Finite element solution of new displacement/pressure poroelastic models in acoustics

A. Bermúdez [a,*], J. L. Ferrín [a] and A. Prieto [a]

[a] *Departamento de Matemática Aplicada, Universidade de Santiago de Compostela,15706 Santiago de Compostela, Spain.*

Abstract

This paper deals with acoustical behavior of porous materials having an elastic solid frame. Firstly an overview to poroelastic models is presented, and then we focus our attention on non-dissipative poroelastic materials with open pores. Assuming periodic structure we compute the coefficients in the model by using homogenization techniques which require solving boundary-value problems in the elementary cell. Next we propose a finite element method in order to compute the response to an harmonic excitation of a three-dimensional enclosure containing a free fluid and a poroelastic material. The finite element used for the fluid is the lowest order face element introduced by Raviart and Thomas that eliminates the spurious modes whereas, for displacements in porous medium, the "mini element" is used in order to achieve stability of the method.

Key words: elastic frame, porous medium, finite element method
1991 MSC: 74Qxx, 81Txx, 74S05

1 Introduction

Acoustic behavior of porous media is of utmost importance because they exhibit good properties as sound absorbers. Porous materials consists of a solid matrix which is completely saturated by a fluid. Their acoustic behavior depends not only on the fluid but on the rigidity of the solid skeleton as well.

* Corresponding author.
Email addresses: mabermud@usc.es (A. Bermúdez), maferrin@usc.es (J. L. Ferrín), maprieto@usc.es (A. Prieto).

When considering macroscopic models for acoustic behavior of porous media, they can be classified depending on whether the solid part is rigid or elastic. When the solid matrix is rigid, which is the simplest case, the porous material has been considered as an equivalent fluid with equivalent density and bulk modulus. These parameters can be obtained through empirical or experimental laws. Delany and Bazley [12] presented a first model in 1970, which has been widely used to describe sound propagation in fibrous materials. This model, was subsequently improved in works by Morse and Ingard [19], Attenborough [5] or Allard and Champoux [2], among others.

For the more realistic case, but also more difficult, of elastic skeleton, theory for mechanical behavior of poroelastic materials was established by Biot [8], when the porous elastic solid is saturated by a viscous fluid. Adaption of this theory to acoustics was made, among others, by Allard [1]. However, when analyzing Biot's model, coefficients are not properly defined and, in general, their determination is not clear although several experimental procedures have been provided, as it can be seen in Biot and Willis [9].

This gives motivation to undertake a derivation of models, rigorously from a mathematical point of view, by using homogenization theory. It can be done for both cases of rigid or elastic solid matrix. For rigid matrix, Darcy's model is obtained. Ene and Sanchez-Palencia [14] were the first who gave a derivation of this model from the Stokes system, using a formal multi-scale method, while Tartar [22] made that derivation rigorously in the case of 2D periodic porous media. This methodology allows us not only to obtain the homogenized model but also the mathematical expression of coefficients appearing in it.

Derivation of macroscopic models for poroelastic materials depends strongly on connectivity of the fluid part. When the domain occupied by the fluid is connected, the material is named open pore material; otherwise, it is named closed pore material. Fundamental references are papers by Gilbert and Mikelić [16] and by Clopeau *et al* [10] where the classical dissipative Biot's model was derived by homogenization using two-scale convergence methods. They also contain a number of references to papers on dissipative Biot's law. Moreover, the same procedure has been applied, for the first time, in Ferrín and Mikelić [15] to derive macroscopic models for non-dissipative poroelastic material with open or closed pore.

Concerning numerical simulation, an increasing number of papers can be found for the two cases of rigid and elastic skeleton. The finite element method has become popular to solve such problems because of its easy implementation and its effectiveness in handling complex geometries. The lowest order finite method introduced by Raviart and Thomas has been applied in Bermúdez *et al* [6] to the case of porous media with rigid skeleton to solve, in particular, the response problem when using both a Darcy's like model and the Allard-

Champoux model. With respect to the case of elastic skeleton, papers by Easwaran *et al* [13], Panneton and Atalla [20], Göransson [17] or Atalla *et al* [4], among others, are examples of application of finite element methods to sound propagation in poroelastic media by using Biot's model.

In this paper, we only consider the case of elastic frame porous material. The non-dissipative model that we shall take into account has been derived in Ferrín and Mikelić [15] for open and closed pores. The advantages exhibited by this model with respect to classical Biot's model lies in that we know mathematical expressions allowing us to compute their coefficients. Numerical experiments presented in this paper concerns the solution of a source problem associated with an external harmonic excitation.

The outline of the paper is as follows. In Section 2 we recall several models of porous media associated with the two cases of rigid and elastic solid skeleton. In Section 3 we consider the model for open poroelastic materials introduced by Ferrín and Mikelić [15] coupled with the model for acoustic propagation in a fluid written in terms of displacements. Then both are specialized for harmonic waves in order to obtain the frequency response of the system. In Section 4 a weak formulation for this coupled model is presented. Numerical solution by finite element approximation is addressed in Section 5 and a matricial description of the discrete problem is given in Section 6. Section 7 is devoted to computation of coefficients in the model by solving cell problems. Finally, in Section 8, some numerical results are shown both for academic tests and for enclosures containing a real life porous material and air.

2 Porous media models

Different porous media models have been considered in the literature. The main characteristic is the rigidity of the solid skeleton. Thus, when the solid part is rigid, the first model was introduced by Darcy. On the other hand, when the solid part is elastic, the fundamental model has been derived by Biot.

In this section, we recall several models for rigid porous media and also for poroelastic media. Most of them have been derived by using homogenization techniques, assuming periodic distribution of pores. This methodology allows us to obtain mathematical expressions for the coefficients appearing in the models.

In what follows, the porous material will be denoted by Ω_{A}.

2.1 Rigid porous media model

Among the different models existing in the bibliography for rigid porous media, we concentrate ourselves on a Darcy's like model, obtained by homogenization techniques, and another one obtained through empirical laws by Allard and Champoux [2] for harmonic motions. The latter generalizes the former, in the sense that both are equivalent for low frequencies when the porosity is nearly unity.

The first one it is derived from Darcy's law, by adding the inertial effect (see for instance [5]). When the porous medium follows a periodic distribution, it can be obtained from a Stokes model by using homogenization methods (see [14] and [22]). In that case, the only parameter characterizing the porous media, named flow resistivity, can be related to the permeability tensor which can be formally calculated by solving a Stokes problem defined in the fluid part of the unit cell.

A finite element solution of acoustic propagation in a rigid porous media simulated with these two models has been carried out in Bermúdez *et al* [6].

2.2 Poroelastic models

Theoretical basis for mechanical behavior of poroelastic materials has been established by Biot (see [8]). He describes the propagation of elastic waves in fluid-saturated elastic porous media. Adaption of Biot's theory to acoustic propagation can be seen in Allard [1]. The resulting model, named below as "classical" Biot's model, is the basis for most of numerical studies involving poroelastic materials. However, the fact that macroscopic coefficients appearing in this model can only be obtained from empirical experiments (see Biot and Willis [9]) and that it involves two displacements (for solid and fluid) are unsuitable features. These are reasons why a rigorous derivation of Biot-like models starting from the first principles and using homogenization techniques is an interesting task. This has been done by Mikelić and coworkers in several papers and we shall refer to them as generalized Biot's models.

2.2.1 Classical Biot model

The classical Biot model involves two displacements, $\mathbf{u}$ and $\mathbf{U}$, which are the macroscopic displacement of solid and fluid parts of the porous medium,

respectively. It is given by

$$\rho_{11}\frac{\partial^2\mathbf{u}}{\partial t^2}+\rho_{12}\frac{\partial^2\mathbf{U}}{\partial t^2}=N\ \mathrm{div}(D(\mathbf{u}))+(A+N)\nabla\ \mathrm{div}\mathbf{u}+Q\nabla\ \mathrm{div}\mathbf{U}-b\frac{\partial}{\partial t}(\mathbf{u}-\mathbf{U}), \tag{1}$$

$$\rho_{12}\frac{\partial^2\mathbf{u}}{\partial t^2}+\rho_{22}\frac{\partial^2\mathbf{U}}{\partial t^2}=Q\nabla\ \mathrm{div}\mathbf{u}+R\nabla\ \mathrm{div}\mathbf{U}+b\frac{\partial}{\partial t}(\mathbf{u}-\mathbf{U}), \tag{2}$$

where N, A, Q and R are the elastic coefficients which are determined through the experiments described in Biot and Willis [9], $D(\mathbf{u})=\frac{1}{2}(\nabla\mathbf{u}+\nabla\mathbf{u}^T)$ and b is related with the permeability tensor, K, appearing in Darcy's law, by means of $b=\mu\phi^2/K$, with μ being fluid viscosity and ϕ porosity.

2.2.2 Generalized Biot models

Dissipative poroelastic model (open pore)

The dissipative generalized Biot model has been derived, by using homogenization techniques, in Clopeau *et al* [10] for the case where fluid viscosity is of order $O(\varepsilon^2)$, ε being the size of the elementary cell. It is the following:

$$\rho\mathbf{I}\frac{\partial^2\mathbf{u}^0}{\partial t^2}-\sum_{i,j}\mathbf{e}_i\frac{d}{dt}\int_0^t\mathcal{A}_{ij}(t-\tau)\left[\frac{\partial p}{\partial x_j}(x,\tau)+\rho_{\mathrm{F}}\frac{\partial^2\mathbf{u}_j^0}{\partial\tau^2}(x,\tau)\right]d\tau-\ \mathrm{div}_x\left\{A^HD_x(\mathbf{u}^0)\right\}$$
$$-\ \mathrm{div}_x\left\{p\mathcal{B}^H\right\}+\phi\nabla_x p=\rho\mathbf{F}-\sum_{i,j}\mathbf{e}_i\frac{d}{dt}\int_0^t\mathcal{A}_{ij}(t-\tau)\rho_{\mathrm{F}}F_j(x,\tau)d\tau, \tag{3}$$

$$\mathrm{div}_x\left\{\phi\frac{\partial\mathbf{u}^0}{\partial t}+\sum_{i,j}\mathbf{e}_i\int_0^t\mathcal{A}_{ij}(t-\tau)\left[F_j(x,\tau)-\frac{1}{\rho_{\mathrm{F}}}\frac{\partial p}{\partial x_j}(x,\tau)\right]d\tau\right.$$
$$\left.-\sum_{i,j}\mathbf{e}_i\int_0^t\mathcal{A}_{ij}(t-\tau)\frac{\partial^2\mathbf{u}_j^0}{\partial\tau^2}(x,\tau)d\tau\right\}=\mathcal{B}^H:D_x\left(\frac{\partial\mathbf{u}^0}{\partial t}\right)+\hat{c}\frac{\partial p}{\partial t}, \tag{4}$$

where $\mathbf{e}_i$ is the i-th vector of the canonical basis, ρ_{F} is the fluid density, ρ_{S} is the density of the solid skeleton, $\rho = \phi\rho_{\mathrm{F}} + (1-\phi)\rho_{\mathrm{S}}$ and

$$A^H_{klij} := \left(\int_{\mathcal{Y}_s} A\left(\frac{\mathbf{e}_i \otimes \mathbf{e}_j + \mathbf{e}_j \otimes \mathbf{e}_i}{2} + D_y(\mathbf{w}^{ij}) \right) dy \right)_{kl}, \tag{5}$$

$$\mathcal{B}^H := \int_{\mathcal{Y}_s} A D_y(\mathbf{w}^0)\, dy, \tag{6}$$

$$\hat{c} := \int_{\mathcal{Y}_s} \mathrm{div}_y \mathbf{w}^0\, dy \tag{7}$$

$$\mathcal{A}_{ij}(t) := \int_{\mathcal{Y}_f} \mathbf{w}^j_i(y, \frac{\rho_{\mathrm{F}}}{\mu} t) dy, \tag{8}$$

which, in particular, implies

$$\mathcal{A}_{ij}(0) = \int_{\mathcal{Y}_f} \mathbf{w}^j_i(y,0)\, dy \;=\; \phi\delta_{ij}. \tag{9}$$

