A Monte Carlo approach to American options pricing including counterparty risk

Iñigo Arregui[a,b], Beatriz Salvador[a,b] and Carlos Vázquez[a,b,c]

[a]Depto. de Matemáticas, Universidad de A Coruña, Campus de Elviña, 15071 A Coruña, Spain. [b]CITIC, Campus de Elviña, 15071 A Coruña, Spain. [c]ITMATI, Campus Vida, 15782 Santiago de Compostela, Spain.

ARTICLE HISTORY
Compiled June 4, 2018

Abstract
In this work we propose a numerical technique to compute the total value adjustment (XVA) for the pricing of American options when considering counterparty risk. Several linear and nonlinear mathematical models, associated to different choices of the mark-to-market value at default, are deduced and numerically solved, thus leading to approximations of the option price with counterparty risk. The methodology is based on Monte Carlo simulations combined with a dynamic programming strategy. At each time step, an optimal stopping criterion is applied and the decision on either exercising or not the option is taken. We present some numerical tests to illustrate the behavior of the proposed method.

KEYWORDS
American options, counterparty risk, total value adjustment, Monte Carlo method, dynamic programming

1. Introduction

The financial crisis that mainly started in 2007 motivated that financial agents began to focus on the concept of counterparty risk, i.e. the risk associated to the possibility of one of the counterparties (or both) to default. Thus, different adjustments related to counterparty risk started to be taken into account when valuing financial products —such as credit value adjustment (CVA), funding value adjustment (FVA) or debit value adjustment (DVA), for example—, which are now included in the final price of derivative products [8]. The increasing number of adjustments leads to include all of them under the generic name of total value adjustment or XVA. The more recent adjusments like capital value adjustment (KVA) and margin value adjustment (MVA) are out of the scope of the present article and maybe taken into account in future works.

It is well known that European and American options are among the most popular derivative products on assets. In both contracts, the holder has the right (but not the obligation) to buy or sell an asset at a price that has been agreed with the counterparty. While European options can only be exercised by the holder at the end of the maturity period, the holder of an American option can exercise it at any moment along this period. In the case without counterparty risk, classical pricing methodologies are

CONTACT I. Arregui. Email: arregui@udc.es

available since a long time in the literature. However, the consideration of counterparty risk makes the valuation more complex, even for these vanilla options.

As we focus on counterparty risk, when this risk is not taken into account we will refer to the derivative as risk-free derivative (i.e. free of counterparty risk), while we will use term risky derivative when counterparty risk is considered.

Three kinds of methodologies are mainly used to price derivative products with counterparty risk. The first one involves Monte Carlo methods as the XVA is expressed in terms of expectations [3, 11] and it is the most used by banks and financial entities. The second one is based on the solution of backwards stochastic differential equations (BSDEs) [5, 6], while the last one involves the solution of partial differential equations (PDEs) [4, 13]. All of them need the use of numerical methods to approximate the final value of the XVA. The present article focuses on the first approach, although the numerical results will be compared to those ones obtained from the third approach.

In a previous work [1] we have proposed different models based on PDEs for pricing European and American options with one stochastic factor. More precisely, depending on the choice of the mark-to-market value, in the case of European options two possible kinds of models are deduced: linear and nonlinear. In the case of American options, the analogous models are posed in terms of linear and nonlinear complementarity problems. A set of numerical methods (semilagrangian discretization, finite element method, augmented Lagrangian active set method and fixed point techniques) are also proposed so that the XVA and thd price of the risky derivative can be computed in different cases. More recently, if we consider stochastic spreads instead of constant ones as in [1], then PDE models for European options with counterparty risk and two stochastic factors are analyzed and numerically solved in [2].

The goal of the present work is to price American options with counterparty risk by means of Monte Carlo methods. Thus, we mainly follow Longstaff and Schwartz [10] and Glasserman [7] in order to obtain the approximation of the riskless option price and the risky option price. In this way, we finally compute the total value adjustment as the difference between both prices. A dynamic programming technique is implemented: at each time step an optimal stopping problem is solved, an optimal exercise criterion is stated and the expected discounted payoff of the option price under this criterion is computed.

In Section 2, we present the mathematical model and deduce two complementarity problems for the risky options, depending on the chosen mark-to-market value. Also, we recall the complementarity problem associated to the classical Black-Scholes equation for the American option price in the case without counterparty risk, as this one is also involved in the solution of the risky option. Section 3 is devoted to the description of the numerical algorithms implemented to compute the value of the risky option in the linear case, while Section 4 presents their adaption to numerically solve the analogous nonlinear complementarity problem. In Section 5 we present some numerical results. Finally, we pose some conclusions in Section 6.

2. A total value adjustment model for American style options

In this section, we take into account the total value adjustment (XVA) in the pricing of American options with counterparty risk and we present the complementarity problems modelling the derivative value. A more detailed deduction can be found in [1].

Thus, we consider a derivative trade between two defaultable counterparties, the issuer B and the buyer C. From the point of view of the seller, the risky derivative value at time t is denoted by $\hat{V}_t = \hat{V}(t, S_t, J_t^B, J_t^C)$, where S_t represents the price of the underlying asset while J_t^B and J_t^C are two independent jump processes that change from 0 to 1 on default of B and C, respectively. The counterparty risk-free American option price is denoted by $V_t = V(t, S_t)$, which can be computed using the classical Black-Scholes complementarity problem for American options (see [15, 16], for example).

Assume that S_t follows a general geometric Brownian motion, thus satisfying:

$$dS_t = r_R S_t \, dt + \sigma S_t \, dW_t \,, \tag{1}$$

where r_R is the rate paid for the underlying asset in a repurchase agreement, σ is its volatility and W_t is a Wiener process.

Let $\mathcal{A}$ be the operator defined by:

$$\mathcal{A}V = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + r_R S \frac{\partial V}{\partial S} \,.$$

Applying classical hedging arguments to a self-financing portfolio, and using Ito's lemma for jump diffusions, we can deduce the inequality that models the price, $\hat{V}$, of the American option including counterparty risk (see [1] for details):

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}\hat{V} - r\hat{V} \le (\lambda_B + \lambda_C)\hat{V} + s_F M^+ - \lambda_B(M^+ + R_B M^-) - \lambda_C(M^- + R_C M^+) \,,$$

where r is the risk-free interest rate, $R_B, R_C \in [0,1]$ represent the recovery rates on the derivatives position of parties B and C, respectively, λ_B and λ_C are the default intensities of B and C, respectively, s_F is the funding cost of the entity, and M represents the close-out mark-to-market value.

Depending on the choice of the mark-to-market value, two different models are deduced for the risky derivative.

- If $M = V$, the model is governed by a linear complementarity problem:

$$\begin{cases} \mathcal{L}(\hat{V}) \equiv \partial_t \hat{V} + \mathcal{A}\hat{V} - (r + \lambda_B + \lambda_C)\hat{V} \\ \qquad\qquad + (R_B\lambda_B - \lambda_C)V^- + (R_C\lambda_C + \lambda_B)V^+ - s_F V^+ \le 0 \\ \hat{V}(t,S) \ge H(S) \\ \mathcal{L}(\hat{V})\,(\hat{V} - H) = 0 \\ \hat{V}(T,S) = H(S)\,. \end{cases} \tag{2}$$

- If $M = \hat{V}$, the model involves a nonlinear complementarity problem:

$$\begin{cases} \mathcal{N}(\hat{V}) \equiv \partial_t \hat{V} + \mathcal{A}\hat{V} - r\hat{V} \\ \qquad\qquad - (1 - R_B)\lambda_B \hat{V}^- - (1 - R_C)\lambda_C \hat{V}^+ - s_F \hat{V}^+ \le 0 \\ \hat{V}(t,S) \ge H(S) \\ \mathcal{N}(\hat{V})\,(\hat{V} - H) = 0 \\ \hat{V}(T,S) = H(S)\,. \end{cases} \tag{3}$$

Moreover, note that the risk-less value of the American option, V, enters in the formulation of problem (2). The function V is the solution of the linear complementarity problem associated to the classical Black-Scholes operator:

$$\begin{cases} \widetilde{\mathcal{L}}(V) \equiv \partial_t V + \mathcal{A}V - rV \leq 0 \\ V(t,S) \geq H(S) \\ \widetilde{\mathcal{L}}(V)\,(V-H) = 0 \\ V(T,S) = H(S)\,. \end{cases} \tag{4}$$

Note that the complementarity problems (2), (3) and (4) can be solved by numerical methods. In [1], we combine characteristic methods for time discretization, finite elements for spatial discretization, fixed point methods to overcome the nonlinearities and duality methods for solving the discrete linear complementarity problems. Finally, the total value adjustment U can be computed as $U = \hat{V} - V$.

3. Numerical methods. The case $M = V$

In this section we address the XVA computation when the mark-to-market value is equal to the option value without counterparty risk.

Unlike the European option, which can only be exercised at maturity time T, an American option can be exercised at any time $t \in (0,T]$. We denote its exercise value at any time $t \in (0,T]$ as

$$h^*(t, S_t) = H(S_t)\,, \tag{5}$$

where $H(S_t)$ represents the payoff of the option. Note that the price process S_t is Markovian.

