Numerical solution of an optimal investment problem with proportional transaction costs ☆

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Abstract

This paper mainly concerns the numerical solution of a nonlinear parabolic double obstacle problem arising in a finite-horizon optimal investment problem with proportional transaction costs. The problem is initially posed in terms of an evolutive HJB equation with gradient constraints and the properties of the utility function allow to obtain the optimal investment solution from a nonlinear problem posed in one spatial variable. The proposed numerical methods mainly consist of a localization procedure to pose the problem on a bounded domain, a characteristics method for time discretization to deal with the large gradients of the solution, a Newton algorithm to solve the nonlinear term in the governing equation and a projected relaxation scheme to cope with the double obstacle (free boundary) feature. Moreover, piecewise linear Lagrange finite elements for spatial discretization are considered. Numerical results illustrate the performance of the set of numerical techniques by recovering all qualitative properties proved in [3].

Key words: Optimal investment, transaction costs, double obstacle problems, free boundaries, numerical methods, characteristics scheme, finite elements.

1. Introduction

This paper concerns the numerical solution of an optimal investment problem in the presence of proportional transactions costs and a finite time horizon. This problem first mainly assumes the existence of a constant relative risk aversion (CRRA) investor whose wealth is partly invested in risky stocks and the rest in a riskless bank account. In the absence of transaction costs, the tools of continuous stochastic calculus allow to formulate the problem in terms of a Hamilton-Jacobi-Bellman equation in [12], the solution of which can be exactly obtained and consists of maintaining a constant proportion of the wealth in the

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bank account and in the asset during the investment period (Merton line). In order to maintain this proportion, the strategy requires a continuous trading, thus incurring in enormous transaction costs that makes the strategy unfeasible and the assumption of the lack of transaction costs unrealistic. This is the reason why proportional transaction costs were introduced in [11], where the consequent existence of a no transaction region is heuristically argued. Next, in [5] the formulation of the optimal policy as a nonlinear free boundary problem is posed. In this setting, the free boundaries separating the buying and selling regions from the no transaction one are additional unknowns. Since this seminal paper, different authors have characterized the existence of an optimal policy: for example, in [13] by means of viscosity solutions to HJB equations or in [2] with the tools of martingale theory.

More recently, in the setting of stochastic control problems and PDEs, the singular control problem is equivalently posed in terms of a nonlinear double obstacle problem associated to the spatial derivative of a transformed solution of the optimal value function in [3]. From the continuity of its solution, this approach allows to recover the *smooth fit condition*, formerly stated in [13] by means of viscosity solutions and in [5] with an ODE approach. Moreover, the well developed tools for double obstacle problems allow to obtain not only the existence and uniqueness of solution but also important qualitative properties of the solution and the free boundaries.

In the present paper we propose a set of numerical techniques that allow to solve the nonlinear double obstacle problem. Thus, first a localization procedure is proposed to pose a formulation in a suitable bounded domain so that the solution of the new problem is not affected neither by the locations of the new boundaries of the bounded domain nor by the consideration of particular boundary conditions. Next, in order to deal with the convection dominated nonlinear parabolic equation leading to large gradients on the solution (mainly located near the lower spatial boundary), a first order characteristics method for the time discretization is proposed [14]. The nonlinear term in the equation is treated by means of a Newton method and, at each step of it, the double obstacle feature is addressed by means of a projected relaxation numerical technique [7].

In order to illustrate the performance of the proposed numerical methods, the different qualitative properties (theoretically stated in [3]) of the solution and the free boundaries are verified. Among them, the semi-analytical solution of the steady state problem has also been tested. Moreover, from the computed solution of the double obstacle problem, the solution of the departure optimal investment problem in terms of the bank account and asset initial values can be represented as well as the corresponding financial regions: buying, selling and no transaction regions.

The paper is organized as follows. In Section 2 the optimal investment problem with proportional transaction costs is presented. Section 3 is devoted to the statement of the nonlinear double obstacle problem formulation and the main theoretical results stated in [3]. In Section 4, the proposed numerical techniques for the various difficulties of the problem are described. In Section 5, several examples illustrate the performance of the numerical methods and the last section presents some conclusions.

2. The optimal investment problem

We consider an optimal investment problem with transaction costs, the model being the same as in [5] or [13].

Nevertheless, we first present the case without transaction costs as proposed in [12], the solution of which can be analytically obtained. Let us suppose a CRRA investor who holds X_t and Y_t in bank and stock accounts, respectively. In the absence of transaction costs, their evolution is given by [12]:

$$\begin{cases} dX_t = rX_t\,dt, & X_0 = x, \\ dY_t = \alpha Y_t\,dt + \sigma Y_t\,d\mathcal{B}_t, & Y_0 = y, \end{cases}$$

where r is the constant risk free interest rate, $\alpha > r$ is the constant expected rate of return of the stock, $\sigma > 0$ is the constant volatility of the stock, $\mathcal{B}_t$ is a standard Brownian process that takes into account the stochastic component of the real process, and x and y are the initial values in monetary terms of the bank and stock accounts, respectively.

Note that in the model without transaction costs, the optimal strategy is to keep along time a constant proportion of the wealth in the bank account and in the stock [12]. More precisely, the strategy requires that

$$\frac{X_t}{Y_t} = -\frac{\alpha - r - (1-\gamma)\sigma^2}{\alpha - r} = x_M\,.$$

This constant proportion, x_M, is classically known as the *Merton line*. In order to maintain this proportion along time, the portfolio has to be continuously updated in time due to changes in stock value. Thus, the optimal strategy involves in practice no negligible transaction costs.

Therefore, in the present paper the existence of proportional transaction costs associated to buy and sell actions is assumed. That is, the costs are equal to a fixed percentage of the transacted amount. Thus, in the presence of transaction costs, the evolution of X_t and Y_t is governed by the following stochastic equations:

$$\begin{cases} dX_t = rX_t\,dt - (1+\lambda)\,dL_t + (1-\mu)\,dM_t\,, & X_0 = x, \\ dY_t = \alpha Y_t\,dt + \sigma Y_t\,d\mathcal{B}_t + dL_t - dM_t\,, & Y_0 = y, \end{cases} \tag{1}$$

where L_t and M_t denote right–continuous, nonnegative and nondecreasing processes representing cumulative money values for the purpose of buying and selling stock, respectively. Moreover, $\lambda \in [0,\infty)$ and $\mu \in [0,1)$ account for constant proportional transaction costs incurred on purchase and sale of stock, respectively.