In order to calculate these coefficients, we must solve the following boundary-value problems defined in the fluid and in the solid parts of the unit cell:

$$\begin{cases} \mathrm{div}_y \left\{ A\left(\frac{\mathbf{e}_i\otimes\mathbf{e}_j+\mathbf{e}_j\otimes\mathbf{e}_i}{2} + D_y(\mathbf{w}^{ij}) \right) \right\} = \mathbf{0} & \text{in } \mathcal{Y}_s, \\ A(\frac{\mathbf{e}_i\otimes\mathbf{e}_j+\mathbf{e}_j\otimes\mathbf{e}_i}{2} + D_y(\mathbf{w}^{ij}))\boldsymbol{\nu} = \mathbf{0} & \text{on } \partial\mathcal{Y}_s \backslash \partial\mathcal{Y}, \\ \int_{\mathcal{Y}_s} \mathbf{w}^{ij}(y)\, dy = \mathbf{0}, \quad \mathbf{w}^{ij} \text{ is 1-periodic}, \end{cases} \tag{10}$$

$$\begin{cases} -\,\mathrm{div}_y \left\{ A D_y(\mathbf{w}^0) \right\} = \mathbf{0} & \text{in } \mathcal{Y}_s, \\ A D_y(\mathbf{w}^0)\boldsymbol{\nu} = -\boldsymbol{\nu} & \text{on } \partial\mathcal{Y}_s \backslash \partial\mathcal{Y}, \\ \int_{\mathcal{Y}_s} \mathbf{w}^0(y)\, dy = \mathbf{0}, \quad \mathbf{w}^0 \text{ is 1-periodic} \end{cases} \tag{11}$$

and

$$\begin{cases} \dfrac{\partial \mathbf{w}^i}{\partial t} - \Delta \mathbf{w}^i + \nabla \pi^i = \mathbf{0}, \\ \mathrm{div}_y \mathbf{w}^i = 0, \quad \mathbf{w}^i(y,0) = \mathbf{e}_i, \\ \mathbf{w}^i \mid_{\partial\mathcal{Y}_f \backslash \partial\mathcal{Y}} = \mathbf{0}, \quad \{\mathbf{w}^i, \pi^i\} \text{ is 1-periodic.} \end{cases} \tag{12}$$

Dissipative poroelastic model (closed pore)

When the pore of the material is closed, we are led to the following system of

partial differential equations, which has been derived in Clopeau *et al* [11]:

$$\rho\mathbf{I}\frac{\partial^2\mathbf{u}^0}{\partial t^2} - \operatorname{div}_x\left\{A^H D_x(\mathbf{u}^0)\right\} + (\phi\mathbf{I} - \mathcal{B}^H)\nabla_x p = \rho\mathbf{F} \quad \text{in } \Omega\times]0,T[, \tag{13}$$

$$-\hat{c}\frac{\partial^2 p}{\partial t^2} + \operatorname{div}_x\left\{(\phi\mathbf{I} - \mathcal{B}^H)\frac{\partial^2\mathbf{u}^0}{\partial t^2}\right\} = 0 \quad \text{in } \Omega\times]0,T[, \tag{14}$$

where A^H, $\mathcal{B}^H$ and $\hat{c}$, with expressions given by (5), (6) and (7), respectively, are defined from auxiliary functions $\mathbf{w}^{ij}$ and $\mathbf{w}^0$ which are solution of the boundary-value problems (10) and (11), respectively.

Non-dissipative poroelastic model (open pore)

An homogenized model for poroelastic materials with open pore and inviscid fluid has been obtained in Ferrín and Mikelić [15]. It is the following:

$$\rho\mathbf{I}\frac{\partial^2\mathbf{u}^0}{\partial t^2} - \mathcal{A}\left[\nabla_x p + \rho_{\mathrm{F}}\frac{\partial^2\mathbf{u}^0}{\partial t^2}\right] - \operatorname{div}_x\left\{A^H D_x(\mathbf{u}^0)\right\} + (\phi\mathbf{I} - \mathcal{B}^H)\nabla_x p = (\rho\mathbf{I} - \rho_{\mathrm{F}}\mathcal{A})\mathbf{F}, \tag{15}$$

$$\operatorname{div}_x\left\{\phi\frac{\partial\mathbf{u}^0}{\partial t} + \frac{1}{\rho_{\mathrm{F}}}\mathcal{A}\int_0^t\left[\rho_{\mathrm{F}}\mathbf{F} - \nabla_x p\right]d\tau\right\} - \operatorname{div}_x\left\{\mathcal{A}\frac{\partial\mathbf{u}^0}{\partial t}\right\} = \mathcal{B}^H : D_x\left(\frac{\partial\mathbf{u}^0}{\partial t}\right) + \hat{c}\frac{\partial p}{\partial t}, \tag{16}$$

where A^H, $\mathcal{B}^H$ and $\hat{c}$ are given by (5), (6) and (7), respectively, whereas

$$\mathcal{A}_{ij} := \int_{\mathcal{Y}_f}\left(\delta_{ij} - \frac{\partial\xi^j}{\partial y_i}\right)dy, \tag{17}$$

with $\mathbf{w}^{ij}$, $\mathbf{w}^0$ and ξ^j being the solution of the boundary-value problems (10) and (11), defined in the solid part of the unit cell, and

$$\begin{cases} -\Delta_y\xi^i = 0 & \text{in } \mathcal{Y}_f, \\ \dfrac{\partial\xi^i}{\partial\boldsymbol{\nu}} = \mathbf{e}_i\cdot\boldsymbol{\nu} & \text{on } \partial\mathcal{Y}_f\backslash\partial\mathcal{Y}, \\ \xi^i \text{ is 1-periodic in } \mathcal{Y}, & \\ \int_{\mathcal{Y}_f}\xi^i dy = 0 & \end{cases} \tag{18}$$

defined in the fluid part of the unit cell, respectively.

Non-dissipative poroelastic model (closed pore)

Similar to the open pore non-dissipative poroelastic model, when the fluid part

is isolated in each cell, the following homogenized model has been derived in Ferrín and Mikelić [15]:

$$\rho\mathbf{I}\frac{\partial^2\mathbf{u}^0}{\partial t^2} - \operatorname{div}_x\left\{A^H D_x(\mathbf{u}^0)\right\} - \operatorname{div}_x\left\{p\mathcal{B}^H\right\} + \phi\nabla_x p = \rho\mathbf{F}, \tag{19}$$

$$\phi\operatorname{div}_x\frac{\partial\mathbf{u}^0}{\partial t} = \mathcal{B}^H : D_x\left(\frac{\partial\mathbf{u}^0}{\partial t}\right) + \frac{\partial p}{\partial t}\int_{\mathcal{Y}_s}\operatorname{div}_y\mathbf{w}^0\,dy, \tag{20}$$

where A^H, $\mathcal{B}^H$ and $\hat{c}$ are given by (5), (6) and (7), respectively, and the cell problems verified by $\mathbf{w}^{ij}$ and $\mathbf{w}^0$ are (10) and (11), respectively.

3 Statement of the problem

In the rest of the paper we consider a coupled system consisting of an acoustic fluid (i.e. inviscid compressible barotropic) in contact with a porous medium. Both are enclosed in a three-dimensional cavity with rigid walls except one on which an harmonic excitation is applied. Let Ω_{F} and Ω_{A} be the domains occupied by the fluid and the porous medium, respectively (see Fig. 1). The boundary of $\overline{\Omega}_{\mathrm{F}}\cup\overline{\Omega}_{\mathrm{A}}$, denoted by Γ, is the union of two parts, Γ_{W} and Γ_{E}, where Γ_{W} denotes the rigid walls of the cavity. Let $\boldsymbol{\nu}$ be the outward unit normal vector to Γ. The interface between the fluid and the porous medium is denoted by Γ_{I} and $\mathbf{n}$ is the unit normal vector to this interface pointing outwards Ω_{A}. Fig. 1 includes a vertical cut of the domain for a better understanding of notations.

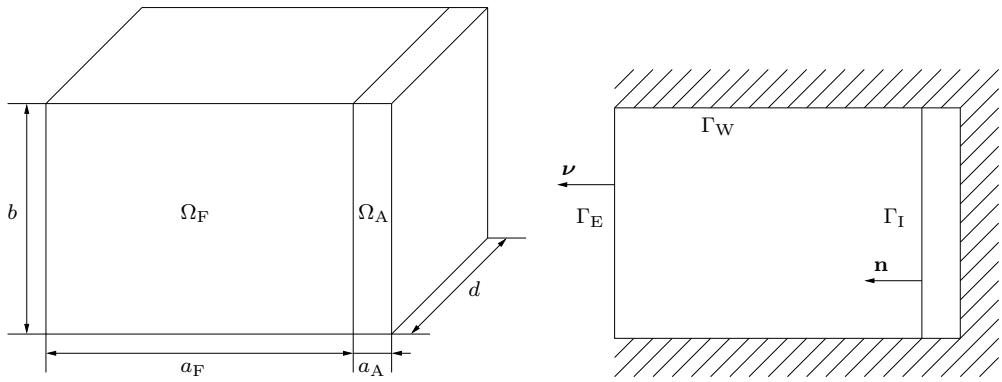

Fig. 1. 3D domain and vertical cut.

In order to study the response of the fluid-porous coupled system subject to an harmonic excitation acting on Γ_{E}, we assume that skeleton of the porous medium is elastic and consider the model for open pore non-dissipative poroelastic media given in the previous Section.

Firstly, we recall that governing equations for free small amplitude motions of an acoustic fluid filling Ω_{F} are given, in terms of displacement and pressure

fields, by

$$\rho_{\mathrm{F}} \frac{\partial^2 \mathbf{U}_{\mathrm{F}}}{\partial t^2} + \mathrm{grad}\ P_{\mathrm{F}} = \mathbf{0} \text{ in } \Omega_{\mathrm{F}}, \tag{21}$$

$$P_{\mathrm{F}} = -\rho_{\mathrm{F}} c^2 \mathrm{div}\ \mathbf{U}_{\mathrm{F}} \text{ in } \Omega_{\mathrm{F}}, \tag{22}$$

where P_{F} is pressure, $\mathbf{U}_{\mathrm{F}}$ is displacement field, ρ_{F} is density and c is acoustic speed.