In a first step, we consider problem (2). Let g be the function defined by:

$$g(V) = (R_B\lambda_B - \lambda_C)V^- + (R_C\lambda_C + \lambda_B)V^+ - s_F V^+\,.$$

Following [12] we can deduce that, in terms of expectations, the risky derivative value at time $t = 0$ for the underlying value S_0 is given by:

$$\hat{V}_0(S_0) = \sup_{\tau \in \mathcal{T}_0} \mathbb{E}_0 \left[e^{-r_0\tau} h^*(\tau, S_\tau) + \int_0^\tau e^{-r_0 u} g(V(u, S(u)))\, du \right],$$

where $r_0 = r + \lambda_B + \lambda_C$ and $\mathcal{T}_t$ is the set of admissible stopping instants in $[t,T]$. In our numerical approach, the value of V solving (4) will be estimated by a classical Monte Carlo technique for American options without counterparty risk.

In order to price the option, we first discretize the time interval by introducing a finite and increasing set of instants, $0 = t_0 < t_1 < t_2 < \ldots < t_M = T \subset [0,T]$.

We will assume that the option can be only exercised in t_i ($i = 0, 1, \ldots, M$). In this way, we are approaching the American option by a Bermudan one. Taking into account the fixed instant times, we denote by $S_i = S(t_i)$, $i = 1, 2, \ldots, M$, the asset price at the i-th exercise opportunity. We approximate those values, solution of the stochastic differential equation (1), by the Euler-Maruyama scheme:

$$S_i = S_{i-1} + r_R S_{i-1}\Delta t + \sigma S_{i-1}\Delta W_i, \qquad i = 1, 2, \ldots, M\,, \tag{6}$$

where $\Delta t = t_i - t_{i-1}$ is the size of the time interval and $\Delta W_i = W_i - W_{i-1}$ is the independent Brownian increment, which follows a normal distribution $\mathcal{N}(0, \sqrt{\Delta t})$.

3.1. A dynamic programming formulation

Considering the previous time discretization for the asset price evolution, the American option with counterparty risk can be priced through a dynamic programming approach. Thus, in a particular time instant $t = t_i$, the risky derivative value is given by

$$\hat{V}_i^*(s) = \sup_{\tau \in \mathcal{T}_{t_i}} \mathbb{E}_{t_i}\left[e^{-r_0(\tau - t_i)} h^*(\tau, S_\tau) + \int_{t_i}^{\tau} e^{-r_0(u-t_i)} g(V(u, S(u)))\, du \;\middle|\; S_i = s \right].$$

If we compute $\hat{V}_i^*(s)$ for $i = M, \ldots, 1, 0$ (thus, from $t = T$ to $t = 0$), we define a strategy for pricing American options.

We know the option value at maturity ($t_M = T$):

$$\hat{V}_M^*(s) = h^*(T, s)$$

for a given underlying value s. At time $t = t_{M-1}$, an investor will choose to exercise the option if and only if the payoff at this instant is greater than the discounted expected value to be received if the investor decides not to exercise. From this consideration, we have:

$$\begin{aligned}\hat{V}_{M-1}^*(s) = \max\Bigg\{ & h^*(t_{M-1}, s), \\ & \mathbb{E}_{t_{M-1}}\left[D_{M-1,M}\hat{V}_M^*(S_M) + \int_{t_{M-1}}^{t_M} e^{-r_0(u-t_{M-1})} g(V(u, S(u)))\, du \;\middle|\; S_{M-1} = s \right] \Bigg\},\end{aligned}$$

where the discount factor is defined by $D_{i-1,i} = e^{-r_0(t_i - t_{i-1})}$. Thus, the recursive formula is given by:

$$\begin{aligned}&\hat{V}_M^*(s) = h^*(T, s)\,, \quad S_M = s\,, \\ &\hat{V}_{i-1}^*(s) = \max\Bigg\{ h^*(t_{i-1}, s), \\ &\qquad \mathbb{E}_{t_{i-1}}\left[D_{i-1,i}\hat{V}_i^*(S_i) + \int_{t_{i-1}}^{t_i} e^{-r_0(u-t_{i-1})} g(V(u, S(u)))\, du \;\middle|\; S_{i-1} = s \right] \Bigg\},\end{aligned} \tag{7}$$

for $i = M, M-1, \ldots, 1$.

Note that we are interested in obtaining the discounted values at $t_0 = 0$, so we consider

$$h_i(s) = D_{0,i} h^*(t_i, s)\,, \qquad \hat{V}_i(s) = D_{0,i} \hat{V}_i^*(s) \quad (i = 0, \ldots, M)\,.$$

Taking into account that $\hat{V}_0(s) = \hat{V}_0^*(s)$ and the recursive expression given in (7), we obtain:

$$\begin{aligned}
\hat{V}_M(s) &= h_M(s) \\
\hat{V}_{i-1}(s) &= D_{0,i-1} \hat{V}_{i-1}^*(s) \\
&= D_{0,i-1} \max \Bigg\{ h^*(t_{i-1}, s), \quad \mathbb{E}_{t_{i-1}} \Bigg[D_{i-1,i} \hat{V}_i^*(S_i) + \\
&\qquad\qquad \int_{t_{i-1}}^{t_i} e^{-r_0(u - t_{i-1})} g(V(u, S(u)))\, du \;\Big|\; S_{i-1} = s \Bigg] \Bigg\} \\
&= \max \Bigg\{ h_{i-1}(s), \quad \mathbb{E}_{t_{i-1}} \Bigg[D_{0,i-1} D_{i-1,i} \hat{V}_i^*(S_i) + \\
&\qquad\qquad \int_{t_{i-1}}^{t_i} D_{0,i-1} e^{-r_0(u - t_{i-1})} g(V(u, S(u)))\, du \;\Big|\; S_{i-1} = s \Bigg] \Bigg\} \\
&= \max \Bigg\{ h_{i-1}(s),\; \mathbb{E}_{t_{i-1}} \Bigg[D_{0,i} \hat{V}_i^*(S_i) + \int_{t_{i-1}}^{t_i} e^{-r_0 u} g(V(u, S(u)))\, du \;\Big|\; S_{i-1} = s \Bigg] \Bigg\},
\end{aligned}$$

for $i = M, M-1, \ldots, 1$. Introducing the discount factor in the payoff and in the functions, the previous expressions can be simplified:

$$\begin{aligned}
\hat{V}_M(s) &= h(T, s)\,, \quad S_M = s \\
\hat{V}_{i-1}(s) &= \max \Bigg\{ h_{i-1}(s), \mathbb{E}_{t_{i-1}} \Bigg[\hat{V}_i(S_i) + \int_{t_{i-1}}^{t_i} e^{-r_0 u} g(V(u, S(u)))\, du \;\Big|\; S_{i-1} = s \Bigg] \Bigg\},
\end{aligned} \tag{8}$$

for $i = M, M-1, \ldots, 1$.

3.2. Optimal stopping rule and continuation value

In the previous section we have approximated the option value in a recursive way. However, it is also important to price the option through stopping rules and exercise region. In that sense, any stopping time τ determines the sub-optimal value

$$\hat{V}_0^\tau(S_0) = \mathbb{E}_0 \left[h_\tau(S_\tau) + \int_0^\tau e^{-r_0 u} g(V(u, S(u)))\, du \right].$$

Our aim is to choose the optimal stopping time, which will be determined by

$$\tau^* = \min \left\{ \tau_i \in \{t_1, \ldots, t_M\} : h_i(S_i) \geq \hat{V}_i(S_i) \right\}, \tag{9}$$

so that the exercise region associated to $\hat{V}_i$ at the i-th exercise date is the set

$$\left\{s:\, h_i(s) = \hat{V}_i(s)\right\}.$$

After defining the optimal stopping rule we introduce the continuation value, which is the value of holding instead of exercising the option. This continuation value can be computed in a recursive way as:

$$\begin{aligned}
&C_M(s) = 0,\\
&C_i(s) = \mathbb{E}_{t_i}\left[\hat{V}_{i+1}(S_{i+1}) + \int_{t_i}^{t_{i+1}} e^{-r_0 u} g(V(u, S(u)))\, du \;\Big|\; S_i = s\right],
\end{aligned}$$

for $i = M-1, \ldots, 0$, where $\hat{V}_i$ is obtained as the solution of the recursive dynamic programming problem. Moreover, according to (8) the option value is given in terms of the continuation and exercise values as follows:

$$\hat{V}_i(s) = \max\{h_i, C_i\}\,, \qquad i = 1, \ldots, M\,.$$

Thus, the optimal stopping rule can be rewritten as

$$\tau^* = \min\left\{\tau_i \in \{t_1, \ldots, t_M\}:\, h_i(S_i) \geq C_i(S_i)\right\}. \tag{10}$$

In terms of the optimal stopping time, the option value is determined by

$$\hat{V}_0^{\tau^*}(S_0) = \mathbb{E}_0\left[h_{\tau^*}(S_{\tau^*}) + \int_0^{\tau^*} e^{-r_0 u} g(V(u, S(u)))\, du\right].$$

3.3. Lower bounds estimator using least-squares regressions

We now introduce the approximations, $\kappa_i(s)$, of the continuation values, $C_i(s)$. Several authors, cf. Longstaff and Schwartz [10] for example, have proposed a least-squares regression to estimate these values from the simulated paths. In this way, the value $C_i(s)$ can be obtained as the regression of

$$\hat{V}_{i+1}(S_{i+1}) + \int_{t_i}^{t_{i+1}} e^{-r_0 u} g(V(u, S(u)))\, du$$

on the current state of the asset price s. Thus, C_i is approximated by a linear combination of known functions of the current state using a least-squares regression that leads to coefficients κ_i.