Thus, the investor's net wealth at time t is given by

$$W_t = \begin{cases} X_t + (1-\mu)Y_t\,, & \text{if } Y_t \geq 0\,, \\ X_t + (1+\lambda)Y_t\,, & \text{if } Y_t < 0\,, \end{cases}$$

and the solvency region can be defined as

$$\mathcal{S} = \{(x,y) \in \mathbb{R}^2 \,/\, x + (1+\lambda)y > 0, \quad x + (1-\mu)y > 0\}\,.$$

Given an initial position $(X_t, Y_t) = (x,y) \in \mathcal{S}$, an investment strategy (L, M) is admissible if and only if $(X_s, Y_s) \in \mathcal{S}$ for all $s \in [t, T]$. Let $A_t(x,y)$ be the set of admissible investment strategies. The investor's problem consists of choosing an admissible strategy so as to maximize, at initial time t, the expected utility of terminal wealth, that is,

$$\sup_{(L,M)\in A_t(x,y)} E_t^{x,y}[U(W_T)]$$

subject to (1), where:

- $E_t^{x,y}$ denotes the conditional expectation at time t given an initial endowment $(X_t, Y_t) = (x,y)$
- U denotes the utility function, that belongs to the class of **H**yperbolic **A**bsolute **R**isk **A**version (HARA) utility functions, and it is defined by

$$U(W) = \begin{cases} \dfrac{W^\gamma}{\gamma}\,, & \text{if } \gamma < 1,\, \gamma \neq 0, \\ \log W\,, & \text{if } \gamma = 0, \end{cases}$$

  where γ denotes a parameter related to HARA utility functions.

As indicated in [5], this particular choice of U leads to an homothetic property of the optimal value function, which is defined by

$$\varphi(x,y,t) = \sup_{(L,M)\in A_t(x,y)} E_t^{x,y}[U(W_T)]\,, \quad (x,y) \in \mathcal{S}, \quad t \in [0,T)\,,$$

so that the forthcoming nonlinear partial differential equation can be reduced to an equation in one spatial-like variable.

3. Mathematical analysis and theoretical results

In order to assume the existence of transaction costs we consider $\lambda + \mu > 0$. In this setting, the optimal value function is the viscosity solution of the following Hamilton–Jacobi–Bellman (HJB) equation [13]:

$$\min\{-\varphi_t - \widehat{\mathcal{L}}\varphi, -(1-\mu)\varphi_x + \varphi_y, (1+\lambda)\varphi_x - \varphi_y\} = 0\,, \quad (x,y) \in \mathcal{S}, \quad t \in [0,T)\,, \tag{2}$$

with the terminal condition

$$\varphi(x,y,T) = \begin{cases} U(x+(1-\mu)y)\,, & \text{if } y>0 \\ U(x+(1+\lambda)y)\,, & \text{if } y\leq 0\,, \end{cases} \tag{3}$$

where

$$\widehat{\mathcal{L}}\varphi = \frac{1}{2}\sigma^2 y^2\, \varphi_{yy} + \alpha y\, \varphi_y + rx\, \varphi_x\,,$$

and hereafter the subindex notation indicates partial derivative with respect to the corresponding variable. The existence of a unique viscosity solution of (2)-(3) has been proven in [5].

Following [3], in order to approximate its solution by numerical methods, we transform the problem into an equivalent one. For this purpose, first we consider that $y>0$ (as short selling is always suboptimal) and we introduce the new function $V(x,t)=\varphi(x,1,t)$ so that

$$\varphi(x,y,t) = \begin{cases} y^\gamma V(\frac{x}{y},t)\,, & \text{if } \gamma<1,\, \gamma\neq 0\,, \\ V(\frac{x}{y},t)+\log y\,, & \text{if } \gamma = 0\,. \end{cases} \tag{4}$$

Moreover, let us define

$$w = \frac{1}{\gamma}\log(\gamma V)\,, \tag{5}$$

so that w satisfies in the domain $\widehat{\Omega}\times(0,T) = (-(1-\mu),+\infty)\times(0,T)$ the following equations:

$$\begin{cases} -w_t - \mathcal{L}_* w = 0, & \text{if } \dfrac{1}{x+1+\lambda} < w_x < \dfrac{1}{x+1-\mu}, \\ -w_t - \mathcal{L}_* w \geq 0, & \text{if } w_x = \dfrac{1}{x+1+\lambda} \text{ or } w_x = \dfrac{1}{x+1-\mu}, \\ w(x,T) = \log(x+1-\mu)\,, & \end{cases}$$

where

$$\begin{aligned} \mathcal{L}_* w &= \frac{1}{2}\sigma^2 x^2 (w_{xx}+\gamma w_x^2) + \beta_2 x w_x + \frac{1}{\gamma}\beta_1 = \\ &= \frac{1}{2}\sigma^2 x^2 (w_{xx}+\gamma w_x^2) - \big(\alpha - r - (1-\gamma)\sigma^2\big)\, x w_x + \alpha - \frac{1}{2}\sigma^2(1-\gamma)\,. \end{aligned}$$

Then, let us define $v(x,t)=w_x(x,t)$. It is proved in [3] that v satisfies the following parabolic double obstacle nonlinear problem in $\widehat{\Omega}\times[0,T)$:

$$\begin{cases} -v_t - \mathcal{L}v = 0, & \text{if } \dfrac{1}{x+1+\lambda} < v < \dfrac{1}{x+1-\mu}, \\ -v_t - \mathcal{L}v \geq 0, & \text{if } v = \dfrac{1}{x+1+\lambda}, \\ -v_t - \mathcal{L}v \leq 0, & \text{if } v = \dfrac{1}{x+1-\mu}, \\ v(x,T) = \dfrac{1}{x+1-\mu}, & \end{cases} \tag{6}$$

where

$$\mathcal{L}v = \frac{1}{2}\sigma^2 x^2 v_{xx} - \left(\alpha - r - (2-\gamma)\sigma^2\right) x v_x - \left(\alpha - r - (1-\gamma)\sigma^2\right) v + \\ + \frac{1}{2}\gamma\sigma^2 \left(x^2 v^2\right)_x .$$

Let us remark that the operator $\mathcal{L}$ presents the nonlinear term $(x^2v^2)_x$. Moreover, the double obstacle problem involves two unknown free boundaries which separate three unknown regions:

- selling region (coincidence with the upper obstacle):

$$\mathbf{SR} = \left\{ (x,t) \in \widehat{\Omega} \times [0,T] \;/\; v(x,t) = \frac{1}{x+1-\mu} \right\}$$

- buying region (coincidence with the lower obstacle):

$$\mathbf{BR} = \left\{ (x,t) \in \widehat{\Omega} \times [0,T] \;/\; v(x,t) = \frac{1}{x+1+\lambda} \right\}$$

- no transaction region (noncoincidence region):

$$\mathbf{NT} = \left\{ (x,t) \in \widehat{\Omega} \times [0,T] \;/\; \frac{1}{x+1+\lambda} < v(x,t) < \frac{1}{x+1-\mu} \right\}.$$

The existence, uniqueness and regularity of solution to problem (6) and some theoretical properties of the solution and the two free boundaries are obtained in [3]. More precisely, the following statements (to be verified numerically in the forthcoming Section 5) are proven:

(P1) $v_t(x,t) \geq 0$ in $\widehat{\Omega} \times [0,T]$

(P2) $v_x + v^2 \leq 0$ in $\widehat{\Omega} \times [0,T]$

(P3) Let be:

$$t_1 = T - \frac{1}{\alpha - r - (1-\gamma)\sigma^2} \log\left(\frac{1+\lambda}{1-\mu}\right) .$$