Secondly, if we denote by $\mathbf{U}_{\mathrm{A}}$ and P_{A} the macroscopic displacement and pressure fields in the porous medium, the equations describing small motions are,

$$(\rho \mathbf{I} - \rho_{\mathrm{F}} \mathcal{A}) \frac{\partial^2 \mathbf{U}_{\mathrm{A}}}{\partial t^2} - \mathrm{div}\left(A^H \left[D(\mathbf{U}_{\mathrm{A}})\right]\right) - \\ - \left(\mathcal{A} + \mathcal{B}^H - \phi \mathbf{I}\right) \mathrm{grad}\ P_{\mathrm{A}} = \mathbf{0} \text{ in } \Omega_{\mathrm{A}}, \tag{23}$$

$$\hat{c} \frac{\partial^2 P_{\mathrm{A}}}{\partial t^2} + \frac{1}{\rho_{\mathrm{F}}} \mathrm{div}\left(\mathcal{A} \mathrm{grad}\ P_{\mathrm{A}}\right) = -\mathrm{div}\left(\left(\mathcal{A} + \mathcal{B}^H - \phi \mathbf{I}\right) \frac{\partial^2 \mathbf{U}_{\mathrm{A}}}{\partial t^2}\right) \text{ in } \Omega_{\mathrm{A}}, \tag{24}$$

where $D(\mathbf{U}) = \frac{1}{2}(\mathrm{grad}\ \mathbf{U} + (\mathrm{grad}\ \mathbf{U})^t)$ and coefficient $\hat{c}$, tensors $\mathcal{A}, \mathcal{B}^H$ and linear operator A^H depend on geometry of cells composing the poroelastic material and also on physical properties of its solid and fluid parts. In fact, one can check that $\mathcal{B}^H$ is a symmetric linear operator and tensor A^H $(A^H[D])_{kl} = A_{klij} D_{ij})$ satisfies $A^H_{klij} = A^H_{lkij} = A^H_{lkji}$.

Since the fluid is supposed to be inviscid, only the normal component of displacements vanishes on Γ_{W}, namely,

$$\mathbf{U}_{\mathrm{F}} \cdot \boldsymbol{\nu} = 0 \quad \text{on } \Gamma_{\mathrm{W}} \cap \partial\Omega_{\mathrm{F}}, \tag{25}$$

whereas for boundary displacement of porous medium we suppose

$$\mathbf{U}_{\mathrm{A}} = \mathbf{0} \quad \text{on } \Gamma_{\mathrm{W}} \cap \partial\Omega_{\mathrm{A}}. \tag{26}$$

Similarly, on interface Γ_{I} between fluid and porous medium we consider the usual interface conditions of continuity of forces and normal displacements, that is,

$$-P_{\mathrm{F}} \mathbf{n} = A^H \left[D(\mathbf{U}_{\mathrm{A}})\right] \mathbf{n} + P_{\mathrm{A}} \left(\mathcal{A} + \mathcal{B}^H - \phi \mathbf{I}\right) \mathbf{n} \text{ on } \Gamma_{\mathrm{I}}, \tag{27}$$

$$\mathbf{U}_{\mathrm{F}} \cdot \mathbf{n} = \mathbf{U}_{\mathrm{A}} \cdot \mathbf{n} \text{ on } \Gamma_{\mathrm{I}}. \tag{28}$$

If a normal displacement U_0 is imposed on Γ_{E}, the above equations describing the motion of coupled system (21)-(28) must be completed with boundary condition

$$\mathbf{U}_{\mathrm{F}} \cdot \boldsymbol{\nu} = U_0 \quad \text{on } \Gamma_{\mathrm{E}}. \tag{29}$$

Finally, in order to close the model (see [4]), we are going to assume that

$$\mathcal{A}\text{grad}p_{\mathrm{A}} \cdot \boldsymbol{\nu} = \mathbf{0} \text{ on } \Gamma_{\mathrm{W}} \cap \partial\Omega_{\mathrm{A}}, \ \mathcal{A}\text{grad}p_{\mathrm{A}} \cdot \mathbf{n} = \mathbf{0} \text{ on } \Gamma_{\mathrm{I}}. \tag{30}$$

We are interested in harmonic vibrations so let us suppose excitation U_0 to be harmonic, i.e.,

$$U_0(x, y, z, t) = \mathrm{Re}\left(e^{i\omega t} u_0(x, y, z)\right). \tag{31}$$

Then all fields are also harmonic:

$$\mathbf{U}_{\mathrm{F}}(x, y, z, t) = \mathrm{Re}\left(e^{i\omega t} \mathbf{u}_{\mathrm{F}}(x, y, z)\right), \tag{32}$$

$$\mathbf{U}_{\mathrm{A}}(x, y, z, t) = \mathrm{Re}\left(e^{i\omega t} \mathbf{u}_{\mathrm{A}}(x, y, z)\right), \tag{33}$$

$$P_{\mathrm{F}}(x, y, z, t) = \mathrm{Re}\left(e^{i\omega t} p_{\mathrm{F}}(x, y, z)\right), \tag{34}$$

$$P_{\mathrm{A}}(x, y, z, t) = \mathrm{Re}\left(e^{i\omega t} p_{\mathrm{A}}(x, y, z)\right). \tag{35}$$

By replacing these expressions in equations (21)-(30), we can define an harmonic source problem associated with the evolutionary source problem, namely,

$$-\omega^2 \rho_{\mathrm{F}} \mathbf{u}_F + \text{grad } p_{\mathrm{F}} = \mathbf{0} \text{ in } \Omega_{\mathrm{F}}, \tag{36}$$

$$p_{\mathrm{F}} = -\rho_{\mathrm{F}} c^2 \text{div } \mathbf{u}_{\mathrm{F}} \text{ in } \Omega_{\mathrm{F}}, \tag{37}$$

$$\begin{aligned} -\omega^2 \left(\rho \mathbf{I} - \rho_{\mathrm{F}} \mathcal{A}\right) \mathbf{u}_{\mathrm{A}} - \ \text{div}\left(A^H [D(\mathbf{u}_{\mathrm{A}})]\right) \\ - \left(\mathcal{A} + \mathcal{B}^H - \phi \mathbf{I}\right) \text{grad } p_{\mathrm{A}} = \mathbf{0} \text{ in } \Omega_{\mathrm{A}}, \end{aligned} \tag{38}$$

$$\begin{aligned} -\omega^2 \hat{c} p_{\mathrm{A}} + \frac{1}{\rho_{\mathrm{F}}} \text{div}(\mathcal{A}\text{grad}p_{\mathrm{A}}) = \\ \omega^2 \text{div}\left(\left(\mathcal{A} + \mathcal{B}^H - \phi \mathbf{I}\right) \mathbf{u}_{\mathrm{A}}\right) \text{ in } \Omega_{\mathrm{A}}, \end{aligned} \tag{39}$$

$$-p_{\mathrm{F}} \mathbf{n} = A^H \left[D(\mathbf{u}_{\mathrm{A}})\right] \mathbf{n} + p_{\mathrm{A}} \left(\mathcal{A} + \mathcal{B}^H - \phi \mathbf{I}\right) \mathbf{n} \text{ on } \Gamma_{\mathrm{I}}, \tag{40}$$

$$\mathbf{u}_{\mathrm{F}} \cdot \mathbf{n} = \mathbf{u}_{\mathrm{A}} \cdot \mathbf{n} \text{ on } \Gamma_{\mathrm{I}}, \tag{41}$$

$$\mathbf{u}_{\mathrm{F}} \cdot \boldsymbol{\nu} = 0 \text{ on } \Gamma_{\mathrm{W}} \cap \partial\Omega_{\mathrm{F}}, \tag{42}$$

$$\mathbf{u}_{\mathrm{A}} = \mathbf{0} \text{ on } \Gamma_{\mathrm{W}} \cap \partial\Omega_{\mathrm{A}}, \tag{43}$$

$$\mathcal{A}\text{grad}p_{\mathrm{A}} \cdot \boldsymbol{\nu} = \mathbf{0} \text{ on } \Gamma_{\mathrm{W}} \cap \partial\Omega_{\mathrm{A}}, \tag{44}$$

$$\mathcal{A}\text{grad}p_{\mathrm{A}} \cdot \mathbf{n} = \mathbf{0} \text{ on } \Gamma_{\mathrm{I}}, \tag{45}$$

$$\mathbf{u}_{\mathrm{F}} \cdot \boldsymbol{\nu} = u_0 \text{ on } \Gamma_{\mathrm{E}}. \tag{46}$$

4 Weak formulation

In order to use finite element methods for numerical solution of (36)-(46), we write a weak formulation. For this purpose, we first introduce appropriate functional spaces. Let $\mathbf{V}$ be the Hilbert space

$$\mathbf{V} = \mathrm{H}(\text{div}, \Omega_{\mathrm{F}}) \times \mathrm{L}^2(\Gamma_{\mathrm{I}}) \times \mathrm{H}^1(\Omega_{\mathrm{A}})^3 \times \mathrm{H}^1(\Omega_{\mathrm{A}})$$

and $\mathbf{V}_0$ its closed subspace

$$\mathbf{V}_0 = \{(\mathbf{v}_\mathrm{F}, q_\mathrm{F}, \mathbf{v}_\mathrm{A},\ q_\mathrm{A}) \in \mathbf{V}\ :\ \mathbf{v}_\mathrm{F}\cdot\boldsymbol{\nu} = 0 \text{ on } (\Gamma_\mathrm{W}\cup\Gamma_\mathrm{E})\cap\partial\Omega_\mathrm{F}, \\ \mathbf{v}_\mathrm{A} = \mathbf{0} \text{ on } \Gamma_\mathrm{W}\cap\partial\Omega_\mathrm{A}\}\,.$$

Because of the choice of this functional framework and due to trace Theorem for space $\mathrm{H}(\mathrm{div},\Omega_\mathrm{F})$, we are led to assume that Dirichlet data function u_0 belongs to functional space $\mathrm{H}_{00}^{-\frac{1}{2}}(\Gamma_\mathrm{N})$.