Following this idea, we introduce how to approximate the continuation values considering counterparty risk. We will write the continuation value as a linear combination of basis functions as follows:

$$C_i(s) = \mathbb{E}_{t_i}\left[\hat{V}_{i+1}(S_{i+1}) + \int_{t_i}^{t_{i+1}} e^{-r_0 u} g(V(u, S(u)))\, du \;\Big|\; S_i = s\right]$$

$$= \sum_{j=1}^{J} b_{ij}\psi_j(s) = b_i^T \psi(s)\,, \tag{11}$$

where $b_i = (b_{i1}, \ldots, b_{iJ})^T$ are the regression coefficients at time t_i and $\psi(s) = (\psi_1(s), \ldots, \psi_J(s))^T$ is the vector of basis functions.

Different bases can be used to approximate the continuation value. We focus on the weighted Laguerre polynomials:

$$\psi_j(x) = e^{-x/2} L_{j-1}(x)\,, \quad j = 1, 2, \ldots$$

where L_j is the $j-$th Laguerre polynomial.

Next, we determine the expression of the regression coefficients b_i using a least-squares optimization technique. Let φ the function to minimize:

$$\varphi(b_i) = \mathbb{E}_{t_i}\left[\left(\psi(S_i)^T b_i - \mathbb{E}_{t_i}\left[\hat{V}_{i+1}(S_{i+1}) + \int_{t_i}^{t_{i+1}} e^{-r_0 u} g(V(u, S(u)))\, du \;\middle|\; S_i = s\right]\right)^2\right].$$

In order to minimize, we vanish the derivatives with respect to b_i, so that we get:

$$\mathbb{E}_{t_i}\left[\psi(S_i)\left(\psi(S_i)^T b_i - \mathbb{E}_{t_i}\left[\hat{V}_{i+1}(S_{i+1}) + \int_{t_i}^{t_{i+1}} e^{-r_0 u} g(V(u, S(u)))\, du \;\middle|\; S_i = s\right]\right)\right] = 0$$

or equivalently,

$$\begin{aligned}
\mathbb{E}_{t_i}[\psi(S_i)\psi(S_i)^T]\, b_i &= \mathbb{E}_{t_i}\left[\psi(S_i)\, \mathbb{E}_{t_i}\left[\hat{V}_{i+1}(S_{i+1}) \;\middle|\; S_i\right]\right] \\
&\quad + \mathbb{E}_{t_i}\left[\psi(S_i)\, \mathbb{E}_{t_i}\left[\int_{t_i}^{t_{i+1}} e^{-r_0 u} g(V(u, S(u)))\, du \;\middle|\; S_i = s\right]\right] \\
&= \mathbb{E}_{t_i}\left[\psi(S_i)\, \hat{V}_{i+1}(S_{i+1})\right] + \mathbb{E}_{t_i}\left[\psi(S_i)\, \mathbb{E}_{t_i}\left[\int_{t_i}^{t_{i+1}} e^{-r_0 u} g(V(u, S(u)))\, du \;\middle|\; S_i = s\right]\right].
\end{aligned}$$

Thus, the expression of b_i is approximated by β_i, which satisfies the linear system:

$$A_i^{\psi}\, \beta_i = d_i^{\psi}\,,$$

where A_i^{ψ} and d_i^{ψ} can be easily estimated by Monte Carlo simulations. For this purpose, let us consider independent paths $(S_{j,1}, S_{j,2}, \ldots, S_{j,M})$ $(j = 1, 2, \ldots, N)$, that can be deduced by (6), and assume that the value $V_{i+1}(S_{j,i+1})$ is known at time t_i. Then, A_i^{ψ} is a $M \times M$ matrix with coefficients:

$$(A_i^{\psi})_{l,k} = \frac{1}{N}\sum_{j=1}^{N} \psi_l(S_{j,i})\, \psi_k(S_{j,i})$$

and d_i^ψ is the M-array with the k-th element given by

$$(d_i^\psi)_k = \frac{1}{N}\sum_{j=1}^{N}\psi_k(S_{j,i})\,\hat{W}_{i+1}(S_{j,i+1}) + \frac{1}{N}\sum_{j=1}^{N}\psi_k(S_{j,i})\int_{t_i}^{t_{i+1}} e^{-r_0 u} g(W(u,S(u)))\,du\,,$$

where $S_{j,i}$ and $S_{j,i+1}$ correspond to the same trajectory. Moreover, W denotes the risk-free value estimated by the classical Longstaff-Schwartz algorithm while $\hat{W}_{i+1}$ is the estimation of the risky value in the previous time step.

Thus, the continuation value C_i can be approximated by:

$$\kappa_i = \beta_i^T \psi(S_i) \tag{12}$$

and the risky derivative value can be replaced by its estimated value

$$\hat{W}_{i+1} = \max\{h_{i+1}(S_{i+1}), \kappa_{i+1}\}\,.$$

Algorithm 1 Regression coefficients β_i (without interpolation)

(1) Simulate N independent paths $\{S_{j,1}, S_{j,2}, \ldots, S_{j,M}\}$ of the asset prices process.
(2) At maturity time t_M, $\hat{W}_M(S_{j,M}) = h_M(S_{j,M})$.
(3) Apply backward induction for $i = M-1, \ldots, 1$.
- Compute the classical Longstaff-Schwartz approximation with $S_0 = S_{j,i}$ for the time interval $[t_i, T]$ to obtain $W_{j,i}$.
- Given the estimated value $\hat{W}_{j,i+1}$ and $W_{j,i}$ ($j = 1, \ldots, N$), compute β_i as the solution of the linear system $A_i^\psi \beta_i = d_i^\psi$.
- Estimate the continuation value $\kappa_i(S_{j,i}) = \beta_i^T \psi(S_{j,i})$ ($j = 1, \ldots, N$).
- Compute $\hat{W}_{j,i}^{k+1} = \max\{h_i(S_{j,i}), \kappa_i(S_{j,i})\}$.

(4) Save the regression coefficients β_i to compute the risky derivative value.

Let us remark that in the previous algorithm (sketched as Algorithm 1) we have to apply an *inner* Monte Carlo method at each step of time and for each asset price path, what makes this solution very expensive from the computational point of view.

With the aim of reducing this computational cost, we introduce a second alternative to solve the same problem (Algorithm 2). In this alternative, we propose to compute the risk-free derivative value, W, for a set of asset prices at each instant time of the discretization used to obtain the risky derivative value. The classical Longstaff-Schwartz algorithm is employed. Then, in each integral, the risk-free derivative value has to be evaluated in the state of the asset price at instant t_i. Instead of the exact value, we propose the use of the interpolated value computed from the set of fixed values previously obtained for different asset prices.

3.4. Low-biased estimator using optimal stopping rule

After obtaining the regression coefficients, we compute the value of the American option with counterparty risk, by simulating a new set of paths independent from the previously used prices. Then, the optimal stopping strategy is determined with the

Algorithm 2 Regression coefficients β_i (with interpolation)

(1) Simulate N independent paths $\{S_{j,1}, S_{j,2}, \ldots, S_{j,M}\}$ of the asset prices process.
(2) Apply forward induction for $i = 0, 1, \ldots, M-1$. Compute the risk-free derivative value for different asset values in the time interval $[t_i, T]$.
(3) At maturity time t_M, $\hat{W}_M(S_{j,M}) = h_M(S_{j,M})$.
(4) Apply backward induction for $i = M-1, \ldots, 1$.
  - Interpolate the risk-free derivative value for the asset price $S_{j,i}$ at time t_i.
  - Given the estimated values $\hat{W}_{j,i+1}$ and $W_{j,i}$ $(j = 1, \ldots, N)$, compute β_i as the solution of the linear system $A_i^{\psi}\beta_i = d_i^{\psi}$.
  - Estimate the continuation value $\kappa_i(S_{j,i}) = \beta_i^T \psi(S_{j,i})$ $(j = 1, \ldots, N)$.
  - Compute $\hat{W}_{j,i}^{k+1} = \max\{h_i(S_{j,i}), \kappa_i(S_{j,i})\}$.
(5) Save the regression coefficients β_i to compute the risky derivative value.

previous algorithm, given the state of the asset price S_i. Thus,

$$\hat{\tau} = \min\left\{\tau_i \in \{t_1, \ldots, t_M\} : h_i(S_i) \geq \kappa_i(S_i)\right\}.$$