Then, an analytical expression for $v(0,t)$ is given in the following way:

- if $\alpha - r - (1-\gamma)\sigma^2 \leq 0$, then

$$v(0,t) = \frac{1}{1-\mu} \tag{7}$$

- if $\alpha - r - (1-\gamma)\sigma^2 > 0$, then

$$v(0,t) = \begin{cases} \dfrac{1}{1+\lambda}, & \text{if } 0 \leq t \leq t_1 \\ \dfrac{1}{1-\mu} e^{-(\alpha - r - (1-\gamma)\sigma^2)(T-t)}, & \text{if } t_1 < t \leq T \end{cases} \tag{8}$$

(P4) There exist two monotonically increasing functions $x_s, x_b : [0,T] \longrightarrow [-(1-\mu), +\infty)$, such that the so called selling and buying regions are characterized as

$$\mathbf{SR} = \left\{ (x,t) \in \widehat{\Omega} \,/\, x \leq x_s(t), \quad t \in [0,T) \right\},$$
$$\mathbf{BR} = \left\{ (x,t) \in \widehat{\Omega} \,/\, x \geq x_b(t), \quad t \in [0,T) \right\},$$

and $x_s(t) < x_b(t), \quad$ for all $t \in [0,T)$. Thus, functions x_s and x_b parameterize the two free boundaries, which are known as selling and buying free boundaries, respectively.

(P5) The function x_s verifies:

- $x_s(t) \leq (1-\mu)x_M, \ t \in [0,T]$
- $\lim_{t\to T^-} x_s(t) = (1-\mu)x_M$
- Moreover, we have:

$$\begin{aligned} x_s(t) > 0\,, &\quad \text{if } \alpha - r - (1-\gamma)\sigma^2 < 0 \\ x_s(t) \equiv 0\,, &\quad \text{if } \alpha - r - (1-\gamma)\sigma^2 = 0 \\ x_s(t) < 0\,, &\quad \text{if } \alpha - r - (1-\gamma)\sigma^2 > 0 \end{aligned}$$

(P6) The function x_b verifies:

- $x_b(t) \geq (1+\lambda)x_M, \ t \in [0,T]$
- $x_b(t) = \infty \quad$ if and only if $t_0 \leq t < T$, with

$$t_0 = T - \frac{1}{\alpha - r} \log\left(\frac{1+\lambda}{1-\mu}\right)$$

- if $\quad \alpha - r - (1-\gamma)\sigma^2 \leq 0$, then $x_b(t) > 0, \ t \in [0,T]$
- if $\quad \alpha - r - (1-\gamma)\sigma^2 > 0$, then

$$\begin{aligned} x_b(t) < 0 &\qquad \text{for } t \in (0, t_1) \\ x_b(t_1) = 0 & \\ x_b(t) > 0 &\qquad \text{for } t \in (t_1, T)\,. \end{aligned}$$

(P7) The steady state problem corresponding to (6) is just obtained by removing the time derivative in the governing equations. In [3], assuming that $\alpha - r - (1-\gamma)\sigma^2 \neq 0$, the following expression for the steady state solution is obtained:

$$v_\infty(x) = \begin{cases} \dfrac{C}{x} + \dfrac{1}{g(x)}\,, & \text{if } x_{s,\infty} < x < x_{b,\infty}, \\ \dfrac{1}{x+1-\mu}\,, & \text{if } x \leq x_{s,\infty}, \\ \dfrac{1}{x+1+\lambda}\,, & \text{if } x \geq x_{b,\infty}, \end{cases} \tag{9}$$

where:

$$g(x) = \begin{cases} \left(\frac{x_{s,\infty}}{x}\right)^{\beta} \left(\frac{x_{s,\infty}\,(x_{s,\infty}+1-\mu)}{(1-C)x_{s,\infty}-(1-\mu)C} - \frac{\gamma\, x_{s,\infty}}{\beta+1}\right) + \frac{\gamma\, x}{\beta+1}\,, & \text{if } \beta \neq -1, \\ \frac{x\,(x_{s,\infty}+1-\mu)}{(1-C)x_{s,\infty}-(1-\mu)C} + \gamma\, x\, \log \frac{x}{x_{s,\infty}}\,, & \text{if } \beta = -1, \end{cases}$$

and

$$x_{s,\infty} = -\frac{a}{a+k}(1-\mu), \qquad x_{b,\infty} = -\frac{a}{a+\frac{k}{k-1}}(1+\lambda),$$

$$a = \frac{\alpha - r - (1-\gamma)\sigma^2}{\frac{1}{2}(1-\gamma)\sigma^2}, \qquad \beta = (1-\gamma)a - 2\gamma C,$$

$$C = -\frac{2(k-1)a^2}{k^2\left(a + \frac{1}{1-\gamma} + \sqrt{(a+\frac{1}{1-\gamma}^{2} + 4\frac{\gamma}{1-\gamma})^2 + 4\frac{\gamma}{1-\gamma}\frac{k-1}{k^2}a^2}\right)},$$

and k is the root of:

$$\frac{a+\frac{k}{k-1}}{a+k}\left(\frac{\frac{\gamma}{\beta+1}+\frac{1}{\frac{a}{k}+C}}{\frac{\gamma}{\beta+1}+\frac{1}{\frac{k-1}{k}a+C}}\right)^{1/(\beta+1)} = \frac{1+\lambda}{1-\mu}\,, \text{ if } \beta \neq -1,$$

$$\frac{a+\frac{k}{k-1}}{a+k}\exp\left(\frac{1}{\gamma}\left(\frac{1}{\frac{a}{k}+C} - \frac{1}{\frac{k-1}{k}a+C}\right)\right) = \frac{1+\lambda}{1-\mu}\,, \text{ if } \beta = -1.$$

Although all previous qualitative properties give an important insight on several features of the unique solution of the problem, as the exact solution cannot be obtained, it results useful to develop appropriate numerical methods to approximate the solution and provide quantitative results under different model parameters conditions.

4. Numerical methods

As mentioned before, our main goal is to propose an adequate set of numerical methods in order to approximate the solution of the double obstacle problem (6), and verify the theoretical properties stated in [3]. Moreover, we can recover the corresponding approximation of the investment value function, φ, which is the unknown of the original optimal investment problem and it is defined in terms of the underlying financial variables. Consequently, we can obtain the buying, selling and no transaction regions.

We first notice that the main difficulties related to the numerical solution of (6) are the following:

1. the problem is posed on an unbounded domain $\widehat{\Omega} = (-(1-\mu)\,,\,+\infty)$

2. the model is governed by a convection–diffusion equation, usually convection dominated
3. the equation presents a nonlinear term, $(x^2v^2)_x$
4. the free boundary feature related to the double obstacle problem.