In order to get a weak formulation of source problem (36)-(46), we first multiply equation (36) by the complex conjugate of a virtual fluid displacement $\mathbf{v}_\mathrm{F}\in\mathrm{H}(\mathrm{div},\Omega_\mathrm{F})$ satisfying Dirichlet condition (42), and then integrate in Ω_F. By using a Green's formula and equation (37) we obtain

$$-\omega^2\int_{\Omega_\mathrm{F}}\rho_\mathrm{F}\mathbf{u}_\mathrm{F}\cdot\bar{\mathbf{v}}_\mathrm{F} + \int_{\Omega_\mathrm{F}}\rho_\mathrm{F}c^2\mathrm{div}\mathbf{u}_\mathrm{F}\ \mathrm{div}\bar{\mathbf{v}}_\mathrm{F} - \int_{\Gamma_\mathrm{I}} p_\mathrm{F}\bar{\mathbf{v}}_\mathrm{F}\cdot\mathbf{n} = 0$$

In an analogous way, equations (38) and (39), when multiplied by the complex conjugate of a virtual porous medium displacement $\bar{\mathbf{v}}_\mathrm{A}$ satisfying Dirichlet condition (43) and a virtual pressure $\bar{q}_\mathrm{A}$ and then integrated in Ω_A yield

$$-\omega^2\int_{\Omega_\mathrm{A}}(\rho\mathbf{I}-\rho_\mathrm{F}\mathcal{A})\,\mathbf{u}_\mathrm{A}\cdot\bar{\mathbf{v}}_\mathrm{A} + \int_{\Omega_\mathrm{A}} A^H[D(\mathbf{u}_\mathrm{A})] : D(\bar{\mathbf{v}}_\mathrm{A}) + \\ \int_{\Omega_\mathrm{A}}\mathrm{div}\left((\mathcal{A}+\mathcal{B}^H-\phi\mathbf{I})^t\bar{\mathbf{v}}_\mathrm{A}\right)p_\mathrm{A} = \int_{\Gamma_\mathrm{I}} A^H\left[D(\mathbf{u}_\mathrm{A})\right]\mathbf{n}\cdot\bar{\mathbf{v}}_\mathrm{A} + \\ \int_{\Gamma_\mathrm{I}}(\mathcal{A}+\mathcal{B}^H-\phi\mathbf{I})^t\bar{\mathbf{v}}_\mathrm{A}\cdot\mathbf{n}p_\mathrm{A},$$

and

$$\int_{\Omega_\mathrm{A}}\hat{c}p_\mathrm{A}\bar{q}_\mathrm{A} + \int_{\Omega_\mathrm{A}}\frac{1}{\rho_\mathrm{F}\omega^2}\mathcal{A}\mathrm{grad}\ p_\mathrm{A}\cdot\mathrm{grad}\ \bar{q}_\mathrm{A} + \int_{\Omega_\mathrm{A}}\mathrm{div}\left((\mathcal{A}+\mathcal{B}^H-\phi\mathbf{I})\mathbf{u}_\mathrm{A}\right)\bar{q}_\mathrm{A} = \\ \int_{\Gamma_\mathrm{I}}\frac{1}{\rho_\mathrm{F}\omega^2}\bar{q}_\mathrm{A}\mathcal{A}\mathrm{grad}\ p_\mathrm{A}\cdot\mathbf{n} + \int_{\Gamma_\mathrm{D}\cap\partial\Omega_\mathrm{A}}\frac{1}{\rho_\mathrm{F}\omega^2}\bar{q}_\mathrm{A}\mathcal{A}\mathrm{grad}p_\mathrm{A}\cdot\boldsymbol{\nu},$$

respectively.

Now, by adding the last three equations and using interface and boundary conditions (40), (44) and (45), we obtain

$$\int_{\Omega_\mathrm{F}}\rho_\mathrm{F}c^2\mathrm{div}\ \mathbf{u}_\mathrm{F}\ \mathrm{div}\ \bar{\mathbf{v}}_\mathrm{F} - \omega^2\int_{\Omega_\mathrm{F}}\rho_\mathrm{F}\mathbf{u}_\mathrm{F}\cdot\bar{\mathbf{v}}_\mathrm{F} - \omega^2\int_{\Omega_\mathrm{A}}(\rho\mathbf{I}-\rho_\mathrm{F}\mathcal{A})\,\mathbf{u}_\mathrm{A}\cdot\bar{\mathbf{v}}_\mathrm{A} + \\ \int_{\Omega_\mathrm{A}}A^H[D(\mathbf{u}_\mathrm{A})] : D(\bar{\mathbf{v}}_\mathrm{A}) + \int_{\Omega_\mathrm{A}}\mathrm{div}\left((\mathcal{A}+\mathcal{B}^H-\phi\mathbf{I})^t\bar{\mathbf{v}}_\mathrm{A}\right)p_\mathrm{A} + \int_{\Omega_\mathrm{A}}\hat{c}p_\mathrm{A}\bar{q}_\mathrm{A} + \\ \int_{\Omega_\mathrm{A}}\frac{1}{\rho_\mathrm{F}\omega^2}\mathcal{A}\mathrm{grad}\ p_\mathrm{A}\cdot\mathrm{grad}\ \bar{q}_\mathrm{A} + \int_{\Omega_\mathrm{A}}\mathrm{div}\left((\mathcal{A}+\mathcal{B}^H-\phi\mathbf{I})\mathbf{u}_\mathrm{A}\right)\bar{q}_\mathrm{A} = \\ \int_{\Gamma_\mathrm{I}}p_\mathrm{F}(\bar{\mathbf{v}}_\mathrm{F}\cdot\mathbf{n}-\bar{\mathbf{v}}_\mathrm{A}\cdot\mathbf{n}).$$

Finally, kinematic constraint (41) is weakly imposed on the interface between the fluid and the porous medium by integrating this equation multiplied by any test function q_{F} defined on Γ_{I}:

$$\int_{\Gamma_{\mathrm{I}}} \bar{q}_{\mathrm{F}}(\mathbf{u}_{\mathrm{A}} \cdot \mathbf{n} - \mathbf{u}_{\mathrm{F}} \cdot \mathbf{n}) = 0.$$

All together allow us to write the following source hybrid problem:

For fixed angular frequency ω, find $(\mathbf{u}_{\mathrm{F}}, p_{\mathrm{F}}, \mathbf{u}_{\mathrm{A}}, p_{\mathrm{A}}) \in \mathbf{V}$ satisfying (42), (43), (46) and furthermore,

$$\begin{aligned}\int_{\Omega_{\mathrm{F}}} \rho_{\mathrm{F}} c^2 \mathrm{div}\ \mathbf{u}_{\mathrm{F}}\ \mathrm{div}\ \bar{\mathbf{v}}_{\mathrm{F}} - \omega^2 \int_{\Omega_{\mathrm{F}}} \rho_{\mathrm{F}} \mathbf{u}_{\mathrm{F}} \cdot \bar{\mathbf{v}}_{\mathrm{F}} - \omega^2 \int_{\Omega_{\mathrm{A}}} (\rho \mathbf{I} - \rho_{\mathrm{F}} \mathcal{A})\, \mathbf{u}_{\mathrm{A}} \cdot \bar{\mathbf{v}}_{\mathrm{A}} + \\ \int_{\Omega_{\mathrm{A}}} A^H[D(\mathbf{u}_{\mathrm{A}})] : D(\bar{\mathbf{v}}_{\mathrm{A}}) + \int_{\Omega_{\mathrm{A}}} \mathrm{div}\left((\mathcal{A} + \mathcal{B}^H - \phi \mathbf{I})^t \bar{\mathbf{v}}_{\mathrm{A}}\right) p_{\mathrm{A}} + \\ \int_{\Omega_{\mathrm{A}}} \frac{1}{\rho_{\mathrm{F}} \omega^2} \mathcal{A} \mathrm{grad}\ p_{\mathrm{A}} \cdot \mathrm{grad}\ \bar{q}_{\mathrm{A}} + \int_{\Omega_{\mathrm{A}}} \mathrm{div}\left((\mathcal{A} + \mathcal{B}^H - \phi \mathbf{I}) \mathbf{u}_{\mathrm{A}}\right) \bar{q}_{\mathrm{A}} \\ \int_{\Omega_{\mathrm{A}}} \hat{c} p_{\mathrm{A}} \bar{q}_{\mathrm{A}} = \int_{\Gamma_{\mathrm{I}}} p_{\mathrm{F}} (\bar{\mathbf{v}}_{\mathrm{F}} \cdot \mathbf{n} - \bar{\mathbf{v}}_{\mathrm{A}} \cdot \mathbf{n}), \quad (47)\end{aligned}$$

$$\int_{\Gamma_{\mathrm{I}}} \bar{q}_{\mathrm{F}}(\mathbf{u}_{\mathrm{A}} \cdot \mathbf{n} - \mathbf{u}_{\mathrm{F}} \cdot \mathbf{n}) = 0, \quad (48)$$

for all $(\mathbf{v}_{\mathrm{F}}, q_{\mathrm{F}}, \mathbf{v}_{\mathrm{F}}, q_{\mathrm{A}}) \in \mathbf{V}_0$.

5 Finite element discretization

Fluid and porous displacement fields belong to different functional spaces, $\mathrm{H}(\mathrm{div}, \Omega_{\mathrm{F}})$ and $\mathrm{H}^1(\Omega_{\mathrm{A}})^3$, respectively, hence different types of finite elements should be used for each of them in order to discretize weak problem (47)-(48).

Let $\mathcal{T}_h$ be a regular family of tetrahedral partitions of $\Omega_{\mathrm{F}} \cup \Omega_{\mathrm{A}}$ such that every tetrahedra is completely contained either in Ω_{F} or in Ω_{A}. We also assume that faces of tetrahedra lying on $\Gamma_{\mathrm{W}} \cup \Gamma_{\mathrm{E}} \cup \Gamma_{\mathrm{I}}$ are completely contained either in Γ_{W}, either in Γ_{E} or in Γ_{I}.

To approximate fluid displacements, the lowest order Raviart-Thomas finite element (see [21] and [7])) is used in order to avoid spurious modes typical of displacement formulations when they are discretized by standard Lagrange finite elements (see [18]). They consist of vector valued functions which, when restricted to each tetrahedron, are incomplete linear polynomials of the form

$$\mathbf{u}^h(x, y, z) = (a + dx, b + dy, c + dz),\ \ a, b, c, d \in \mathbb{C}.$$

These vector fields have constant normal components on each of the four faces of a tetrahedron (see Fig. 2) which define a unique polynomial function of this type. Moreover, the global discrete displacement field $\mathbf{u}^h$ is allowed to have discontinuous tangential components on the faces of tetrahedra of partition $\mathcal{T}_h$. Instead, its constant normal components must be continuous through these faces (these constant values being the degrees of freedom defining $\mathbf{u}^h$). Because of this, div $\mathbf{u}^h$ is globally well defined in Ω_{F}.

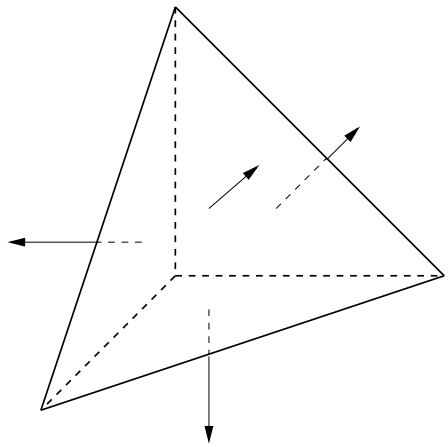

Fig. 2. Raviart-Thomas finite element.

Thus, for fluid displacements we use the Raviart-Thomas space (see [21])

$$\mathbf{R}_h\left(\Omega_{\mathrm{F}}\right) := \left\{\mathbf{u} \in \mathrm{H}(\mathrm{div}, \Omega_{\mathrm{F}}) \ : \ \mathbf{u}|_T \in \mathcal{R}_0(T), \ \forall T \in \mathcal{T}_h, \ T \subset \Omega_{\mathrm{F}}\right\},$$

where

$$\mathcal{R}_0(T) := \left\{\mathbf{u} \in \mathcal{P}_1(T)^2 \ : \ \mathbf{u}(x,y,z) = (a+dx, b+dy, c+dz), \ a,b,c,d \in \mathbb{C}\right\}.$$

To approximate displacements in the porous medium, we use the so called "MINI element" in order to achieve stability in the discrete problem (see [3]). We recall definition of the corresponding discrete space by first defining bubble functions. For fixed $T \in \mathcal{T}_h$, we denote by $\lambda_1^T, \ldots, \lambda_4^T$ barycentric coordinates in tetrahedron T. Then bubble function α, associated with T, is defined by the product

$$\alpha = 256 \prod_{i=1}^{4} \lambda_i^T.$$

This bubble function is a polynomial of degree four, null on surface of tetrahedron T and taking value one at barycenter of T. The approximating space associated with the MINI element consists of continuous vector valued functions whose components, restricted to each tetrahedron, are sum of a bubble function and a polynomial of degree one, i.e.,

$$\mathbf{u}_i^h(x,y,z)|_T = ax + by + cz + d + e\alpha(x,y,z), \ a,b,c,d,e \in \mathbb{C}.$$

The degrees of freedom for functions in this space are the values of the vector field at vertices and barycenters of tetrahedra (see Fig. 3).