By using this stopping strategy, with the second set of paths, the risky American option value is estimated as

$$\begin{aligned}\hat{W}_0(S_0) = \mathbb{E}_0 \Bigg[h_{\hat{\tau}}(S_{\hat{\tau}}) + \int_0^{\hat{\tau}} e^{-r_0 u} \big[(R_B\lambda_B - \lambda_C) W(u, S(u))^- \\ + (R_C\lambda_C + \lambda_B) W(u, S(u))^+ - s_F W(u, S(u))^+ \big] \, du \Bigg]. \end{aligned} \tag{13}$$

Taking into account the expression of the risky derivative value $\hat{V}_0(S_0)$, given by

$$\begin{aligned}\hat{V}_0(S_0) &= \sup_{\tau \in \mathcal{T}_0} \mathbb{E}_0 \Bigg[h(\tau, S_\tau) + \int_0^{\tau} e^{-r_0 u} \big[(R_B\lambda_B - \lambda_C) V(u, S(u))^- \\ &\qquad + (R_C\lambda_C + \lambda_B) V(u, S(u))^+ - s_F V(u, S(u))^+ \big] \, du \Bigg] \\ &\geq \mathbb{E}_0 \Bigg[h_{\hat{\tau}}(S_{\hat{\tau}}) + \int_0^{\hat{\tau}} e^{-r_0 u} \big[(R_B\lambda_B - \lambda_C) W(u, S(u))^- \\ &\qquad + (R_C\lambda_C + \lambda_B) W(u, S(u))^+ - s_F W(u, S(u))^+ \big] \, du \Bigg] = \hat{W}_0(S_0), \end{aligned}$$

we deduce that the estimator defined in (13) is a low-biased estimator which provides a lower bound of the theoretical value. The algorithm that provides the low estimator is shown as Algorithm 3.

Algorithm 3 Derivative value estimation

(1) Load regression coefficients β_i ($i = 1, \ldots, M$).
(2) Simulate N independent paths $\{S_{j,1}, S_{j,2}, \ldots, S_{j,M}\}$ of the asset prices process from the first one used.
(3) Apply forward induction for $i = 1, \ldots, M-1$ and $j = 1, \ldots, N$.
 - Compute the continuation value $\kappa_i(S_{j,i}) = \beta_i^T \psi(S_{j,i})$ ($j = 1, \ldots, N$).
 - Compute the payoff functions $h_i(S_{j,i})$.

(4) At maturity time t_M, $\hat{W}_M(S_{j,M}) = h_M(S_{j,M})$ and $C_M(S_{j,M}) = 0$.
(5) Compute $\hat{W}_{j,0}(S_0) = h_{i^*}(S_{j,i^*})$ ($i^* = \min\{i \in \{1, \ldots, M\} : h_i(S_{j,i}) \geq \kappa_i(S_{j,i})\}$).
(6) Calculate the estimated value of the option: $\hat{W}_0(S_0) = \frac{1}{N} \sum_{j=1}^N \hat{W}_{j,0}$.

3.5. Duality. Upper bounds estimator using martingales

As we have seen in Section 3.4, the estimator of the American option, obtained using least square regression, was a lower estimator on the real American value. In this section an upper estimator using martingales is considered. For this purpose, we follow the works of Haugh and Kogan [9] and Rogers [14]. Both have established dual formulations which represent the price of an American option through a suitable minimization problem. The duality technique minimizes over a class of supermartingales or martingales and leads to a high-biased approximation, therefore obtaining upper bounds on prices.

As we have seen in (8), the discounted value $\hat{V}_i(S_i)$ satisfies the recursive formulation

$$
\begin{aligned}
&\hat{V}_M(s) = h(T, s)\,, \quad S_M = s \\
&\hat{V}_{i-1}(s) = \max\Bigg\{ h_{i-1}(s), \mathbb{E}_{t_{i-1}}\Bigg[\hat{V}_i(S_i) + \int_{t_{i-1}}^{t_i} e^{-r_0 u} \left[(R_B\lambda_B - \lambda_C) V(u, S(u))^- \right. \\
&\qquad\qquad \left. + (R_C\lambda_C + \lambda_B) V(u, S(u))^+ - s_F V(u, S(u))^+ \right] du \;\Bigg|\; S_{i-1} = s \Bigg]\Bigg\}\,,
\end{aligned}
$$

for $i = M, M-1, \ldots, 1$. From the previous recursive formula, the following inequality is obtained:

$$
\begin{aligned}
\hat{V}_i(S_i) &\geq \mathbb{E}_{t_i}\Bigg[\hat{V}_{i+1}(S_{i+1}) + \int_{t_i}^{t_{i+1}} e^{-r_0 u} \left[(R_B\lambda_B - \lambda_C) V(u, S(u))^- \right. \\
&\qquad \left. + (R_C\lambda_C + \lambda_B) V(u, S(u))^+ - s_F V(u, S(u))^+ \right] du \;\Bigg|\; S_i \Bigg] \\
&\geq \mathbb{E}_{t_i}\left[\hat{V}_{i+1}(S_{i+1}) \;\middle|\; S_i \right],
\end{aligned}
$$

for $i = 0, \ldots, M-1$. Thus, we can conclude that $\hat{V}_i$ is a supermartingale [12].

On the other hand, the American option price satisfies:

$$
\hat{V}_i(S_i) \geq h_i(S_i), \quad i = 0, \ldots, M\,.
$$

Thus, the value function process $\hat{V}_i(S_i)$ $(i = 0, \dots M)$ is the minimal supermartingale dominating $h_i(S_i)$ at each exercise time t_i.

Let $\mathcal{M} = \{\mathcal{M}_i, i = 0, \dots, M\}$ be a martingale, with $\mathcal{M}_0 = 0$. By the optimal stopping theorem of martingales, the expected value of a martingale at a stopping time is equal to the expected value of its initial value. Then, for any stopping time $\tau \in \{t_1, t_2, \dots, t_M\}$, we have $\mathbb{E}[\mathcal{M}_\tau] = \mathcal{M}_0 = 0$ and we can deduce:

$$\begin{aligned}\mathbb{E}_0\left[h_\tau(S_\tau) + \int_0^\tau e^{-r_0 u} g(V(u, S(u)))\, du\right] &= \mathbb{E}_0\left[h_\tau(S_\tau) + \int_0^\tau e^{-r_0 u} g(V(u, S(u)))\, du - \mathcal{M}_\tau\right] \\ &\leq \mathbb{E}_0\left[\max_{i=1,\dots,M}\left(h_i(S_i) + \int_0^{t_i} e^{-r_0 u} g(V(u, S(u)))\, du - \mathcal{M}_i\right)\right]. \end{aligned} \tag{14}$$

Moreover, in terms of the infimum over martingales $\mathcal{M}$ with initial value $\mathcal{M}_0 = 0$, we obtain

$$\begin{aligned}&\mathbb{E}_0\left[h_\tau(S_\tau) + \int_0^\tau e^{-r_0 u} g(V(u, S(u)))\, du\right] \\ &\qquad \leq \inf_{\mathcal{M}} \mathbb{E}_0\left[\max_{i=1,\dots,M}\left(h_i(S_i) + \int_0^{t_i} e^{-r_0 u} g(V(u, S(u)))\, du - \mathcal{M}_i\right)\right], \end{aligned} \tag{15}$$

which holds for any stopping time τ. Thus, the American option price written in terms of the supremum over τ leads to the following inequality:

$$\begin{aligned}\hat{V}_0(S_0) &= \sup_\tau \mathbb{E}_0\left[h_\tau(S_\tau) + \int_0^\tau e^{-r_0 u} g(V(u, S(u)))\, du\right] \\ &\leq \inf_{\mathcal{M}} \mathbb{E}_0\left[\max_{i=1,\dots,M}\left(h_i(S_i) + \int_0^{t_i} e^{-r_0 u} g(V(u, S(u)))\, du - \mathcal{M}_i\right)\right] \end{aligned} \tag{16}$$

for every martingale $\mathcal{M}$. The minimization problem on the right hand side is known as dual problem.

Next, let us consider the stochastic process defined by:

$$\mathcal{M}_0 = 0\,, \qquad \mathcal{M}_i = \sum_{k=1}^{i} \Delta_k\,, \quad i = 1, \dots, M\,, \tag{17}$$

where $\Delta_k = \hat{V}_k(S_k) - \mathbb{E}_{t_{k-1}}[\hat{V}_k(S_k) \mid S_{k-1}]$. We can easily prove that this process is a martingale, so that it satisfies (16). Furthermore, we can also prove [7]:

$$\hat{V}_0(S_0) = \mathbb{E}_0\left[\max_{i=1,\dots,M}\left\{h_i(S_i) + \int_0^{t_i} e^{-r_0 u} g(V(u, S(u)))\, du - \mathcal{M}_i\right\}\right]. \tag{18}$$

Thus, inequality (16) in the Appendix holds for our particular choice of martingale.

In the Appendix we detail the computation of an estimated martingale, $\hat{\mathcal{M}}$, close to the optimal one, $\mathcal{M}$, in order to obtain the following estimated value of $\hat{V}_0$:

$$\hat{W}_0(S_0) = \mathbb{E}_0\left[\max_{i=1,\dots,M}\left\{h_i(S_i) + \int_0^{t_i} e^{-r_0 u} g(W(u,S(u)))\,du - \hat{\mathcal{M}}_i\right\}\right], \tag{19}$$

which is the so called duality estimator. Algorithm 4 sketches the computation of this dual estimator.