First of all, although it is not strictly necessary, we introduce a new time variable, $\tau = T - t$ (time to finite horizon), so that the problem (6) can be written more classically as the following initial value problem:

$$\begin{cases} v_\tau - \mathcal{L}v = 0, & \text{if } \dfrac{1}{x+1+\lambda} < v < \dfrac{1}{x+1-\mu}, \\ v_\tau - \mathcal{L}v \geq 0, & \text{if } v = \dfrac{1}{x+1+\lambda}, \\ v_\tau - \mathcal{L}v \leq 0, & \text{if } v = \dfrac{1}{x+1-\mu}, \\ v(x,0) = \dfrac{1}{x+1-\mu}. \end{cases} \tag{10}$$

Notice that the first equation in (10) can be shortly written as:

$$v_\tau - a_0 x^2 v_{xx} - a_1 x v_x - a_2 v + a_3 (x^2 v^2)_x = 0\,, \tag{11}$$

where the coefficients are

$$a_0 = \frac{\sigma^2}{2},\; a_1 = -\left(\alpha - r - (2-\gamma)\sigma^2\right),\; a_2 = -\left(\alpha - r - (1-\gamma)\sigma^2\right),\; a_3 = -\frac{\gamma\sigma^2}{2}.$$

4.1. Localization on a bounded domain

In order to overcome the difficulty associated to the unbounded domain, as in many financial problems [9], we perform a localization procedure by replacing $\widehat{\Omega}$ by the bounded interval $\Omega = (x^*, N)$, where $x^* > -(1-\mu)$ and $N < +\infty$. Notice that this approximation by a problem posed on a bounded domain is already used in [3] to obtain the existence and regularity of solution. For simplicity, we avoid the notation that includes the dependence of Ω (and therefore the associated solution in Ω) on the values of x^* and N, i.e. we shortly denote $\Omega = \Omega_{x^*,N}$ and $v = v_{x^*,N}$. Moreover, we introduce appropriate boundary conditions at both boundaries of the interval Ω. More precisely, as the solution will naturally be in contact with the upper obstacle on the left part (*selling region*) of the domain, at $x = x^*$ we impose the following Dirichlet boundary condition:

$$v(x^*) = \frac{1}{x^*+1-\mu}.$$

On the other hand, we cannot ensure that the solution will be in contact with the lower obstacle on the buying region at all instants. In fact, the existence of $x_b(t)$ is not ensured for any time (there exist values of t for which the solution does not reach the lower obstacle). So, having in view this argument, a possible reasonable choice is to impose a Neumann boundary condition at $x = N$ equal to the slope of the lower obstacle, i.e.

$$v_x(N) = -\frac{1}{(N+1+\lambda)^2}\,.$$

4.2. Time and space discretizations

As we will use finite elements for the spatial discretization, we first rewrite the main equation of (10) with the diffusion term in conservative form, that is

$$v_\tau + (2a_0 - a_1)xv_x - a_0(x^2 v_x)_x - a_2 v + a_3(x^2 v^2)_x = 0, \qquad (12)$$

or equivalently,

$$\frac{Dv}{D\tau} - a_0(x^2 v_x)_x - a_2 v + a_3(x^2 v^2)_x = 0\,, \qquad (13)$$

where

$$Dv/D\tau = v_\tau + (2a_0 - a_1)xv_x$$

represents the *material* or *total derivative.*

As mentioned before, in order to deal with the presence of large gradients in the solution, we propose a time discretization procedure which approximates the material derivative by means of a characteristics method. This method has been first used in a financial application for pricing European and American vanilla options in [14]. Thus, for $M > 1$ let $\Delta\tau = T/M$ be the time step and let $\tau^m = m\Delta\tau$, with $m = 0, 1, \ldots, M$, be the time discretization points. Moreover, for a given $v^m = v(\cdot, \tau^m)$, let v^{m+1} be the solution of the following time discretized problem:

$$\begin{cases} \dfrac{v^{m+1} - (v^m \circ \chi^m)}{\Delta\tau} - a_0(x^2 v_x^{m+1})_x - a_2 v^{m+1} + a_3(x^2 (v^{m+1})^2)_x = 0, & \text{if } l < v^{m+1} < u, \\ \dfrac{v^{m+1} - (v^m \circ \chi^m)}{\Delta\tau} - a_0(x^2 v_x^{m+1})_x - a_2 v^{m+1} + a_3(x^2 (v^{m+1})^2)_x \geq 0, & \text{if } v^{m+1} = l, \\ \dfrac{v^{m+1} - (v^m \circ \chi^m)}{\Delta\tau} - a_0(x^2 v_x^{m+1})_x - a_2 v^{m+1} + a_3(x^2 (v^{m+1})^2)_x \leq 0, & \text{if } v^{m+1} = u, \\ v^{m+1}(x^*) = \dfrac{1}{x^* + 1 - \mu}, \\ v_x^{m+1}(N) = -\dfrac{1}{(N + 1 + \lambda)^2}, \end{cases} \qquad (14)$$

where

$$l(x) = \frac{1}{x + 1 + \lambda}, \qquad u(x) = \frac{1}{x + 1 - \mu} \qquad (15)$$

denote the lower and upper obstacles, respectively.

In order to compute the term χ^m, for given values of τ^{m+1} and x, we solve the final value ODE problem:

$$\frac{d\chi}{d\tau} = (2a_0 - a_1)\chi, \qquad \chi(\tau^{m+1}) = x\,, \qquad (16)$$

so that $\chi^m = \chi(\tau^m)$ can be exactly computed and is given by $\chi^m = x\exp(-(2a_0 - a_1)\Delta\tau)$. Notice that χ^m is independent of m, which is an advantage from the computational point of view.

Moreover, there exists an equivalent formulation of (14) in terms of a variational inequality. More precisely, let us consider the space

$$V = \left\{ v : \Omega \to \mathrm{I\!R} \,/\, v(x^*) = \frac{1}{x^* + 1 - \mu} \right\},$$

and the convex subset

$$K = \{v \in V \,/\, l \leq v \leq u\} \ .$$

Then, if we multiply the main equation in (14) by $w - v^{m+1}$ and integrate over Ω, we get:

$$\begin{aligned}
&\int_\Omega v^{m+1}(w - v^{m+1}) - a_0\Delta\tau \int_\Omega (x^2 v_x^{m+1})_x (w - v^{m+1}) - \\
&\quad - a_2\Delta\tau \int_\Omega v^{m+1}(w - v^{m+1}) + a_3\Delta\tau \int_\Omega (x^2(v^{m+1})^2)_x (w - v^{m+1}) \geq \\
&\quad \geq \int_\Omega (v^m \circ \chi^m)(w - v^{m+1}) \,, \qquad \forall w \in K.
\end{aligned}$$

Moreover, using Green theorem we obtain

$$\begin{aligned}
&\int_\Omega v^{m+1}(w - v^{m+1}) + a_0\Delta\tau \int_\Omega x^2 v_x^{m+1}(w - v^{m+1})_x - \\
&\quad - a_0\Delta\tau \left[x^2 v_x^{m+1}(w - v^{m+1})\right]_{\partial\Omega} - a_2\Delta\tau \int_\Omega v^{m+1}(w - v^{m+1}) - \\
&\quad - a_3\Delta\tau \int_\Omega x^2(v^{m+1})^2 (w - v^{m+1})_x + a_3\Delta\tau \left[x^2(v^{m+1})^2(w - v^{m+1})\right]_{\partial\Omega} \geq \\
&\quad \geq \int_\Omega (v^m \circ \chi^m)(w - v^{m+1}) \,, \qquad \forall w \in K \,.
\end{aligned}$$