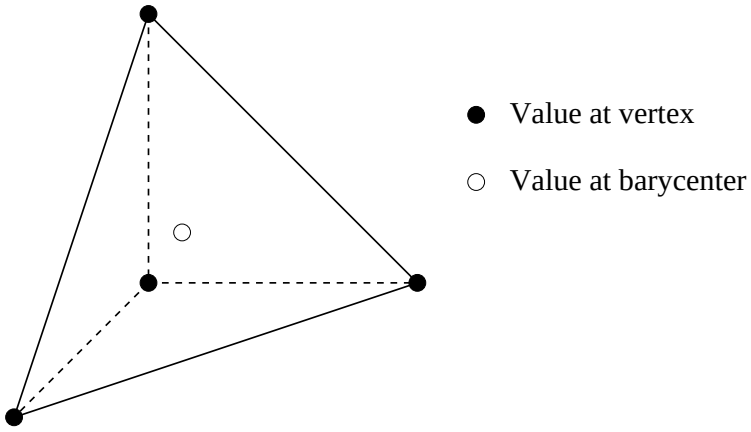

Fig. 3. MINI finite element.

Then, for porous displacements, we use the MINI space

$$\mathbf{M}_h\left(\Omega_{\mathrm{A}}\right) := \left\{\mathbf{u} \in \mathrm{H}^1(\Omega_{\mathrm{A}})^3 \ : \ \mathbf{u}|_T \in (\mathcal{P}_1(T) \oplus \mathcal{P}^b(T))^3, \ \forall T \in \mathcal{T}_h, \ T \subset \Omega_{\mathrm{A}}\right\},$$

where

$$\mathcal{P}^b(T) = \{a\alpha : a \in \mathbb{C}\}.$$

To approximate porous medium pressure, continuous piecewise linear finite elements are used. They consist of scalar valued functions which, when restricted to each tetrahedron, are polynomials of the form

$$p^h(x, y, z)|_T = ax + by + cz + d, \ a, b, c, d \in \mathbb{C}.$$

Thus, porous medium pressure is approximated in the finite-dimensional space,

$$\mathbf{L}_h\left(\Omega_{\mathrm{A}}\right) := \left\{p \in \mathrm{H}^1(\Omega_{\mathrm{A}}) \ : \ p|_T \in \mathcal{P}_1(T), \ \forall T \in \mathcal{T}_h, \ T \subset \Omega_{\mathrm{A}}\right\}.$$

We recall that the degrees of freedom defining p^h are its values at vertices of tetrahedra.

Finally, in order to approximate the interface pressure we use piecewise constant functions on the triangles of the mesh lying on the interface Γ_I. In other words, for interface pressure we use the space

$$\mathbf{C}_h\left(\Gamma_{\mathrm{I}}\right) := \left\{p \in \mathrm{L}^2(\Gamma_{\mathrm{I}}) \ : \ p|_{\partial T} \in \mathcal{P}_0(\partial T), \ \forall T \in \mathcal{T}_h, \ \partial T \cap \Gamma_{\mathrm{I}} \neq \emptyset\right\}.$$

The degrees of freedom of this finite element space are the (constant) values on triangles in Γ_{I}.

Consequently, the discrete analogue to $\mathbf{V}$ is

$$\mathbf{V}_h = \mathbf{R}_h\left(\Omega_{\mathrm{F}}\right) \times \mathbf{M}_h\left(\Omega_{\mathrm{A}}\right) \times \mathbf{L}_h\left(\Omega_{\mathrm{A}}\right) \times \mathbf{C}_h\left(\Gamma_{\mathrm{I}}\right) \tag{49}$$

while the corresponding to $\mathbf{V}_0$ is

$$\mathbf{V}_{0h} = \{(\mathbf{v}_{\mathrm{F}}, q_{\mathrm{F}}, \mathbf{v}_{\mathrm{A}}, q_{\mathrm{A}}) \in \mathbf{R}_h(\Omega_{\mathrm{F}}) \times \mathbf{M}_h(\Omega_{\mathrm{A}}) \times \mathbf{L}_h(\Omega_{\mathrm{A}}) \times \mathbf{C}_h(\Gamma_{\mathrm{I}}) \ : \\ \mathbf{v}_{\mathrm{F}} \cdot \boldsymbol{\nu} = 0 \text{ on } (\Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{N}}) \cap \partial\Omega_{\mathrm{F}}, \mathbf{v}_{\mathrm{A}} = \mathbf{0} \text{ on } \Gamma_{\mathrm{D}} \cap \partial\Omega_{\mathrm{A}}\}.$$

With these finite element spaces we can define the approximate problem to (47)-(48) by

For an angular frequency ω fixed, find $(\mathbf{u}_{\mathrm{F}}^h, p_{\mathrm{F}}^h, \mathbf{u}_{\mathrm{A}}^h, p_{\mathrm{A}}^h) \in \mathbf{V}_h$ satisfying

$$\begin{aligned} \mathbf{u}_{\mathrm{F}}^h \cdot \boldsymbol{\nu} &= 0 \ \textit{on faces in } \Gamma_{\mathrm{W}} \cap \partial\Omega_{\mathrm{F}}, \\ \mathbf{u}_{\mathrm{A}}^h &= \mathbf{0} \ \textit{at vertices in } \Gamma_{\mathrm{W}} \cap \partial\Omega_{\mathrm{A}}, \\ \mathbf{u}_{\mathrm{F}}^h \cdot \boldsymbol{\nu} &= u_0 \ \textit{on faces in } \Gamma_{\mathrm{E}}, \end{aligned}$$

and furthermore,

$$\int_{\Omega_{\mathrm{F}}} \rho_{\mathrm{F}} c^2 \mathrm{div}\ \mathbf{u}_{\mathrm{F}}^h\ \mathrm{div}\ \bar{\mathbf{v}}_{\mathrm{F}}^h - \omega^2 \int_{\Omega_{\mathrm{F}}} \rho_{\mathrm{F}} \mathbf{u}_{\mathrm{F}}^h \cdot \bar{\mathbf{v}}_{\mathrm{F}}^h - \omega^2 \int_{\Omega_{\mathrm{A}}} (\rho \mathbf{I} - \rho_{\mathrm{F}} \mathcal{A})\, \mathbf{u}_{\mathrm{A}}^h \cdot \bar{\mathbf{v}}_{\mathrm{A}}^h + \\ \int_{\Omega_{\mathrm{A}}} A^H [D(\mathbf{u}_{\mathrm{A}}^h)] : D(\bar{\mathbf{v}}_{\mathrm{A}}^h) + \int_{\Omega_{\mathrm{A}}} \mathrm{div}\left((\mathcal{A} + \mathcal{B}^H - \phi \mathbf{I})^t \bar{\mathbf{v}}_{\mathrm{A}}^h\right) p_{\mathrm{A}}^h + \int_{\Omega_{\mathrm{A}}} \hat{c} p_{\mathrm{A}}^h \bar{q}_{\mathrm{A}}^h + \\ \int_{\Omega_{\mathrm{A}}} \frac{1}{\rho_{\mathrm{F}} \omega^2} \mathcal{A} \mathrm{grad}\ p_{\mathrm{A}}^h \cdot \mathrm{grad}\ \bar{q}_{\mathrm{A}}^h + \int_{\Omega_{\mathrm{A}}} \mathrm{div}\left((\mathcal{A} + \mathcal{B}^H - \phi \mathbf{I}) \mathbf{u}_{\mathrm{A}}^h\right) \bar{q}_{\mathrm{A}}^h = \\ \int_{\Gamma_{\mathrm{I}}} p_{\mathrm{F}}^h (\bar{\mathbf{v}}_{\mathrm{F}}^h \cdot \mathbf{n} - \bar{\mathbf{v}}_{\mathrm{A}}^h \cdot \mathbf{n}) \quad (50)$$

and

$$\int_{\Gamma_{\mathrm{I}}} \bar{q}_{\mathrm{F}}^h (\mathbf{u}_{\mathrm{A}}^h \cdot \mathbf{n} - \mathbf{u}_{\mathrm{F}}^h \cdot \mathbf{n}) = 0, \quad (51)$$

for all $(\mathbf{v}_{\mathrm{F}}^h, q_{\mathrm{F}}^h, \mathbf{v}_{\mathrm{F}}^h, q_{\mathrm{A}}^h) \in \mathbf{V}_{0h}$.

6 Matricial description

In the previous Section, a discrete formulation of our source problem has been established. Now a matrix description is given and, if it is assumed to be well posed, then we show that it is equivalent to another reduced linear system whose unknowns are the degrees of freedom of the interface pressure.

Let U_{F}^h and V_{F}^h denote the column vectors of nodal components of fluid displacement fields $\mathbf{u}_{\mathrm{F}}^h$ and $\mathbf{v}_{\mathrm{F}}^h$, in the standard finite element basis associated with $\mathbf{R}_h(\Omega_{\mathrm{F}})$, excluding those corresponding to faces on $(\Gamma_{\mathrm{W}} \cup \Gamma_{\mathrm{N}}) \cap \partial\Omega_{\mathrm{F}}$. Similarly, let $(U_{\mathrm{A}}^h, P_{\mathrm{A}}^h)$ and $(V_{\mathrm{A}}^h, Q_{\mathrm{A}}^h)$ denote the column vectors of nodal components of the pair fields $(\mathbf{u}_{\mathrm{A}}^h, p_{\mathrm{A}}^h)$ and $(\mathbf{v}_{\mathrm{A}}^h, q_{\mathrm{A}}^h)$, in the standard finite element basis associated with $\mathbf{M}_h(\Omega_{\mathrm{A}}) \times \mathbf{L}_h(\Omega_{\mathrm{A}})$, excluding those corresponding to

vertices in $\Gamma_{\mathrm{W}} \cap \partial\Omega_{\mathrm{A}}$ for U_{A}^h and V_{A}^h. Lastly, let us call P_{F}^h and Q_{F}^h the vectors of nodal components of the interface pressure fields p_{F}^h and q_{F}^h, in the space of finite elements $\mathbf{C}_h(\Gamma_{\mathrm{I}})$.