Algorithm 4 Dual estimator using martingales

(1) Load regression coefficients β_i, $i = 1, \dots, M$ given by Algorithms 1 or 2
(2) Simulate N independent paths $\{S_{j,1}, S_{j,2}, \dots, S_{j,M}\}$ of the asset prices process.
(3) Set the initial martingale $\hat{\mathcal{M}}_0 = 0$
(4) For each $j = 1, \dots, N$, apply forward induction for $i = 1, \dots, M$.
- Compute the continuation values κ_i.
- Estimate the American option price, $\hat{W}_i(S_{j,i}) = \max\{h_i(S_{j,i}), \kappa_i(S_{j,i})\}$.
- Simulate N_S subpaths $\{S_{1,i}, S_{2,i}, \dots, S_{N_S,i}\}$ starting from $S_{j,i-1}$.
- Compute the estimation of the martingale differential $\hat{\Delta}_i$
- Obtain the martingales $\hat{\mathcal{M}}_i = \hat{\mathcal{M}}_{i-1} + \hat{\Delta}_i$

(5) Set $\hat{W}_{0,j}(S_0) = \max\limits_{i=1,\dots,M}\left(h_i(S_{j,i}) + \int_0^{t_i} e^{-r_0 u} g(V(u,S(u)))du - \hat{\mathcal{M}}_{j,i}\right)$.
(6) Compute the dual estimated value as $\hat{W}_0(S_0) = \frac{1}{N}\sum_{j=1}^N \hat{W}_{0,j}(S_0)$.

3.6. Confidence intervals

We take into account the lower and upper estimators developed in the previous sections to propose confidence intervals that contain the American option price.

We denote by $\underline{V}$ and $\overline{V}$ the lower and upper estimators, respectively, both computed with N paths. Then, the $(1-\alpha)$ confidence interval is given by

$$\left(\underline{V} - z_{\alpha/2}\frac{s_{\underline{V}}(N)}{\sqrt{N}}\ ,\ \overline{V} + z_{\alpha/2}\frac{s_{\overline{V}}(N)}{\sqrt{N}}\right),$$

where $s_{\underline{V}}(N)$ and $s_{\overline{V}}(N)$ denote the respective sample standard deviations and $z_{\alpha/2}$ represents the $(1-\alpha/2)$ quantile of the normal distribution.

4. The nonlinear problem ($M = \hat{V}$)

In the previous section we have deduced how to price the American option value considering counterparty risk, when the mark-to-market is equal to the risk-free derivative value. Two alternative algorithms have been proposed, transforming the classical Longstaff-Schwartz scheme. More precisely, Algorithm 1 consists of two nested Monte Carlo methods while Algorithm 2 combines a Monte Carlo method with an interpolation technique.

Now, when the mark-to-market value is equal to the price of the derivative with conterparty risk ($M = \hat{V}$), in the corresponding complementarity problem (3) we identify a nonlinear dependence on the solution $\hat{V}$. In this case, Feynman-Kac theorem [12] provides the risky American option value at time $t = 0$, which satisfies:

$$\hat{V}_0(S_0) = \sup_{\tau \in \mathcal{T}_0} \mathbb{E}_0 \left[e^{-r\tau} h^*(\tau, S_\tau) + \int_0^\tau e^{-ru} f(\hat{V}(u, S(u)))\, du \right],$$

where function f is defined by:

$$f(\hat{V}) = -(1 - R_B)\lambda_B \hat{V}^- - (1 - R_C)\lambda_C \hat{V}^+ - s_F \hat{V}^+ .$$

Recall that the asset prices follow the geometric Brownian motion process defined in (1). Once again, to simulate a continuously exercisable American option the period of time is discretized in $M + 1$ time steps. Thus, the asset price value at each time step is approximated by Euler-Maruyama scheme like in (6).

Now, using dynamic programming formulation, the American option value can be written in a recursive formula

$$\begin{aligned}
&\hat{V}_M(s) = h(T, s)\,, \quad S_M = s \\
&\hat{V}_{i-1}(s) = \max \left\{ h_{i-1}(s), \mathbb{E}_{t_{i-1}} \left[D_{0,i} \hat{V}_i^*(S_i) + \int_{t_{i-1}}^{t_i} e^{-ru} f(\hat{V}(u, S(u)))\, du \;\Big|\; S_{i-1} = s \right] \right\},
\end{aligned}$$

for $i = M, M-1, \ldots, 1$, the discount factor being defined as

$$D_{i-1,i} = e^{-r(t_i - t_{i-1})}.$$

Introducing the discount factor in each term, the recursive formula becomes:

$$\begin{aligned}
&\hat{V}_M(s) = h(T, s)\,, \quad S_M = s \\
&\hat{V}_{i-1}(s) = \max \left\{ h_{i-1}(s), \mathbb{E}_{t_{i-1}} \left[\hat{V}_i(S_i) + \int_{t_{i-1}}^{t_i} e^{-ru} f(\hat{V}(u, S(u)))\, du \;\Big|\; S_{i-1} = s \right] \right\},
\end{aligned}$$

for $i = 1, \ldots, M$.

Next, we write the continuation value, which is also approximated by a regression function, as follows:

$$\begin{aligned}
C_i(s) &= \mathbb{E}_{t_i} \left[\hat{V}_{i+1}(S_{i+1}) + \int_{t_i}^{t_{i+1}} e^{-ru} f(\hat{V}(u, S(u)))\, du \;\Big|\; S_i = s \right] \\
&= \sum_{j=1}^{J} b_{ij} \psi_j(s) = b_i^T \psi(s)\,. \qquad (20)
\end{aligned}$$

Let us remark that the main difference with respect to the case where the mark-to-market is equal to the risk-free derivative value arises in the continuation value, which leads to a different expression of d_i^ψ. Furthermore, the continuation value at time t_i

is defined in terms of the risky derivative value in the previous time step, which has been previously computed, and the risky derivative value at the same instant of time.

In order to deal with the nonlinear feature of this problem, we propose a fixed point algorithm to compute coefficients β_i as the estimators of b_i.

Algorithm 5 Regression coefficients β_i with fixed point iteration

(1) Simulate N independent paths $\{S_{j,1}, S_{j,2}, \ldots, S_{j,M}\}$ of the asset prices process.
(2) At maturity time t_M, $\hat{W}_M(S_{j,M}) = h_M(S_{j,M})$.
(3) Set the tolerance ϵ.
(4) For $i = M-1, \ldots, 1$, perform a fixed point algorithm:
- Initialize $\ell = 0$ and set $\hat{W}^0_{j,i} = \hat{W}_{j,i+1}$.
- Given the estimated value $\hat{W}_{j,i+1}$ $(j = 1, \ldots, N)$, compute A^ψ.
- Iterate the following steps while $e > \epsilon$
  - Compute $d_i^{\psi,\ell}$ in terms of $\hat{W}^\ell_{j,i}$.
  - Compute β_i as the solution of the linear system $A_i^\psi \beta_i^\ell = d_i^{\psi,\ell}$.
  - Estimate the continuation value $\kappa_i(S_{j,i}) = \beta_i^T \psi(S_{j,i})$ for $j = 1, \ldots, N$.
  - Compute $\hat{W}^{\ell+1}_{j,i} = \max\{h_i(S_{j,i}), \kappa_i(S_{j,i})\}$.
  - $e = \|\hat{W}^{\ell+1}_{j,i} - \hat{W}^\ell_{j,i}\|$ and set $\ell = \ell + 1$

(5) Save the regression coefficients β_i to compute the risky derivative value.

Therefore, to obtain the lower estimator of the risky derivative value at time $t = 0$ we apply Algorithm 3, using the β coefficients obtained with Algorithm 5.

Using a similar procedure that in the linear complementarity problem, when $M = V$, an upper estimator of the derivative value can be obtained. In this case, after computing the regression coefficients β_i by Algorithm 5, we apply Algorithm 4 to obtain the estimator of the American option value. Remark that function $g(V)$ in Algorithm 4 is replaced by function $f(\hat{V})$. Again the confidence intervals are obtained like in Section 3.6.

5. Numerical results

In this section, some numerical results are presented. Our aim is to compare the Monte Carlo algorithms here proposed with the numerical methods previously applied in [1] to the corresponding PDE formulations.

In all examples, the initially chosen financial parameters are: $K = 15$, $r = 0.04$, $r_R = 0.06$, $\sigma = 0.25$, $R_B = R_C = 0.3$, $\lambda_B = \lambda_C = 0.04$, $s_F = (1 - R_B)\lambda_B$ and $T = 0.5$. We will also show the sensitivity of the option price with respect to parameters λ_B, λ_C, R_B and R_C by shifting these initial values.

For the numerical simulation with Monte Carlo techniques, we have used 500 paths and 1000 time steps. In particular, for Algorithm 1 we have additionally considered 8 inner paths, while for Algorithm 4 we use $N_S = 50$. Moreover, we consider a basis consisting of three Laguerre polynomials in the regression formula (11).