Thus, we search a function $v^{m+1} \in K$ such that:

$$\begin{aligned}
&(1 - a_2\Delta\tau) \int_\Omega v^{m+1}(w - v^{m+1}) + a_0\Delta\tau \int_\Omega x^2 v_x^{m+1}(w - v^{m+1})_x - \\
&\quad - a_3\Delta\tau \int_\Omega x^2(v^{m+1})^2(w - v^{m+1})_x - \\
&\quad - a_0\Delta\tau \left[x^2 v_x^{m+1}(w - v^{m+1})\right]_{\partial\Omega} + a_3\Delta\tau \left[x^2(v^{m+1})^2(w - v^{m+1})\right]_{\partial\Omega} \geq \\
&\quad \geq \int_\Omega (v^m \circ \chi^m)(w - v^{m+1}) \,, \qquad \forall w \in K \,. \qquad (17)
\end{aligned}$$

We can now discretize (17) by means of a finite element method. For this purpose, we consider a uniform finite element mesh with stepsize h and nodes x_i, $i = 1, \ldots, I$. More precisely, we use continuous piecewise linear Lagrange finite elements, defined by the functional space

$$V_h = \{v_h \in \mathcal{C}(x^*, N)/v_h \,|_{[x_i, x_{i+1}]} \in \mathcal{P}_1, \, i = 1, \ldots, I - 1\},$$

and the convex set

$$K_h = \{v_h \in V_h / l(x_i) \leq v_h(x_i) \leq u(x_i), \, i = 2, \ldots, I - 1\},$$

where $\mathcal{P}_1$ denotes the space of polynomials of degree less or equal than one. Therefore, the discrete problem formulation consists of finding $v_h^{m+1} \in K_h$, such that

$$
\begin{aligned}
&(1-a_2\Delta\tau)\int_\Omega v_h^{m+1}(w_h - v_h^{m+1}) + a_0\Delta\tau\int_\Omega x^2(v_h^{m+1})_x(w_h - v_h^{m+1})_x - \\
&\quad - a_3\Delta\tau\int_\Omega x^2(v_h^{m+1})^2(w_h - v_h^{m+1})_x - \\
&\quad - a_0\Delta\tau\left[x^2(v_h^{m+1})_x(w_h - v_h^{m+1})\right]_{\partial\Omega} + a_3\Delta\tau\left[x^2(v_h^{m+1})^2(w_h - v_h^{m+1})\right]_{\partial\Omega} \geq \\
&\quad \geq \int_\Omega (v_h^m \circ \chi^m)(w_h - v_h^{m+1}), \qquad \forall w_h \in K_h\,. \qquad (18)
\end{aligned}
$$

The solution of the discrete problem (18) is equivalent to the computation of the vector $V^{m+1} = (V_1^{m+1}, \ldots, V_I^{m+1})^T$, such that :

$$
\begin{cases}
l_i \leq V_i^{m+1} \leq u_i \\
[(M+A)V^{m+1} + B(V^{m+1})]_i \geq q_i\,, & \text{if } V_i^{m+1} = l_i, \\
[(M+A)V^{m+1} + B(V^{m+1})]_i = q_i\,, & \text{if } l_i < V_i^{m+1} < u_i, \\
[(M+A)V^{m+1} + B(V^{m+1})]_i \leq q_i\,, & \text{if } V_i^{m+1} = u_i,
\end{cases}
\qquad (i = 1, 2, \ldots, I),
$$

where:

- M and A are the classical finite element mass and stiffness matrices [1]
- B is the vector associated to the nonlinear term $x^2(v^{m+1})^2$
- q is the second member vector
- $l_i = l(x_i)$ and $u_i = u(x_i)$.

4.3. Numerical solution of the discrete problem

In order to linearize the previous system of nonlinear inequalities we propose a Newton method. For this purpose, let us introduce the notation

$$F(V^{m+1}) = (M+A)V^{m+1} + B(V^{m+1}) - q,$$

so that Newton algorithm can be written in the following way:

Set $V^{m+1,0} = V^m$

For $\ell = 0, 1, 2, ...$

Compute $\quad F(V^{m+1,\ell}) = (M+A)V^{m+1,\ell} + B(V^{m+1,\ell}) - q,$

and $\quad J_F(V^{m+1,\ell}) = M + A + J_B(V^{m+1,\ell}).$

Solve the linear complementarity problem

$$
\begin{cases}
\left[J_F(V^{m+1,\ell})V^{m+1,\ell+1}\right]_i \geq \left[J_F(V^{m+1,\ell})V^{m+1,\ell} - F(V^{m+1,\ell})\right]_i\,, & \text{if } V_i^{m+1,\ell+1} = l_i, \\
\left[J_F(V^{m+1,\ell})V^{m+1,\ell+1}\right]_i = \left[J_F(V^{m+1,\ell})V^{m+1,\ell} - F(V^{m+1,\ell})\right]_i\,, & \text{if } l_i < V_i^{m+1,\ell+1} < u_i, \\
\left[J_F(V^{m+1,\ell})V^{m+1,\ell+1}\right]_i \leq \left[J_F(V^{m+1,\ell})V^{m+1,\ell} - F(V^{m+1,\ell})\right]_i\,, & \text{if } V_i^{m+1,\ell+1} = u_i,
\end{cases}
\qquad (19)
$$

where J_F and J_B are the jacobian matrices of the nonlinear functions F and B, respectively. The algorithm iterates in index ℓ until convergence according to a given relative error tolerance ϵ_N.

So, at each iteration, we have to solve a system of linear inequalities. Let us rewrite this system in generic form as:

$$\begin{cases} (Ay)_i \geq b_i \,, & \text{if } y_i = l_i \\ (Ay)_i = b_i \,, & \text{if } l_i < y_i < u_i \\ (Ay)_i \leq b_i \,, & \text{if } y_i = u_i \,, \end{cases}$$

where A, b and y denote the matrix, the second member and the unknown vectors of order I, respectively. Different families of methods can be used to solve this kind of problem [6]. We propose a classical projected relaxation algorithm. This method consists of building a sequence of vectors, starting with y^0. At step k, with a given vector y^k, for $i = 1, \ldots, I$ we compute:

$$\begin{aligned} \widehat{y}_i^{k+1} &= \frac{1}{a_{ii}} \left(b_i - \sum_{j=1}^{i-1} a_{ij} y_j^{k+1} - \sum_{j=i+1}^{I} a_{ij} y_j^k \right) \\ y_i^{k+1} &= \min \left[\max \left(l_i \,,\ y_i^k + \omega(\widehat{y}_i^{k+1} - y_i^k) \right), u_i \right] . \end{aligned}$$

The algorithm iterates in k until the error between two consecutive iterations is below a given tolerance ϵ_R. Projected gradient or duality methods [7] can be alternatively applied.

