}$ = 615	$n_{0,z}$ = 709	$n_{0,z}$ = 223	$n_{0,z}$ = 551	$n_{0,z}$ = 1173
Q	$n_{1,z}$ = 631	$n_{1,z}$ = 1144	$n_{1,z}$ = 2927	$n_{1,z}$ = 739	$n_{1,z}$ = 2752	$n_{1,z}$ = 1712
	$n_{0,z}$ = 344	$n_{0,z}$ = 569	$n_{0,z}$ = 698	$n_{0,z}$ = 208	$n_{0,z}$ = 163	$n_{0,z}$ = 395
R	$n_{1,z}$ = 188	$n_{1,z}$ = 327	$n_{1,z}$ = 441	$n_{1,z}$ = 92	$n_{1,z}$ = 137	$n_{1,z}$ = 414
	$n_{0,z}$ = 300	$n_{0,z}$ = 428	$n_{0,z}$ = 597	$n_{0,z}$ = 280	$n_{0,z}$ = 173	$n_{0,z}$ = 279
S	$n_{1,z}$ = 242	$n_{1,z}$ = 354	$n_{1,z}$ = 546	$n_{1,z}$ = 164	$n_{1,z}$ = 266	$n_{1,z}$ = 425

TABLE 4 Source sample size ($n_{0,z}$) and target sample size ($n_{1,z}$) in each stratum depending on level of *Studies* (1-6) and *CNAE* group of belonging (B-S).

	Studies					
CNAE	1	2	3	4	5	6
B	10.34427	0.05311245	0.06555036	0.2250906	10.32863	11.46244
C	0.7827234	0.1570725	0.2240537	0.379917	1.036544	1.006386
D	10.23072	8.540002	11.09998	10.90716	11.87939	11.15287
E	4.034317	1.54504	3.425392	6.90935	8.561086	9.706537
F	0.1979207	0.1042053	0.1779684	0.7073216	2.416007	2.031799
G	10.59052	0.7064822	2.42826	5.206829	5.918734	10.74743
H	1.049699	0.8470396	2.464503	6.063806	6.250508	4.231824
I	7.865014	0.7109818	1.295192	4.747013	3.477528	2.816318
J	6.05535	1.539357	1.665482	0.6700639	1.620418	0.6657766
K	11.36344	11.8833	12.25746	11.09947	11.41423	6.882228
L	3.848873	1.101402	2.196453	3.877995	7.955811	5.016808
M	2.164264	1.776538	1.112405	1.194964	1.119651	0.4577655
N	0.4322325	0.1414672	0.193088	0.2586702	0.814847	0.4869169
O	10.67815	2.554323	11.71623	11.92024	10.88268	11.16583
P	10.27285	3.853947	10.83386	9.703337	8.053916	1.865665
Q	10.0757	4.310891	1.978999	9.339508	7.023242	3.641688
R	11.75589	4.868713	10.85612	10.14804	8.112695	9.912113
S	3.342381	0.8564468	5.826307	2.773484	4.111988	5.707007

TABLE 5 Bandwidth selector ($h_{BOOT,Z}$) in each stratum depending on level of *Studies* (1-6) and *CNAE* group of belonging (B-S).