For solving the PDE formulations, we consider a combination of characteristics method for time discretization, finite elements for spatial discretization, a fixed point technique for the nonlinearity and ALAS algorithm for the solution of the resulting obstacle problem [1]. For this purpose, we use a spatial mesh with 601 asset nodes and 200 time steps.

5.1. Example with mark-to-market $M = V$

Table 1 presents some numerical results obtained when the mark-to-market is $M = V$. More precisely, for different underlying prices, the numerical solution of the linear complementarity problem (2), the lower (13) and upper (19) estimators and the 99% confidence interval are shown jointly with the excercise value.

The numerical solution of (2) is computed with the techinques described in [1]. We can appreciate that it lies in the confidence interval, except in the first critical case for $S = 0$ where Monte Carlo approximation is very close to the exercise value. For the larger underlying prices ($S \geq 25$), all values become naturally close to zero, as expected.

Table 1. American option value considering counterparty risk and $M = V$ (Algorithms 1 and 4).

S	Pay-off	Complementarity problem approximation	Lower estimator	Upper estimator	Confidence interval
0.0	15.0	15.00000000	14.99910003	14.99919002	(14.99910003 , 14.99919002)
2.5	12.5	12.50000000	12.49809274	12.50113807	(12.49633159 , 12.50151772)
5.0	10.0	10.00000000	10.00054706	10.00902565	(9.99385387 , 10.01043362)
7.5	7.5	7.50000000	7.49881347	7.52529594	(7.49403593 , 7.52565934)
10.0	5.0	5.00000000	5.00256029	5.05869954	(4.99576292 , 5.05925524)
12.5	2.5	2.51600182	2.44617335	2.67576415	(2.41916259 , 2.67817891)
15.0	0.0	0.87745370	0.85460499	1.06069986	(0.84971491 , 1.06301242)
17.5	0.0	0.22907711	0.18793087	0.49197404	(0.17322978 , 0.49335142)
20.0	0.0	0.04701583	0.04818037	0.31265812	(0.04221081 , 0.31332163)
22.5	0.0	0.00810599	0.00652163	0.15992887	(0.00431714 , 0.16022584)
25.0	0.0	0.00124366	0.00057994	0.01382764	(-0.00003049 , 0.01403672)
27.5	0.0	0.00017746	0.00002964	0.00015364	(-0.00001945 , 0.00159181)
30.0	0.0	0.00002432	0.00001639	0.00003384	(-0.00002584 , 0.00009372)

A similar behavior is observed with Algorithm 2, where the risk-free price V is interpolated from the values previously obtained in a thin mesh for the asset, instead of being computed by an inner Monte Carlo algorithm (see Table 2).

Table 2. American option value considering counterparty risk and $M = V$ (Algorithms 2 and 4).

S	Pay-off	Complementarity problem approximation	Lower estimator	Upper estimator	Confidence interval
0.0	15.0	15.00000000	14.99910003	14.99919002	(14.99910003 , 14.99919002)
2.5	12.5	12.50000000	12.49878788	12.50074403	(12.49701990 , 12.50109953)
5.0	10.0	10.00000000	9.99825352	10.00895521	(9.99516292 , 10.01031047)
7.5	7.5	7.50000000	7.49814808	7.52543188	(7.48526493 , 7.52874533)
10.0	5.0	5.00000000	5.00220584	5.05834509	(4.98403716 , 5.06774399)
12.5	2.5	2.51600182	2.46371870	2.67394114	(2.30502657 , 2.68898177)
15.0	0.0	0.87745370	0.84859226	1.06127342	(0.70565317 , 1.08186773)
17.5	0.0	0.22907711	0.20054059	0.49115613	(0.13612061 , 0.51495675)
20.0	0.0	0.04701583	0.03959987	0.31201452	(0.01426014 , 0.32890688)
22.5	0.0	0.00810599	0.00823601	0.16045927	(-0.00358441 , 0.16930795)
25.0	0.0	0.00124366	0.00052809	0.01281424	(-0.00048394 , 0.01380686)
27.5	0.0	0.00017746	0.00000000	0.00003246	(0.00000000 , 0.00003439)
30.0	0.0	0.00002432	0.00000000	0.00000442	(0.00000000 , 0.00000468)

In order to compare the efficency of algorithms 1 and 2, we have measured the ellapsed CPU time in both cases. In all examples, tests have been performed by using Matlab on an Intel(R) Xeon(R) CPU E3-1241 3.50 GHz computer. Algorithm 1 takes 55134 seconds for lower estimator and 37390 for upper estimator. However, Algorithm 2 needs 5.4863 seconds to obtain the regression coefficients. Note that when using Algorithm 2, a high computational time is employed to obtain the risk-free derivative

Table 3. American option value considering counterparty risk and $M = \hat{V}$ (Algorithms 5 and 4).

S	Pay-off	Complementarity problem approximation	Lower estimator	Upper estimator	Confidence interval
0.0	15.0	15.00000000	14.99970000	14.99979001	(14.99970000 , 14.99979001)
2.5	12.5	12.50000000	12.49987807	12.50229450	(12.49825098 , 12.50281445)
5.0	10.0	10.00000000	9.99848975	10.01481249	(9.99246462 , 10.01684125)
7.5	7.5	7.50000000	7.50417414	7.53459612	(7.49600267 , 7.53920122)
10.0	5.0	5.00000000	4.96081214	5.07280533	(4.90694567 , 5.08156060)
12.5	2.5	2.52410327	2.32223198	2.69563668	(2.15396736 , 2.71232058)
15.0	0.0	0.89012915	0.81039447	1.08655935	(0.66794198 , 1.10885981)
17.5	0.0	0.23345859	0.23289102	0.50408124	(0.15736865 , 0.52928497)
20.0	0.0	0.04802108	0.04619341	0.31320212	(0.01305985 , 0.32997245)
22.5	0.0	0.00828968	0.01042668	0.14064556	(-0.00236734 , 0.14839822)
25.0	0.0	0.00127290	0.00036507	0.02547380	(-0.00037522 , 0.02682979)
27.5	0.0	0.00018174	0.00000000	0.00000000	(0.00000000 , 0.00000000)
30.0	0.0	0.00002492	0.00000000	0.00000000	(0.00000000 , 0.00000000)

value on the thin mesh used to interpolate. More precisely, it takes 215250 seconds to obtain the lower and upper estimators of the risk-free derivative price for the whole set of asset nodes. Furthermore, Algorithms 3 and 4 take 0.0759 and 2.1875 seconds, respectively, for the computation of the risky American option price.

All these computational times correspond to the approximation of the option price for just one asset price. We can observe that the interpolation of the risk-less option values implies a larger time in obtaining the lower and upper estimators for a unique initial asset price. Nevertheless, once the values of the riskless derivative on the fine mesh are available, the computation of the option price for several asset prices by Algorithm 2 (interpolation) is much more efficient than Algorithm 1 (inner Longstaff-Schwartz scheme). Indeed, only 6 additional seconds per asset price are required in Algorithm 2.

Alternatively, the numerical solution of the complementarity problem (2) results clearly more efficient, as only 6.89 seconds are needed to approximate the solution on a mesh of 601 nodes (each node represents an initial asset price) and 200 time steps.

5.2. Example with mark-to-market $M = \hat{V}$

Table 3 shows the results obtained in the example with mark-to-market $M = \hat{V}$, which corresponds to PDE formulation (3). The associated Monte Carlo technique has been described in Section 4. In this example, Algorithm 5 takes 6.2608 seconds, while the numerical methods [1] employed to approximate the solution of the nonlinear complementarity problem take 270 seconds with a 601 nodes mesh and 200 time steps. We point out the good agreement between the values computed from the PDE formulation and the confidence intervals obtained with the proposed Monte Carlo technique.

We can also analyze the influence of different parameters on the value of the option. Table 4 shows, for an initial price S=20, the numerical solution of the complementarity problem, the Monte Carlo lower and upper estimators, and the confidence intervals computed for different values of the intensity of default λ_B. As expected, we appreciate that for increasing values of this parameter both estimators decrease. We have observed the same effect when we have fixed λ_B and taken different increasing values for the intensity of default λ_C.

A similar behavior, in the opposite sense, is observed when we increase the recovery rates R_B or R_C. Tables 5 and 6 show the obtained results for $S_0 = 20$.

Table 4. American option value considering counterparty risk and $M = \hat{V}$ (Algorithms 5 and 4). Effect of the intensity of default. $S_0 = 20$, $\lambda_C = 0.04$, $R_B = R_C = 0.30$.

λ_B	Complementarity problem approximation	Lower estimator	Upper estimator	Confidence interval
0.04	0.04802108	0.04942329	0.31842569	(0.01289458 , 0.33546431)
0.10	0.04715281	0.04930002	0.31715650	(0.01287656 , 0.33380921)
0.30	0.04439205	0.04895565	0.30504576	(0.01282437 , 0.32139431)

Table 5. American option value considering counterparty risk and $M = \hat{V}$ (Algorithms 5 and 4). Effect of the recovery rate. $S_0 = 20$, $\lambda_B = \lambda_C = 0.30$, $R_C = 0.30$.