5. Numerical results

In this Section we present some numerical tests, in order to illustrate the good performance of the numerical techniques. We mainly verify the qualitative properties concerning the solution and the free boundaries which have been theoretically proven in [3].

Test 1

In this first example we consider the following set of financial parameters:

$$\sigma = 0.25, \quad r = 0.03, \quad \alpha = 0.10, \quad \gamma = 0.50, \quad \lambda = 0.08, \quad \mu = 0.02,$$

and solve the problem for the time interval $t \in [0, 4]$. Notice that with the previous values, we have $\alpha - r - (1-\gamma)\sigma^2 = 0.03875 > 0$ so that **(P5)** and **(P6)** state the following properties for the selling (x_s) and buying (x_b) free boundaries:

$$\begin{cases} x_s(t) < 0, \text{ for } t \in [0, T], \\ x_b(t) < 0 \text{ for } 0 \leq t < t_1 = 1.4925 \text{ and } x_b(t) > 0 \text{ for } t > t_1. \end{cases}$$

Concerning the numerical method, we consider the truncated domain $\Omega = (-0.95\,,\, 10)$ and we use a uniform finite element mesh with 801 nodes, so that $h = 0.0136875$. Moreover, we consider 400 time steps, so that $\Delta t = 0.01$. We point out that a good convergence behavior has been observed in different tests as soon as the mesh parameters Δt and h tend to zero. Other numerical parameters that we have used are the relaxation parameter $\omega = 1.80$, the tolerance for the relative error in the relaxation algorithm $\epsilon_R = 10^{-15}$ and the tolerance for the relative error in the Newton algorithm $\epsilon_N = 10^{-4}$.

First, we have verified that the theoretical properties **(P1)** and **(P2)**:

$$v_x + v^2 \leq 0, \qquad\qquad v_t \geq 0,$$

are satisfied for all the nodal points and at every discrete time.

Figure 1 (a) shows the numerical approximation of the investment pricing at instant $t = 0$, while in (b) a zoom of (a) shows the solution in the *no transaction* (NT) region.

Figure 2 (a) shows the numerical approximation and the analytical solution $v(0, t)$. Notice that in this case the analytical solution is provided by expression (8). The relative error in Figure 2 (b) shows a good agreement between the expression (8) and the computed solution.

The computed approximations of functions x_b and x_s are shown in Figures 3 and 4. Both functions verify the theoretical properties **(P4)-(P6)** described in Section 3. More precisely, Figure 3 shows the monotonicity of both functions, in financial terms this mainly indicates that as soon as the finite horizon is approaching the optimality of buying risky assets increases and the one of selling them decreases. Also notice that the value of $x_b(t)$ tends to N (upper bound of the spatial domain) when t tends to $t_0 = 2.6119$. We have also verified that as soon as N increases the point t at which $x_b(t) = N$ approaches to t_0. This is the way we can numerically verify that the free boundary x_b blows up at t_0 as it has been proven. Figures 4 (a) and (b) show the theoretically stated upper bound for the selling boundary and lower bound for the buying boundary, respectively. Moreover, the computed value for which $x_b(t)$ vanishes coincides with the theoretical value $t_1 = 1.4925$.

Figure 5 (a) shows the decomposition of the solvency region into the buying, selling and no transaction regions at time $t = 0$, as well as the (dotted) Merton line corresponding to the solution of the case without transaction costs and which is always contained in the no transaction region. Figure 5 (b) shows the optimal investment value function, φ, over the solvency region. As formally stated in [5], the no transaction region is a wedge and the regions above and below are the buying and selling regions, respectively. Both free boundaries are straight lines through the origin in the plane defined by the stock and bank account coordinates. In the selling and buying regions, the optimal strategy of the investor consists in making an instantaneous finite transaction to get the respective boundary with the no transaction region. By doing this buying or selling transaction, the investor portfolio moves up or down to achieve the free boundary in a normal direction to it. After that, the future transactions take place at the achieved free boundary, so that the processes L_t and M_t can be understood as local times to the respective boundaries.

Moreover, we point out that the function φ is numerically obtained after several change of variables steps. First, from the approximated solution of the double obstacle problem, v, the function w is numerically computed from expression

$$w(x,t) = A(t) + \log(x_s(t) + 1 - \mu) + \int_{x_s(t)}^{x} v(y,t)\,dy, \tag{20}$$

where

$$A(t) = \int_t^T \frac{rx^2 + (\alpha + r)(1-\mu)x + (\alpha - 0.5\sigma^2(1-\gamma))(1-\mu)^2}{(x+1-\mu)^2} \,|_{x=x_s(\tau)}\ d\tau\,. \tag{21}$$

Also notice that for $x \le x_s(t)$, according to (20), we can more straightforwardly compute

$$w(x,t) = A(t) + \log(x + 1 - \mu).$$

Next, from w we can compute

$$V = \frac{1}{\gamma}\exp(\gamma w)$$

according to (5) and then φ is recovered from expression (4).

Finally, the numerical approximation of the steady state solution and the semi-analytical solution (9) in **(P7)** at Section 3 have been compared. The approximation of the steady state solution has been obtained as the limit of the evolutive one when time tends to infinity (numerically, when the difference between the approximations for two consecutive time steps is below a prescribed tolerance). In order to solve the nonlinear equations involved in expression (9) to obtain the semi-analytical steady state solution, Newton and fixed point iteration algorithms have been used. Then, using a discrete L^2-norm and the previous numerical parameters for the evolutive problem, we have obtained a relative error $\epsilon = 0.0018$ between the computed steady state solution from the PDE problem and the one obtained from expression (9) in **(P7)**. Moreover, the relative errors in the computed values of $x_{s,\infty}$ and $x_{b,\infty}$ are 0.0021 and 0.069, respectively.

Test 2

In this second numerical tests, the following financial parameters have been considered:

$$\sigma = 0.25, \quad r = 0.05, \quad \alpha = 0.08, \quad \gamma = 0.50, \quad \lambda = 0.06, \quad \mu = 0.02. \tag{22}$$

The main difference with respect to Test 1 is that now $\alpha - r - (1-\gamma)\sigma^2 = -0.00125 < 0$, so that theoretical results indicated in **(P5)-(P6)** particularly establish that x_s and x_b are both positive.

On the other hand, we have considered the truncated domain $\Omega = (-0.95\,,\,12)$, that we have discretized with 201 nodes, so that $h = 0.06475$. The values of $\Delta\tau$, ϵ_N, ϵ_R and ω are the same ones we used in Test 1.

Figures 6, 7 and 8 show the numerical approximation of the solution, the evolution of the free boundaries with respect to time, the solvency region at time $t = 0$ and the value function at $t = 0$, respectively. The obtained numerical results are again in full agreement with the theoretical ones stated in [3] and summarized in Section 3 of the present paper.

6. Conclusions

A set of numerical methods has been proposed to approximate the solution of a finite–horizon optimal problem with transaction costs, for which no analytical solution exists. The model can be transformed into a double obstacle problem associated to a second order nonlinear parabolic partial differential equation on an unbounded domain.