Then the discretization problem can be written in matrix form as

$$\begin{pmatrix} R_{\mathrm{F}} - \omega^2 M_{\mathrm{F}} & 0 & 0 & -D_{\mathrm{F}} \\ 0 & R_{u,\mathrm{A}} - \omega^2 M_{u,\mathrm{A}} & C^* & D_{\mathrm{A}} \\ 0 & C & \omega^{-2} R_{p,\mathrm{A}} + M_{p,\mathrm{A}} & 0 \\ -D_{\mathrm{F}}^* & D_{\mathrm{A}}^* & 0 & 0 \end{pmatrix} \begin{pmatrix} U_{\mathrm{F}}^h \\ U_{\mathrm{A}}^h \\ P_{\mathrm{A}}^h \\ P_{\mathrm{F}}^h \end{pmatrix} = \begin{pmatrix} B^h \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{52}$$

where

$$\begin{aligned}
(V_{\mathrm{F}}^h)^* R_{\mathrm{F}} U_{\mathrm{F}}^h &= \int_{\Omega_{\mathrm{F}}} \rho_{\mathrm{F}} c^2 \mathrm{div}\ \mathbf{u}_{\mathrm{F}}^h\ \mathrm{div}\ \bar{\mathbf{v}}_{\mathrm{F}}^h, \\
(V_{\mathrm{A}}^h)^* R_{u,\mathrm{A}} U_{\mathrm{A}}^h &= \int_{\Omega_{\mathrm{A}}} A^H [D(\mathbf{u}_{\mathrm{A}}^h)] : D(\bar{\mathbf{v}}_{\mathrm{A}}^h), \\
(Q_{\mathrm{A}}^h)^* R_{p,\mathrm{A}} P_{\mathrm{A}}^h &= \int_{\Omega_{\mathrm{A}}} \frac{1}{\rho_{\mathrm{F}}} \mathcal{A} \mathrm{grad}\ p_{\mathrm{A}}^h \cdot \mathrm{grad}\ \bar{q}_{\mathrm{A}}^h, \\
(V_{\mathrm{F}}^h)^* M_{\mathrm{F}} U_{\mathrm{F}}^h &= \int_{\Omega_{\mathrm{F}}} \rho_{\mathrm{F}} \mathbf{u}_{\mathrm{F}}^h \cdot \bar{\mathbf{v}}_{\mathrm{F}}^h, \\
(V_{\mathrm{A}}^h)^* M_{u,\mathrm{A}} U_{\mathrm{A}}^h &= \int_{\Omega_{\mathrm{A}}} (\rho \mathbf{I} - \rho_{\mathrm{F}} \mathcal{A})\, \mathbf{u}_{\mathrm{A}}^h \cdot \bar{\mathbf{v}}_{\mathrm{A}}^h, \\
(Q_{\mathrm{A}}^h)^* M_{p,\mathrm{A}} P_{\mathrm{A}}^h &= \int_{\Omega_{\mathrm{A}}} \hat{c} p_{\mathrm{A}}^h \bar{q}_{\mathrm{A}}^h, \\
(Q_{\mathrm{A}}^h)^* C U_{\mathrm{A}}^h &= \int_{\Omega_{\mathrm{A}}} \mathrm{div} \left((\mathcal{A} + \mathcal{B}^H - \phi \mathbf{I}) \mathbf{u}_{\mathrm{A}}^h \right) \bar{q}_{\mathrm{A}}^h, \\
(V_{\mathrm{F}}^h)^* D_{\mathrm{F}} P_{\mathrm{F}}^h &= \int_{\Gamma_{\mathrm{I}}} p_{\mathrm{F}}^h \bar{\mathbf{v}}_{\mathrm{F}}^h \cdot \mathbf{n}, \\
(V_{\mathrm{F}}^h)^* D_{\mathrm{A}} P_{\mathrm{F}}^h &= \int_{\Gamma_{\mathrm{I}}} p_{\mathrm{F}}^h \bar{\mathbf{v}}_{\mathrm{A}}^h \cdot \mathbf{n},
\end{aligned}$$

and the right hand side B^h comes from boundary data u_0. R_{F} and M_{F} are the standard stiffness and mass matrices of the fluid, while $R_{u,\mathrm{A}}$, $M_{u,\mathrm{A}}$ and $R_{p,\mathrm{A}}$, $M_{p,\mathrm{A}}$ are the corresponding ones for the porous medium in the case of displacement and pressure, respectively. Notice that every matrix depending on fluid fields is highly sparse because only a maximum of seven entries per row can be different from zero (this corresponds to the number of faces of two adjacent tetrahedra).

Now, we are going to choose ω such that it is not an eigenvalue of the discrete problem. Specifically, we assume that the entire matrix of linear system (52)

and matrices $R_{\mathrm{F}} - \omega^2 M_{\mathrm{F}}$ and

$$K_{\mathrm{A}} = \begin{pmatrix} R_{u,\mathrm{A}} - \omega^2 M_{u,\mathrm{A}} & C^* \\ C & \omega^{-2} R_{p,\mathrm{A}} + M_{p,\mathrm{A}} \end{pmatrix} \tag{53}$$

are non-singular. On the other hand, we notice that matrix $\omega^{-2} R_{p,\mathrm{A}} + M_{p,\mathrm{A}}$ is clearly positive definite and hence non-singular.

In order to improve resolution of linear system (52), we are going to take into account the non-singularity of diagonal matrices. Firstly, we can rewrite this system as

$$(R_{\mathrm{F}} - \omega^2 M_{\mathrm{F}}) U_{\mathrm{F}}^h - D_{\mathrm{F}} P_{\mathrm{F}}^h = B^h, \tag{54}$$

$$(R_{u,\mathrm{A}} - \omega^2 M_{u,\mathrm{A}}) U_{\mathrm{A}}^h + C^* P_{\mathrm{A}}^h + D_{\mathrm{A}} P_{\mathrm{F}}^h = 0, \tag{55}$$

$$C U_{\mathrm{A}}^h + (\omega^{-2} R_{p,\mathrm{A}} + M_{p,\mathrm{A}}) P_{\mathrm{A}}^h = 0, \tag{56}$$

$$-D_{\mathrm{F}}^* U_{\mathrm{F}}^h + D_{\mathrm{A}}^* U_{\mathrm{A}}^h = 0. \tag{57}$$

Since matrix $R_{\mathrm{F}} - \omega^2 M_{\mathrm{F}}$ is non-singular, because of the choice of frequency ω, and $\omega^{-2} R_{p,\mathrm{A}} + M_{p,\mathrm{A}}$ is positive definite, we can obtain from (54) and (56)

$$U_{\mathrm{F}}^h = (R_{\mathrm{F}} - \omega^2 M_{\mathrm{F}})^{-1} (B^h + D_{\mathrm{F}} P_{\mathrm{F}}^h), \tag{58}$$

$$P_{\mathrm{A}}^h = -(\omega^{-2} R_{p,\mathrm{A}} + M_{p,\mathrm{A}})^{-1} C U_{\mathrm{A}}^h. \tag{59}$$

If we take into account that K_{A} is non-singular and $\omega^{-2} R_{p,\mathrm{A}} + M_{p,\mathrm{A}}$ is positive definite, we can conclude that matrix $R_{u,\mathrm{A}} - \omega^2 M_{u,\mathrm{A}} - C^* (\omega^{-2} R_{p,\mathrm{A}} + M_{p,\mathrm{A}})^{-1} C$ is also non-singular. Then, it results from (55) and (59) that

$$U_{\mathrm{A}}^h = -(R_{u,\mathrm{A}} - \omega^2 M_{u,\mathrm{A}} - C^* (\omega^{-2} R_{p,\mathrm{A}} + M_{p,\mathrm{A}})^{-1} C)^{-1} D_{\mathrm{A}} P_{\mathrm{F}}^h. \tag{60}$$

Collecting equations (58) and (60), we can write a simple linear system from (57) where the unique unknown is nodal interface pressure vector P_{F}^h, namely,

$$\Big\{ D_{\mathrm{A}}^* (R_{u,\mathrm{A}} - \omega^2 M_{u,\mathrm{A}} - C^* (\omega^{-2} R_{p,\mathrm{A}} + M_{p,\mathrm{A}})^{-1} C)^{-1} D_{\mathrm{A}} \\ + D_{\mathrm{F}}^* (R_{\mathrm{F}} - \omega^2 M_{\mathrm{F}})^{-1} D_{\mathrm{F}} \Big\} P_{\mathrm{F}}^h = D_{\mathrm{F}}^* (R_{\mathrm{F}} - \omega^2 M_{\mathrm{F}})^{-1} B^h,$$

where the involved matrix is non-singular because this system is equivalent to the full system (52) which has unique solution.

7 Numerical solution of cell problems

In this section we solve the cell problems allowing us to obtain values of the macroscopic coefficients for the poroelastic model considered in this paper.

More precisely, we are going to solve problems (10), (11) and (18) for cell shown in Fig. 4. This is a first step to determine coefficients A^H, $\mathcal{B}^H$, $\mathcal{A}$ and $\hat{c}$ appearing in the poroelastic model (15)-(16).

We use a finite element method to solve these boundary-value problems. More precisely, we employ continuous piecewise linear finite elements on tetrahedral meshes to approximate w^{ij}, w^0 and ξ^i. These meshes can be seen in Fig. 4.

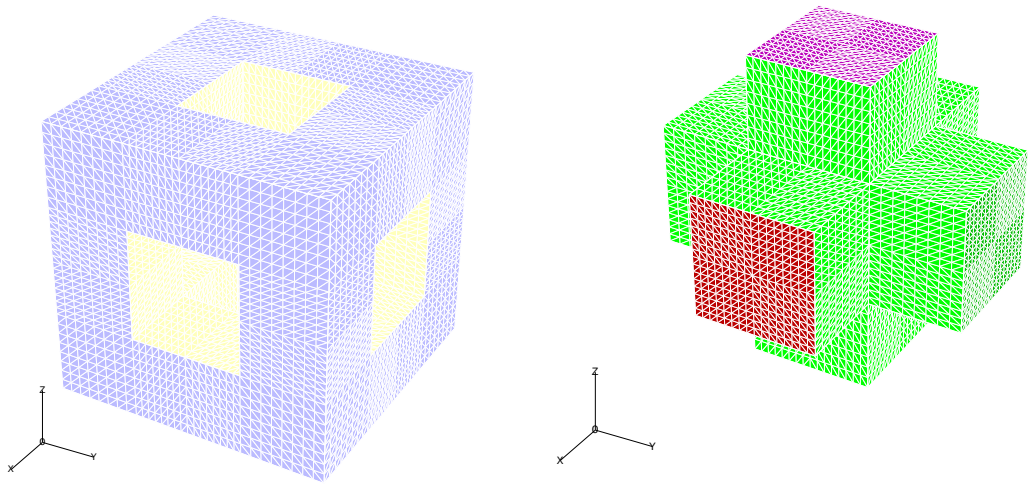

Fig. 4. Meshes of the fluid and solid part of the unit cell ($\mathcal{Y}_f$ and $\mathcal{Y}_s$, respectively)

We assume that the solid part of the poroelastic material is glasswool of type R, with the following properties:

Young modulus	8.7×10^{10} N/m^2
Poisson coefficient	0.15
Density	2500 kg/m^3

Assuming that the poroelastic material is completely saturated by air, we have obtained the following macroscopic coefficients for generalized Biot model (15)-(16):

$$\mathcal{A} = \begin{pmatrix} 0.7576 & 0.3222\text{E-}4 & 0.8560\text{E-}4 \\ 0.8526\text{E-}4 & 0.7581 & 0.8646\text{E-}4 \\ 0.5362\text{E-}4 & 0.2928\text{E-}4 & 0.7573 \end{pmatrix},$$

$$\hat{c} = \text{-}0.4955\text{E-}11,$$

$$\mathcal{B}^H = \begin{pmatrix} \text{-0.2050} & \text{-0.1385E-3} & \text{-0.5167E-3} \\ \text{-0.1385E-3} & \text{-0.2061} & \text{0.4572E-3} \\ \text{-0.5167E-3} & \text{0.4572E-3} & \text{-0.2047} \end{pmatrix},$$

$$A^H = 10^{11} \times \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{12} & M_{22} & M_{23} \\ M_{13} & M_{23} & M_{33} \end{pmatrix},$$

where

$$M_{11} = \begin{pmatrix} \text{0.1589} & \text{-0.3337E-4} & \text{-0.8505E-4} \\ \text{-0.3337E-4} & \text{0.1195E-1} & \text{-0.1311E-3} \\ \text{-0.8505E-4} & \text{-0.1311E-3} & \text{0.1213E-1} \end{pmatrix},$$

$$M_{12} = \begin{pmatrix} \text{-0.1057E-4} & \text{0.1813E-1} & \text{-0.5855E-4} \\ \text{0.1813E-1} & \text{-0.2185E-4} & \text{-0.1051E-5} \\ \text{-0.5855E-4} & \text{-0.1051E-5} & \text{0.2270E-4} \end{pmatrix},$$

$$M_{13} = \begin{pmatrix} \text{0.1115E-5} & \text{-0.1254E-4} & \text{0.1834E-1} \\ \text{-0.1254E-4} & \text{0.8837E-6} & \text{0.1045E-5} \\ \text{0.1834E-1} & \text{0.1045E-5} & \text{0.4812E-5} \end{pmatrix},$$

$$M_{22} = \begin{pmatrix} \text{0.1190E-1} & \text{-0.1144E-3} & \text{0.2185E-3} \\ \text{-0.1144E-3} & \text{0.1586} & \text{0.7522E-5} \\ \text{0.2185E-3} & \text{0.7522E-5} & \text{0.1204E-1} \end{pmatrix},$$

$$M_{23} = \begin{pmatrix} \text{0.1190E-5} & \text{0.4927E-4} & \text{-0.1044E-3} \\ \text{0.4927E-4} & \text{-0.1069E-5} & \text{0.1851E-1} \\ \text{-0.1044E-3} & \text{0.1851E-1} & \text{-0.1966E-5} \end{pmatrix},$$

$$M_{33} = \begin{pmatrix} \text{0.1212E-1} & \text{0.5964E-2} & \text{-0.1349E-3} \\ \text{0.5964E-2} & \text{0.1203E-1} & \text{-0.2908E-3} \\ \text{-0.1349E-3} & \text{-0.2908E-3} & \text{0.1584} \end{pmatrix}.$$

Moreover, porosity of the porous sample is $\phi = 0.648$.

Finally, by progressive refinement of meshes one can see that matrices of macroscopic parameters $\mathcal{A}$ and $\mathcal{B}^H$ tend to isotropic matrices. Similarly, tensor A^H has the same structure as elasticity tensor for isotropic elastic materials.

Thus, due to the symmetry properties of the cell, macroscopic behavior of the studied porous medium is isotropic.

8 Numerical results

In this Section we present some numerical results obtained with a computer code developed by us which implements the numerical methods proposed in this paper. This code allows us to compute the response diagram of the enclosure shown in Fig. 1, consisting of a fluid and a poroelastic medium.

In order to validate our method, we are going to build a simple example which can be reduced to an one-dimensional problem and then solved exactly. We recall that equations satisfied by pressure and displacement fields in the poroelastic media are

$$\omega^2 \left(\rho \mathbf{I} - \rho_{\mathrm{F}} \mathcal{A}\right) \mathbf{u}_{\mathrm{A}} + \operatorname{div}\left(A^H[D(\mathbf{u}_{\mathrm{A}})]\right) + \left(\mathcal{A} + \mathcal{B}^H - \phi \mathbf{I}\right) \operatorname{grad} p_{\mathrm{A}} = \mathbf{0} \text{ in } \Omega_{\mathrm{A}},$$
$$-\omega^2 \hat{c} p_{\mathrm{A}} + \frac{1}{\rho_{\mathrm{F}}} \operatorname{div}(\mathcal{A} \operatorname{grad} p_{\mathrm{A}}) = \omega^2 \operatorname{div}\left(\left(\mathcal{A} + \mathcal{B}^H - \phi \mathbf{I}\right) \mathbf{u}_{\mathrm{A}}\right) \text{ in } \Omega_{\mathrm{A}}.$$

If we assume that every linear operator is a multiple of identity operator, we can find a solution $(\mathbf{u}_{\mathrm{A}}, p_{\mathrm{A}})$ of the form

$$\mathbf{u}_{\mathrm{A}}(x, y, z) = u_{\mathrm{A}}(z) \mathbf{e}_3,$$
$$p_{\mathrm{A}}(x, y, z) = p_{\mathrm{A}}(z),$$

and rewrite the above three-dimensional problem as an one-dimensional problem, namely,

$$\omega^2 \left(\rho - \rho_{\mathrm{F}} a\right) u_{\mathrm{A}} + s u_{\mathrm{A}}'' + \left(a + b - \phi\right) p_{\mathrm{A}}' = 0 \text{ in } (0, a_{\mathrm{A}}),$$
$$-\omega^2 \hat{c} p_{\mathrm{A}} + \frac{a}{\rho_{\mathrm{F}}} p_{\mathrm{A}}'' = \omega^2 \left(a + b - \phi\right) u_{\mathrm{A}}' \text{ in } (0, a_{\mathrm{A}}),$$

where the prime denotes derivative with respect to z and we have supposed that $A^H\left[D(\mathbf{u}_{\mathrm{A}})\right] \mathbf{e}_3 = s u_{\mathrm{A}}' \mathbf{e}_3$, $\mathcal{A} = a\mathbf{I}$, $\mathcal{B}^H = b\mathbf{I}$. After some algebraic manipulations, we obtain

$$-\frac{a}{\omega^2 \rho_{\mathrm{F}}} p_{\mathrm{A}}'''' + \left(\hat{c} - \frac{(a + b - \phi)^2}{s} - \frac{a(\rho - \rho_{\mathrm{F}} a)}{\omega^2 s \rho_{\mathrm{F}}}\right) p_{\mathrm{A}}'' + \frac{\omega^2 (\rho - \rho_{\mathrm{F}} a) \hat{c}}{s} p_{\mathrm{A}} = 0,$$
$$u_{\mathrm{A}} = \frac{s}{\omega^2 (a + b - \phi)(\rho - \rho_{\mathrm{F}} a)} \left(-\frac{a}{\omega^2 \rho_{\mathrm{F}}} p_{\mathrm{A}}''' + \left(\hat{c} - \frac{(a + b - \phi)^2}{s}\right) p_{\mathrm{A}}'\right).$$

Let us assume a similar assumption for fluid displacement and interface pressure, i.e., $\mathbf{u}_{\mathrm{F}}(x, y, z) = u_{\mathrm{F}}(z) \mathbf{e}_3$ and $p_{\mathrm{F}}(x, y, z) = p_{\mathrm{F}}(z)$. We also suppose that

$\Omega_{\rm F} = (0,b)\times(0,d)\times(-a_{\rm F},0)$ and $\Omega_{\rm A} = (0,b)\times(0,d)\times(0,a_{\rm A})$. Then the coupled fluid-poroelastic problem can be written as

$$
\begin{aligned}
&-\omega^2\rho_{\rm F}u_F + p'_{\rm F} = 0, && (61)\\
&p_{\rm F} = -\rho_{\rm F}c^2u'_{\rm F}, && (62)\\
&-\frac{a}{\omega^2\rho_{\rm F}}p''''_{\rm A} + \left(\hat{c} - \frac{(a+b-\phi)^2}{s} - \frac{a(\rho-\rho_{\rm F}a)}{\omega^2 s\rho_{\rm F}}\right)p''_{\rm A} + \frac{\omega^2(\rho-\rho_{\rm F}a)\hat{c}}{s}p_{\rm A} = 0, && (63)\\
&u_{\rm A} = \frac{s}{\omega^2(a+b-\phi)(\rho-\rho_{\rm F}a)}\left(-\frac{a}{\omega^2\rho_{\rm F}}p'''_{\rm A} + \left(\hat{c} - \frac{(a+b-\phi)^2}{s}\right)p'_{\rm A}\right), && (64)\\
&-p_{\rm F}(0) = su'_{\rm A}(0) + (a+b-\phi)\,p_{\rm A}(0), && (65)\\
&u_{\rm F}(0) = u_{\rm A}(0), && (66)\\
&u_{\rm F}(-a_{\rm F}) = 0, && (67)\\
&u_{\rm A}(a_{\rm A}) = 0, && (68)\\
&ap'_{\rm A}(a_{\rm A}) = 0, && (69)\\
&ap'_{\rm A}(0) = 0, && (70)\\
&u_{\rm F}(-a_{\rm F}) = u_0. && (71)
\end{aligned}
$$

The general solution of ordinary differential system (61)-(63) is

$$
\begin{aligned}
&u_{\rm F}(z) = C_1e^{-ik_{\rm F}z} + C_2e^{ik_{\rm F}z},\ z\in(-a_{\rm F},0), && (72)\\
&p_{\rm A}(z) = C_3e^{-ik_{\rm A,1}z} + C_4e^{ik_{\rm A,1}z} + C_5e^{-ik_{\rm A,2}z} + C_6e^{ik_{\rm A,2}z},\ z\in(0,a_{\rm A}), && (73)
\end{aligned}
$$

where $k_{\rm F} = \frac{\omega}{c}$ and $\{k_{\rm A,1}, -k_{\rm A,1}, k_{\rm A,2}, -k_{\rm A,2}\}$ are the four roots of polynomial equation

$$
-\frac{a}{\omega^2\rho_{\rm F}}k_{\rm A}^4 + \left(\hat{c} - \frac{(a+b-\phi)^2}{s} - \frac{a(\rho-\rho_{\rm F}a)}{\omega^2 s\rho_{\rm F}}\right)k_{\rm A}^2 + \frac{\omega^2(\rho-\rho_{\rm F}a)\hat{c}}{s} = 0.
$$

If we take into account boundary and interface conditions (65)-(71) and expressions (72) and (73), amplitudes $C_j,\ 1\le j\le 6$ can be calculated by solving a linear system of equations.

We have considered that fluid is air with $\rho_{\rm F} = 1.225$ kg/m^3 and $c = 343$ m/s, whereas properties of the porous material are summarized in $s = 9.18633\times 10^{10}$N/m^2, $\phi = 0.95$, $a = 0.67857$, $b = -0.05$, $\hat{c} = -6.59172\times 10^{-6}$ms^2/kg and $\rho = 1.26163\times 10^2$ kg/m^3. With respect to dimensions of the enclosure, length and width are 1 m while height is 1 m for the first layer of free fluid and 1 m for the second layer of porous material whereas the normal displacement on $\Gamma_{\rm E}$ is $u_0 = 60$.