R_B	Complementarity problem approximation	Lower estimator	Upper estimator	Confidence interval
0.10	0.04005223	0.04732412	0.29536326	(0.01512059 , 0.31200451)
0.30	0.04107955	0.04766287	0.30351023	(0.01513649 , 0.32043366)
0.90	0.04435412	0.04790897	0.31169431	(0.01514366 , 0.32841816)

Table 6. American option value considering counterparty risk and $M = \hat{V}$ (Algorithms 5 and 4). Effect of the recovery rate. $S_0 = 20$, $\lambda_B = \lambda_C = 0.30$, $R_B = 0.30$.

R_C	Complementarity problem approximation	Lower estimator	Upper estimator	Confidence interval
0.10	0.04005223	0.04732412	0.29615655	(0.01512059 , 0.31268850)
0.30	0.04107955	0.04766287	0.30497545	(0.01513649 , 0.32197302)
0.90	0.04435412	0.04790897	0.31058947	(0.01514366 , 0.32782823)

6. Conclusions

In this work we have addressed the computation of American option prices, when counterparty risk is taken into account. More precisely, for the most usual couple of choices of the mark-to-market value of the American option at default, we express the option price in terms of expectations involving the optimal stopping times. Moreover, when the mark-to-market is equal to the option price without counterparty risk we propose two algorithms: a first one requiring two nested Monte Carlo loops, and a second one considering a suitable interpolation technique for the risk-free option price. When the mark-to-market value at default is equal to the risky option price, a fixed point iteration is considered. The proposed techniques involve the computation of lower and upper estimators to build up a confidence interval for the American option price. These estimators are obtained by extending some previous results from [10] and [7].

Although we have only considered constant spreads, we note that the proposed methodology can be extended to a nonconstant spreads setting, as well as to other financial products with early exercise feature such as callable bonds or Bermudan swaptions, for example. Moreover, from the computational perspective the use of parallel computing techniques (like those ones related to multi-CPUs or GPUs) would allow a high speed up of the involved algorithms. These parallel computing tools result very efficient for the here considered Monte Carlo based techniques.

Acknowledgments

This work has been financially supported by the Spanish government (Ministerio de Economía y Competitividad, project MTM2016-76497-R) and Xunta de Galicia (grant GRC2014/044, including FEDER funds). Second author is partially supported by Spanish Predoc Fellowship BES-2014-070782.

The authors are very grateful to both reviewers for their valuable comments and suggestions, which contributed to improve the manuscript.

References

[1] I. Arregui, B. Salvador, and C. Vázquez, *PDE models and numerical methods for total value adjustment in European and American options with counterparty risk*, Appl. Math. Comput. 308 (2017), pp. 31–53.

[2] I. Arregui, B. Salvador, D. Ševčovič, and C. Vázquez, *Total value adjustment for european options with two stochastic factors. Mathematical model, analysis and numerical simulation*, Preprint submitted for publication .

[3] D. Brigo and A. Capponi, *Bilateral counterparty risk valuation with stochastic dynamical models and applications to CDSs* (2009).

[4] C. Burgard and M. Kjaer, *PDE representations of derivatives with bilateral counterparty risk and funding costs*, The Journal of Credit Risk 7 (2011), pp. 1–19.

[5] S. Crépey, *Bilateral counterparty risk under funding constraints-part i: Pricing*, Mathematical Finance 25 (2015), pp. 1–22.

[6] S. Crépey, *Bilateral counterparty risk under funding constraints-part ii: CVA*, Mathematical Finance 25 (2015), pp. 23–50.

[7] P. Glasserman, *Monte Carlo Methods in Financial Engineering*, Springer, 2003.

[8] J. Gregory, *Counterparty credit risk and credit value adjustment*, Wiley Finance, 2012.

[9] M.B. Haugh and L. Kogan, *Pricing american options: a duality approach*, Operations Research 52 (2004), pp. 258–270.

[10] F.A. Longstaff and E.S. Schwartz, *Valuing American options by simulation: A simple least-squares approach*, Rev. Financ. Stud. 14 (2001), pp. 113–147.

[11] A. Pallavicini, D. Perini, and D. Brigo, *Funding, collateral and hedging: A consistent framework included CVA, DVA, collateral, netting rules and re-hypotecation*, SSRN Electronic Journal (2011), pp. 18–24.

[12] A. Pascucci, *PDE and Martingale Methods in Option Pricing*, Springer, 2011.

[13] V. Piterbarg, *Funding beyond discounting: collateral agreements and derivatives pricing*, Risk Magazine 23 (2010), pp. 97–102.

[14] L.C.G. Rogers, *Monte Carlo valuation of American options*, Mathematical Finance 12 (2002), pp. 271–286.

[15] P. Wilmott, S. Howison, and J. Dewynne, *The mathematics of financial derivatives. A students introduction*, Cambridge University Press, Cambridge, 1996.

[16] P. Wilmott, S. Howison, and J. Dewynne, *Option pricing: Mathematical models and computation*, Oxford Financial Press, Oxford, 1996.

Appendix: Duality, upper bounds estimator using martingales

As indicated in Section 3.4, by using least square regression, a lower estimator of the American option price with counterparty risk can be obtained. In this Appendix we detail the statement of an upper estimator using martingales, by following Haugh and Kogan [9] and Rogers [14], where dual formulations which represent the American option price without counterparty risk through a minimization problem are obtained. The duality minimizes over a class of supermartingales or martingales and leads to a upper-biased estimator, thus obtaining upper bounds on prices.

As we have seen, the discounted value $\hat{V}_i(S_i)$ satisfies the recursive formulation (8). From this recursive formula, the following inequality is obtained:

$$\hat{V}_i(S_i) \geq \mathbb{E}_{t_i}\Big[\hat{V}_{i+1}(S_{i+1}) + \int_{t_i}^{t_{i+1}} e^{-r_0 u}\left[(R_B\lambda_B - \lambda_C)V(u,S(u))^- \right.$$
$$\left. +(R_C\lambda_C + \lambda_B)V(u,S(u))^+ - s_F V(u,S(u))^+\right] du \mid S_i\Big] \quad i = 0,\dots,M-1,$$

which indicates that $\hat{V}_i(S_i)$ is a supermartingale. On other hand, we also have

$$\hat{V}_i(S_i) \geq h_i(S_i), \quad i = 0,\dots,M.$$

So, the value function process $\hat{V}_i(S_i), \quad i = 0,\dots M$ is the minimal supermartingale dominating $h_i(S_i)$ at each exercise time t_i.

Let $\mathcal{M} = \{\mathcal{M}_i, i = 0,\dots,M\}$ be a martingale, with $\mathcal{M}_0 = 0$. By the optional stopping times theorem of martingales, the expected value of a martingale at a stopping time is equal to the expected value of its initial value. Then, for any stopping time $\tau \in \{t_1, t_2, \dots, t_M\}$, $\mathbb{E}[\mathcal{M}_\tau] = \mathcal{M}_0 = 0$, and we have

$$\mathbb{E}_0\left[h_\tau(S_\tau) + \int_0^\tau e^{-r_0 u} g(V(u,S(u)))du\right] = \mathbb{E}_0\left[h_\tau(S_\tau) + \int_0^\tau e^{-r_0 u} g(V(u,S(u)))\,du - \mathcal{M}_\tau\right]$$
$$\leq \mathbb{E}_0\left[\max_{i=1,\dots,M}\left(h_i(S_i) + \int_0^{t_i} e^{-r_0 u} g(V(u,S(u)))\,du - \mathcal{M}_i\right)\right]. \quad (21)$$

Moreover, in terms of the infimum over the martingales $\mathcal{M}$ with initial value $\mathcal{M}_0 = 0$, we obtain

$$\mathbb{E}_0\left[h_\tau(S_\tau) + \int_0^\tau e^{-r_0 u} g(V(u,S(u)))\,du\right]$$
$$\leq \inf_{\mathcal{M}} \mathbb{E}_0\left[\max_{i=1,\dots,M}\left(h_i(S_i) + \int_0^{t_i} e^{-r_0 u} g(V(u,S(u)))\,du - \mathcal{M}_i\right)\right], \quad (22)$$

which remains true for any stopping time τ. So, the American option price written in terms of the supremum over τ leads to (16). The minimization problem on the right hand side of (16) is known as the dual problem.

Next, we prove the equality for the particular martingale defined as follows:

$$\mathcal{M}_0 \;=\; 0, \qquad \mathcal{M}_i \;=\; \sum_{k=1}^{i} \Delta_k, \quad i=1,\dots,M, \tag{23}$$

where $\Delta_k = \hat{V}_k(S_k) - \mathbb{E}_{t_{k-1}}[\hat{V}_k(S_k) \mid S_{k-1}]$.