We have designed a set of numerical methods (including characteristics, finite elements, Newton linearization and projected relaxation techniques) to cope with the different difficulties of the numerical solution, and the computed approximations verify all the qualitative properties which have been theoretically stated in [3]. Moreover, the numerical solution allows to approximate the optimal investment value function in the presence of proportional transaction costs and to determine the no transaction, buying and selling regions.

Indeed, other portfolio optimization problems with transactions costs admit an equivalent double obstacle formulation and the techniques here proposed are suitable to approximate the solution. One example could be the extension including a consumption term treated in [4], although the techniques are not straightforwardly adapted. In fact, the main difficulty arises from the presence of an exponential term in variable w in the PDE associated to the unknown v, so that a first possible approach is a fixed point iteration, solving for an explicit w the PDE and updating it with the new computed v. The design of an implicit scheme seems to be a difficult task. Another example is the dual approach and the use of shadow prices for optimal investment with transaction costs developed in [8], where the solution of a double obstacle problem associated to a steady state equation defines the shadow prices process.

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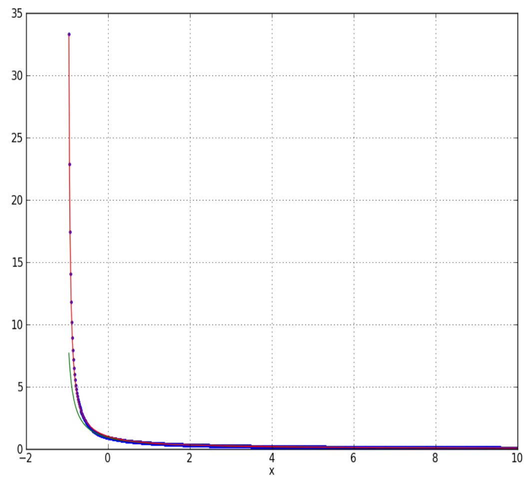

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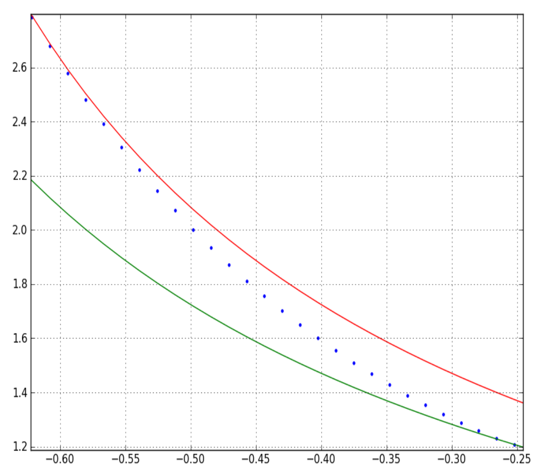
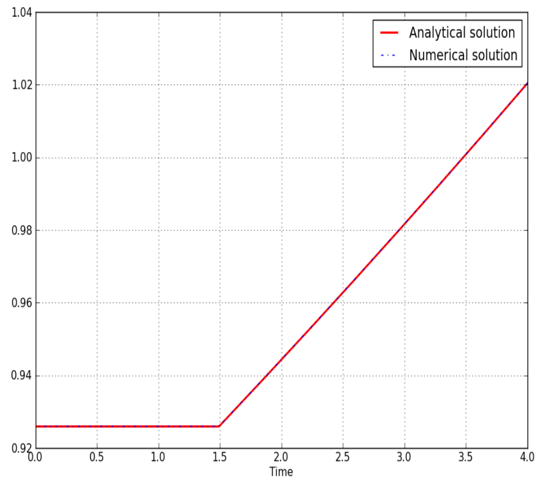


(b)

Figure 1: Numerical solution at $t = 0$ for Test 1. (a) In the computational domain. (b) In the no transaction region.



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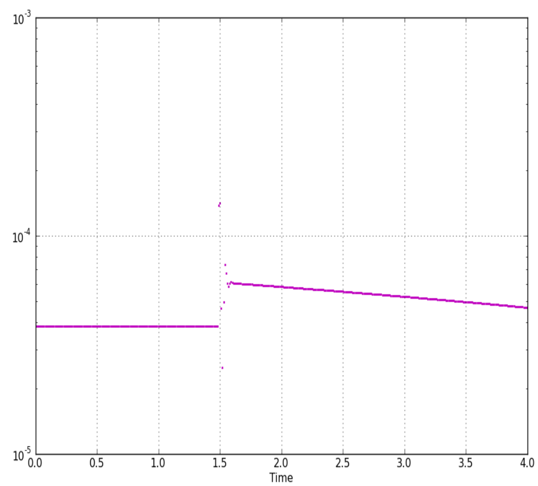

(b)

Figure 2: Comparison of numerical and analytical solutions at $x = 0$ in Test 1. (a) Both solutions. (b) Relative error.

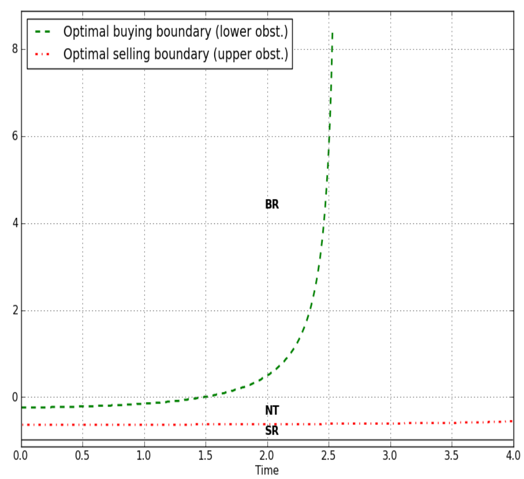

Figure 3: Monotone evolution of the free boundaries in Test 1 (both in the same scale)

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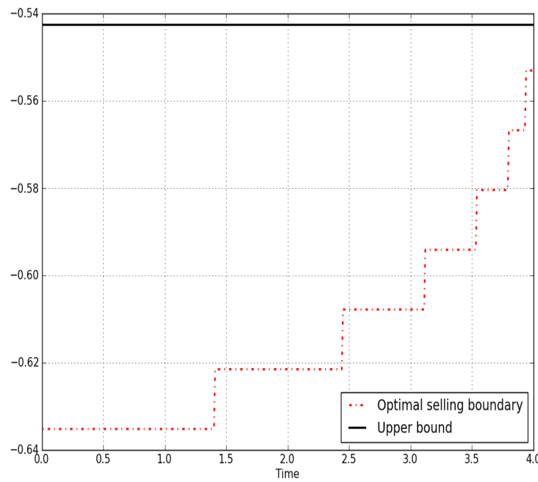