We have computed the solution to this problem with three different uniform meshes, named mesh 1, mesh 2 and mesh 3 of 2548, 8140 and 18788 degrees of freedom, respectively. In Fig. 5, we show the L^2-norm of the relative

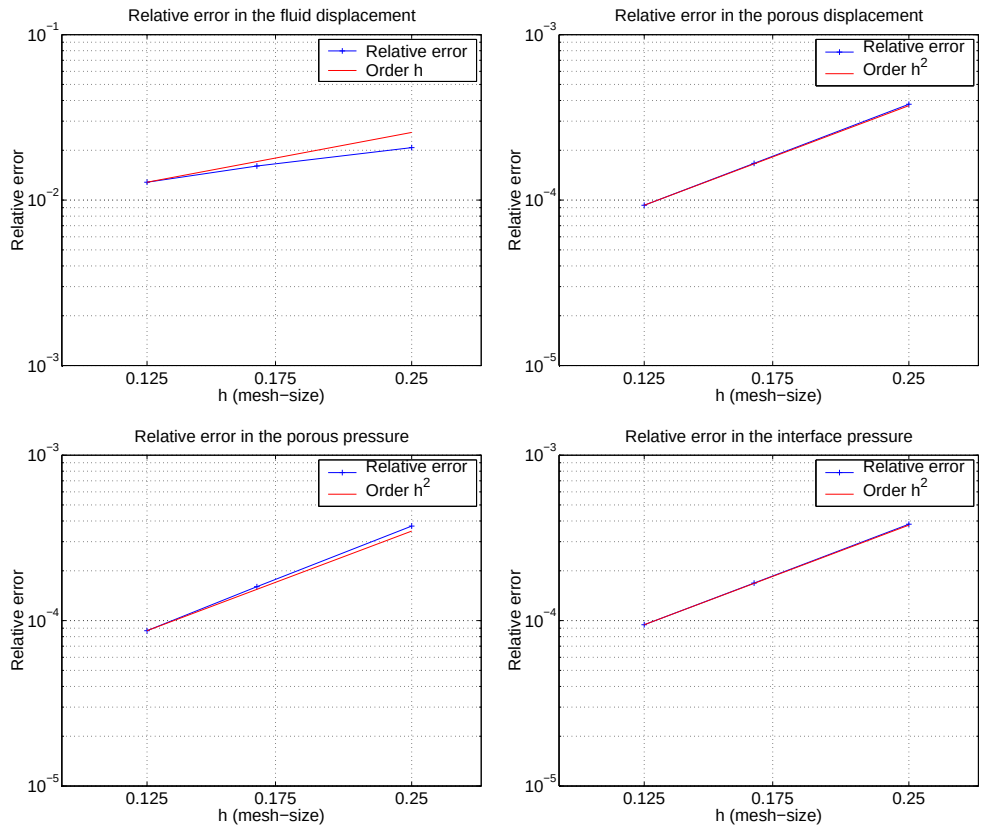

Fig. 5. Curves of convergence for fluid and porous fields.

errors for fluid displacement, $\left\|u_{\mathrm{F}}^h - u_{\mathrm{F}}\right\|_{0,\Omega_{\mathrm{F}}} / \left\|u_{\mathrm{F}}\right\|_{0,\Omega_{\mathrm{F}}}$, porous displacement, $\left\|u_{\mathrm{A}}^h - u_{\mathrm{A}}\right\|_{0,\Omega_{\mathrm{A}}} / \left\|u_{\mathrm{A}}\right\|_{0,\Omega_{\mathrm{A}}}$, porous pressure, $\left\|p_{\mathrm{A}}^h - p_{\mathrm{A}}\right\|_{0,\Omega_{\mathrm{A}}} / \left\|p_{\mathrm{A}}\right\|_{0,\Omega_{\mathrm{A}}}$, and interface pressure, $\left\|p_{\mathrm{F}}^h - p_{\mathrm{F}}\right\|_{0,\Omega_{\mathrm{F}}} / \left\|p_{\mathrm{F}}\right\|_{0,\Omega_{\mathrm{F}}}$, against mesh-size, h. As it can be seen, convergence of order 2 is achieved for poroelastic fields and interface pressure. In addition, convergence of order 1 is achieved for fluid displacement.

As a real life test, we are going to compute the solution of the coupled problem using the data obtained in the previous Section by solving cell problems. Fig. 6 shows the *response curves* for enclosure in Fig. 1, when solved with mesh 1 having 2548 degrees of freedom. In this curve the logarithm of "energy",

$$\log_{10}\left(\frac{1}{2}\int_{\Omega_{\mathrm{F}}} \rho_{\mathrm{F}} c^2 |\mathrm{div}\ \mathbf{u}_{\mathrm{F}}|^2 + \int_{\Omega_{\mathrm{A}}} A^H[D(\mathbf{u}_{\mathrm{A}})] : D(\bar{\mathbf{u}}_{\mathrm{A}}) + \int_{\Omega_{\mathrm{A}}} \mathrm{div}\left((\mathcal{A} + \mathcal{B}^H - \phi\mathbf{I})\mathbf{u}_{\mathrm{A}}\right)\bar{p}_{\mathrm{A}}\right),$$

is plotted for angular frequencies ω ranging from 50 to 2000 rad/s. Several response peaks can be observed in this curve. In fact, response peaks of the curve shows the resonance frequencies for the coupled system shown in Fig. 1.

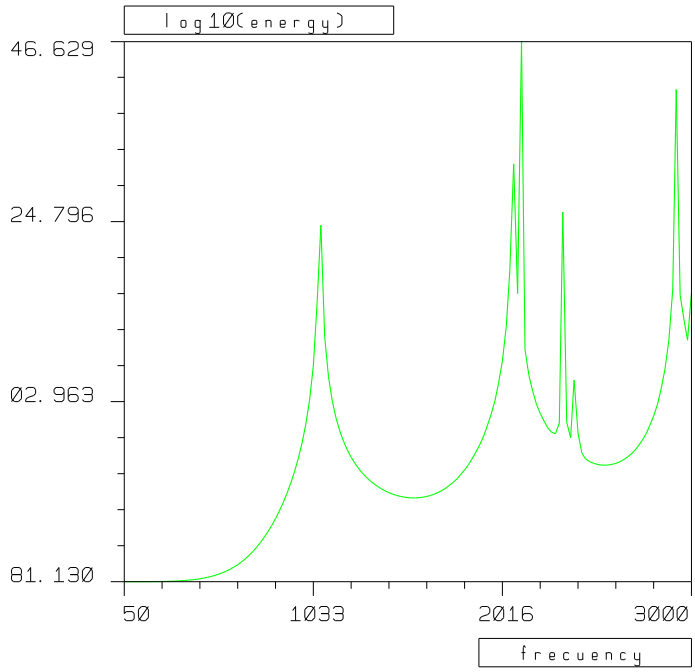

Fig. 6. Response curve.

9 Conclusions

We have considered a mathematical model for acoustic propagation in periodic non-dissipative porous media with elastic solid frame and open pore. Parameters of this model have been computed by solving some partial differential equations in the unit cell obtained by homogenization methods. Then a three-dimensional finite element method has been proposed and implemented for numerical solution of the coupling between a fluid and the above porous medium. In order to validate the proposed methodology and to assess convergence properties, the computer code has been used for a test example having analytical solution. Then, as an application, we have computed the response curve for an enclosure containing air and a layer of porous material.

Acknowledgements

Part of this work was supported by Ministerio de Ciencia y Tecnología (Spain) through project No. DPI2001-1613-C02-02 and Xunta de Galicia under grant PGIDT02PXIC20701PN. For obtaining the numerical results, a Compaq HPC 320 computer from CESGA (Centro de Supercomputación de Galicia) has been used.

References

[1] J.-F. Allard. *Propagation of sound in porous media: modelling sound absorbing materials.* Elsevier, New York, 1993.

[2] J.-F. Allard and Y. Champoux. New empirical equations for sound propagation in rigid frame fibrous materials. *J. Acoust. Soc. Am.*, 91(6):3346–3353, 1992.

[3] D. N. Arnold, F. Brezzi, and M. Fortin. A stable finite element for the Stokes equations. *Calcolo*, 21(4):337–344 (1985), 1984.

[4] N. Atalla, R. Panneton, and P. Debergue. A mixed displacement pressure formulation for poroelastic materials. *J. Acoust. Soc. Am.*, 104(3):1444–1452, 1998.

[5] K. Attenborough. Acoustical characteristics of rigid fibrous absorbents and granular materials. *J. Acoust. Soc. Am.*, 73(3):785–799, 1983.

[6] A. Bermúdez, J. L. Ferrín, and A. Prieto. A finite element solution of acoustic propagation in rigid porous media. To appear in *Int. J. Numer. Meth. Engrg.*

[7] A. Bermúdez and R. Rodríguez. Finite element computation of the vibration modes of a fluid-solid system. *Comput. Methods Appl. Mech. Engrg.*, 119(3-4):355–370, 1994.

[8] M. A. Biot. Theory of propagation of elastic waves in a fluid-saturated porous solid. I. Low-frequency range. *J. Acoust. Soc. Am.*, 28:168–178, 1956.

[9] M. A. Biot and D. G. Willis. The elastic coefficients of the theory of consolidation. *J. Appl. Mech.*, 24:594–601, 1957.

[10] Th. Clopeau, J. L. Ferrín, R. P. Gilbert, and A. Mikelić. Homogenizing the acoustic properties of the seabed. Part II. *Math. Comput. Modelling*, 33(8-9):821–841, 2001.

[11] Th. Clopeau, J. L. Ferrín, and A. Mikelić. Derivation of the diphasic Biot's law for an elastic solid matrix containing isolated fluid drops. In *Multiscale problems in science and technology (Dubrovnik, 2000)*, pages 151–160. Springer, Berlin, 2002.

[12] M. A. Delany and E. N. Bazley. Acoustic properties of fibrous absorbent materials. *Appl. Acoust.*, 3:105–116, 1970.

[13] V. Easwaran, W. Lauriks, and J. P. Coyette. Displacement-based finite element method for guided wave propagation problems: application to poroelastic media. *J. Acoust. Soc. Am.*, 100(5):2989–3002, 1996.

[14] H. I. Ene and E. Sánchez-Palencia. Équations et phénomènes de surface pour l′ écoulement dans un modèle de milieu poreux. *J. Mécanique*, 14:73–108, 1975.

[15] J. L. Ferrín and A. Mikelić. Homogenizing the acoustic properties of a porous matrix containing an incompressible inviscid fluid. *Math. Methods Appl. Sci.*, 26(10):831–859, 2003.

[16] R. P. Gilbert and A. Mikelić. Homogenizing the acoustic properties of the seabed. Part I. *Nonlinear Anal.*, 40(1-8, Ser. A: Theory Methods):185–212, 2000.

[17] P. Göransson. A 3-D, simmetric, finite element formulation of the Biot equations with application to acoustic wave propagation through an elastic porous medium. *Int. J. Numer. Meth. Engrg.*, 41:167–192, 1998.

[18] L. Kiefling and G. C. Feng. Fluid-structure finite element vibrational analysis. *AIAA J.*, 14(2):199–203, 1976.

[19] P. M. Morse and K. U. Ingard. *Theoretical acoustics.* McGraw-Hill, New York, 1978.

[20] R. Panneton and N. Atalla. An efficient finite element scheme for solving the three-dimensional poroelasticiey problem in acoustics. *J. Acoust. Soc. Am.*, 101(6):3287–3298, 1997.

[21] P. A. Raviart and J. M. Thomas. A mixed finite element method for 2nd order elliptic problems. In *Mathematical aspects of finite element methods (Proc. Conf., Consiglio Naz. delle Ricerche (C.N.R.), Rome, 1975)*, pages 292–315. Lecture Notes in Math., Vol. 606. Springer, Berlin, 1977.

[22] E. Sanchez-Palencia. *Nonhomogeneous media and vibration theory.* Lecture Notes in Physics **127**. Springer-Verlag, Berlin, 1980.