In a first step, we prove that the previously defined $\mathcal{M}$ satisfies the martingale property. Thus, taking into account the definition of Δ_k, we have

$$\mathbb{E}_{t_{i-1}}[\Delta_i \mid S_{i-1}] = \mathbb{E}_{t_{i-1}}\Big[\hat{V}_i(S_i) - \mathbb{E}_{t_{i-1}}[\hat{V}_i(S_i) \mid S_{i-1}] \mid S_{i-1}\Big] = 0.$$

For this purpose, first we have

$$\mathbb{E}_{t_{i-1}}[\mathcal{M}_i \mid S_{i-1}] = \mathbb{E}_{t_{i-1}}\Big[\sum_{k=1}^{i} \Delta_k \mid S_{i-1}\Big] = \sum_{k=1}^{i-1} \Delta_k = \mathcal{M}_{i-1}, \tag{24}$$

which shows that $\mathcal{M}$ satisfies the martingale property.

Next, we use backward induction to prove that

$$\begin{aligned}\hat{V}_i(S_i) = \mathbb{E}_{t_i}\Bigg[&\max\Big\{h_i(S_i) + \int_{t_i}^{t_i} e^{-r_0 u} g(V(u,S(u)))du,\\ &\qquad h_{i+1}(S_{i+1}) + \int_{t_i}^{t_{i+1}} e^{-r_0 u} g(V(u,S(u)))du - \Delta_{i+1},\\ &\qquad h_{i+2}(S_{i+2}) + \int_{t_i}^{t_{i+2}} e^{-r_0 u} g(V(u,S(u)))du - \Delta_{i+2} - \Delta_{i+1}, \dots,\\ &\qquad h_M + \int_{t_i}^{t_M} e^{-r_0 u} g(V(u,S(u)))du - \Delta_M - \ldots - \Delta_{i+1}\Big\} \;\Big|\; S_i\Bigg]\end{aligned} \tag{25}$$

For the maturity time t_M, we have $\hat{V}_M(S_M) = h_M(S_M) = \mathbb{E}[h_M(S_M) \mid S_M]$. So, the equality (25) is satisfied.

Next, we assume that (25) is satisfied at time t_i. Next, we obtain

$$\begin{aligned}\hat{V}_{i-1}(S_{i-1}) &= \max\left\{h_{i-1}(S_{i-1}), \mathbb{E}_{t_i}\left[\hat{V}_i(S_i) + \int_{t_{i-1}}^{t_i} e^{-r_0 u} g(V(u,S(u)))du \;\Big|\; S_{i-1}\right]\right\}\\ &= \mathbb{E}_{t_{i-1}}\left[\max\Big\{h_{i-1}(S_{i-1}), \mathbb{E}_{t_i}\left[\hat{V}_i(S_i) + \int_{t_{i-1}}^{t_i} e^{-r_0 u} g(V(u,S(u)))du \;\Big|\; S_{i-1}\right]\Big\} \;\Big|\; S_{i-1}\right]\\ &= \mathbb{E}_{t_{i-1}}\left[\max\Big\{h_{i-1}(S_{i-1}), \hat{V}_i(S_i) + \int_{t_{i-1}}^{t_i} e^{-r_0 u} g(V(u,S(u)))du - \Delta_i\} \;\Big|\; S_{i-1}\right]\\ &= \mathbb{E}_{t_{i-1}}\Bigg[\max\Big\{h_{i-1}(S_{i-1}), h_i(S_i) + \int_{t_{i-1}}^{t_i} e^{-r_0 u} g(V(u,S(u)))du - \Delta_i, \dots,\\ &\qquad h_{i+1}(S_{i+1}) + \int_{t_i}^{t_{i+1}} e^{-r_0 u} g(V(u,S(u)))du - \Delta_{i+1} - \Delta_i, \dots,\end{aligned}$$

$$h_M(S_M) + \int_{t_{M-1}}^{t_M} e^{-r_0 u} g(V(u,S(u)))du - \Delta_M - \ldots - \Delta_i \Big\} \;\Big|\; S_{i-1} \Bigg],$$

so that (25) also holds for t_{i-1}. Finally, at $t = t_0$ the American option value is given by

$$\begin{aligned}\hat{V}_0(S_0) &= \mathbb{E}_0\left[\hat{V}_1(S_1) + \int_0^{t_1} e^{-r_0 u} g(V(u,S(u)))\,du \;\Big|\; S_0\right] \\ &= \hat{V}_1(S_1) + \int_0^{t_1} e^{-r_0 u} g(V(u,S(u)))du - \Delta_1. \end{aligned} \tag{26}$$

Moreover, according to (25)

$$\begin{aligned}\hat{V}_1(S_1) = \mathbb{E}_{t_1}\Bigg[\max\Big\{ & h_1(S_1) + \int_{t_1}^{t_1} e^{-r_0 u} g(V(u,S(u)))\,du, \\ & h_2(S_2) + \int_{t_1}^{t_2} e^{-r_0 u} g(V(u,S(u)))\,du - \Delta_2, \\ & h_3(S_3) + \int_{t_1}^{t_3} e^{-r_0 u} g(V(u,S(u)))\,du - \Delta_3 - \Delta_2, \ldots, \\ & h_M(S_M) + \int_{t_1}^{t_M} e^{-r_0 u} g(V(u,S(u)))\,du - \Delta_M - \ldots - \Delta_2 \Big\} \;\Big|\; S_1\Bigg]. \end{aligned} \tag{27}$$

Then, we have

$$\begin{aligned}\hat{V}_0(S_0) = \mathbb{E}_{t_1}\Bigg[\max\Big\{ & h_1(S_1) + \int_{0}^{t_1} e^{-r_0 u} g(V(u,S(u)))\,du - \Delta_1, \\ & h_2(S_2) + \int_{0}^{t_2} e^{-r_0 u} g(V(u,S(u)))\,du - \Delta_2 - \Delta_1, \\ & h_3(S_3) + \int_{0}^{t_3} e^{-r_0 u} g(V(u,S(u)))\,du - \Delta_3 - \Delta_2 - \Delta_1, \ldots, \\ & h_M(S_M) + \int_{0}^{t_M} e^{-r_0 u} g(V(u,S(u)))\,du - \Delta_M - \ldots - \Delta_2 \Big\} \;\Big|\; S_1\Bigg]. \end{aligned} \tag{28}$$

Finally, we get

$$\hat{V}_0(S_0) = \mathbb{E}_0\left[\max_{i=1,\ldots,M}\left\{h_i(S_i) + \int_0^{t_i} e^{-r_0 u} g(V(u,S(u)))\,du - \mathcal{M}_i\right\}\right], \tag{29}$$

which proves the inequality (16) for the martingale defined by (23).

Moreover, by (29) we have obtained an upper estimator for the American options price with counterparty risk.

Next, for practical purposes, the goal is to find a computable estimated martingale $\hat{\mathcal{M}}$ close to the optimal martingale $\mathcal{M}$, to obtain an estimated value of $\hat{V}_0$ in the form

$$\hat{W}_0(S_0) = \mathbb{E}_0\left[\max_{i=1,\ldots,M}\left\{h_i(S_i) + \int_0^{t_i} e^{-r_0 u} g(W(u,S(u)))du - \hat{\mathcal{M}}_i\right\}\right], \tag{30}$$

which is the so called duality estimator.

Next, we construct the martingale $\hat{\mathcal{M}}_i$. Thus, we follow the definition given in (23) to find the suitable martingale.

$$\hat{\mathcal{M}}_0 = 0\,, \qquad\qquad \hat{\mathcal{M}}_i = \sum_{k=1}^{i} \hat{\Delta}_k, \quad i = 1,\ldots,M, \tag{31}$$

where $\hat{\Delta}_k$ is given by $\hat{\Delta}_i = \hat{W}_i(S_i) - \mathbb{E}_{t_{i-1}}[\hat{W}_i(S_i) \mid S_{i-1}]$. Then, $\hat{\mathcal{M}}$ satisfies the general martingale property.

Note that $\hat{\Delta}_k$ is now expressed in terms of the estimated value of the American options, which was given by

$$\hat{W}_0 = \max\{h_i(S_i), \kappa_i(S_i)\}. \tag{32}$$

where κ_i was defined in (12). In (12) the vector β_i and the function bases ψ are the same as for least square method.

Next, we explain how to estimate the martingale value. For this purpose, we assume that we have simulated the main Monte Carlo paths $\{S_{j,i}, j = 1,\ldots,N\}$. Then, for each S_{i-1} we simulate N_S successors $\{\tilde{S}_{i,k}, k = 1,\ldots,N_S\}$, and estimate the conditional expectation $\mathbb{E}_{t_{i-1}}[\hat{W}_i(S_i) \mid S_{i-1}]$ by

$$\mathbb{E}_{t_{i-1}}[\hat{W}_i(S_i) \mid S_{i-1}] = \frac{1}{N_S}\sum_{k=1}^{N_S} \hat{W}_i(\tilde{S}_{k,i}), \tag{33}$$

where $\hat{W}_i(\tilde{S}_{k,i})$ is calculated as in (32). Then, the estimated value $\hat{\Delta}_i$ is given by

$$\hat{\Delta}_i = \hat{W}_i(S_i) - \frac{1}{M}\sum_{k=1}^{M} \hat{W}_i(\tilde{S}_{k,i}) \tag{34}$$

which gives the upper-biased estimator.