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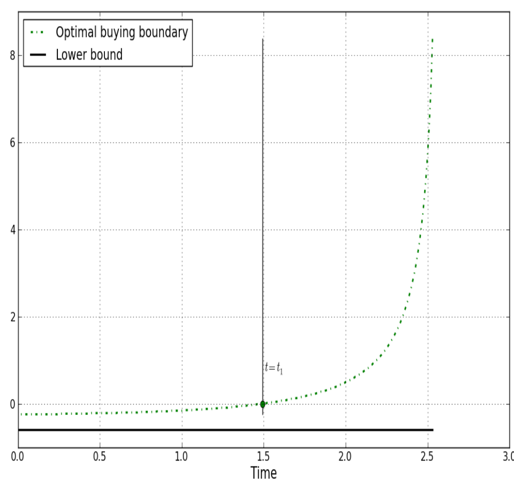
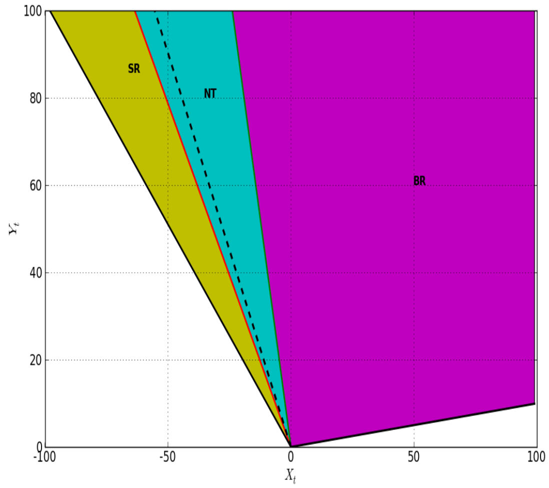


(b)

Figure 4: Bounds for the free boundaries in Test 1 (in different scales). (a) Upper bound for x_s. (b) Lower bound for x_b and value of $t_1 = 1.495$ such that $x_b(t_1) = 0$.



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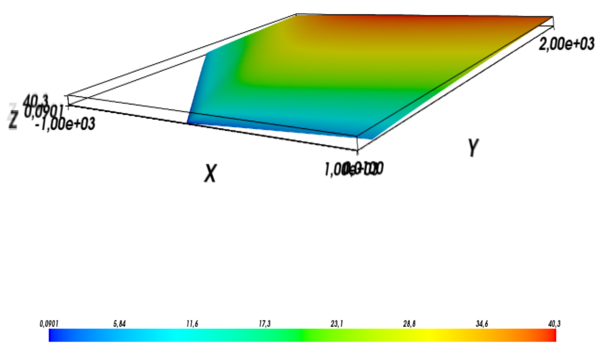
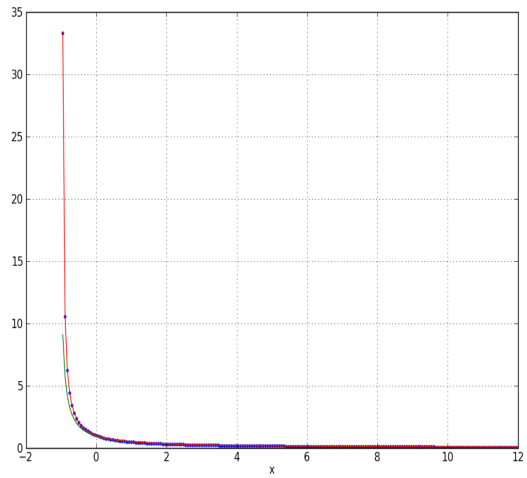


(b)

Figure 5: Results for Test 1 in original financial variables. (a) Decomposition of the solvency region. (b) Optimal investment value function at $t = 0$



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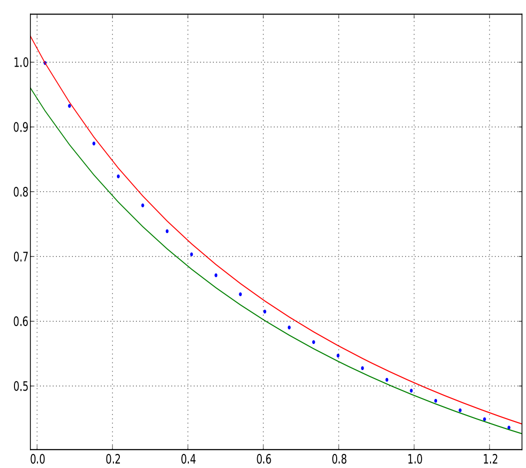


(b)

Figure 6: Numerical solution at $t = 0$ for Test 2. (a) In the computational domain. (b) In the no transaction region

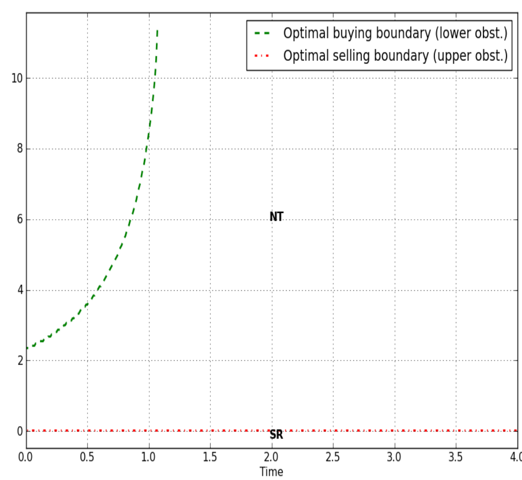
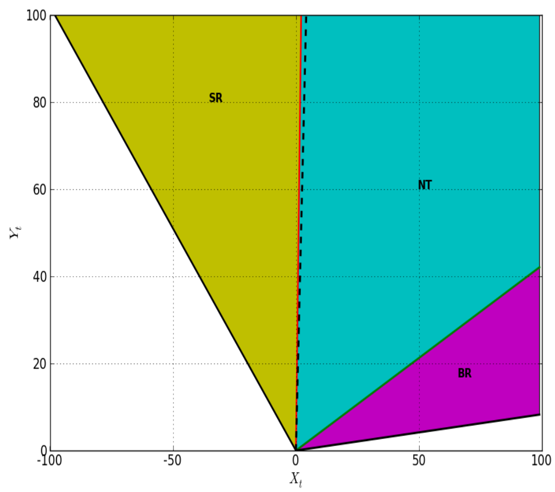


Figure 7: Monotone evolution of the free boundaries in Test 2



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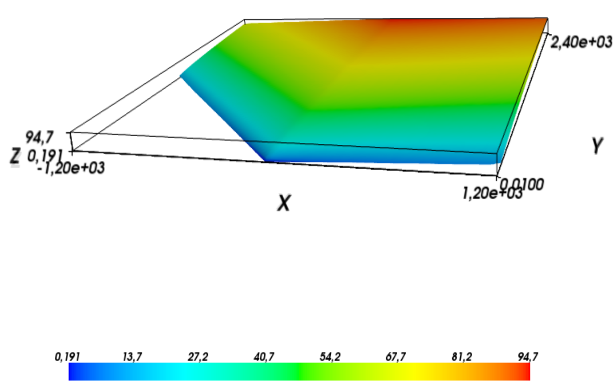

(b)

Figure 8: Results for Test 2 in original financial variables. (a) Decomposition of the solvency region. (b) Optimal investment value function at $t = 0$