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AN OPTIMAL FINITE-ELEMENT/PML METHOD FOR THE SIMULATION OF ACOUSTIC WAVE PROPAGATION PHENOMENA

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Abstract. We introduce an optimal bounded Perfectly Matched Layer (PML) technique based on a particular absorbing function with unbounded integral. This technique allows us to avoid spurious reflections, even though the thickness of the layer is finite. Moreover, it is easy to implement in a finite-element method and overcomes the dependency of parameters for the discrete problem. The efficiency and accuracy of this approach is illustrated with some numerical tests.

Key words: Perfectly Matched Layer, finite-element method, time-harmonic scattering, acoustic wave propagation.

1 INTRODUCTION

The first problem to be tackled for the numerical solution of any scattering problem in an unbounded domain is to truncate the computational domain without perturbing too much the solution of the original problem. In an ideal framework, the method should satisfy, at least, three properties: efficiency, easiness of implementation, and robustness.

Several numerical techniques have been developed with this purpose: boundary element methods, infinite element methods, Dirichlet-to-Neumann operators based on truncating Fourier expansions, absorbing boundary conditions, etc. The potential advantages of each of them have been widely studied in the literature[2,16,23,28].

We focus our attention on the last mentioned technique: local absorbing boundary conditions (ABCs) can be used to preserve the computational efficiency of the numerical method. Those of Bayliss and Turkel[5], Engquist and Majda[17], and Feng[18] are among the most widely used. However, in spite of the simple implementation of lowest order ABCs, good accuracy is only achieved for higher order ones[30], because these conditions are not fully non-reflecting on the truncated boundary of the computational domain. As a consequence, high accuracy using ABCs leads to a substantial computational cost and increases the difficulty of implementation. Recently, a promising way has been open: high order ABCs not involving high derivatives[19,24].

An alternative approach to deal with the truncation of unbounded domains is the so called *Perfectly Matched Layer* (PML) method, which was introduced by Berenger[8,9,10]. It is based on simulating an absorbing layer of damping material surrounding the domain of interest, like a thin sponge which absorbs the scattered field radiated to the exterior of this domain. This method is known as 'perfectly matched' because the interface between the physical domain and the absorbing layer does not produce spurious reflections inside the domain of interest.

This method has been applied to different problems. It was initially settled for Maxwell's equations in electromagnetism[7,8] and subsequently used for the scalar Helmholtz equation[20,29,32], advective acoustics[1,6,22], elasticity[4,15], poroelastic media[33], shallow water waves[27], other hyperbolic problems[26], etc. We focus our attention on wave propagation time-harmonic scattering problems in linear acoustics, i.e., on the scalar Helmholtz equation.

In practice, since the PML has to be truncated at a finite distance of the domain of interest, its external boundary produces artificial reflections. Theoretically, these reflections are of minor importance because of the exponential decay of the acoustic waves inside the PML. In fact, for Helmholtz-type scattering problems, Lassas and Somersalo[25] proved, using boundary integral equation techniques, that the approximate solution obtained by the PML method converges exponentially to the exact solution in the computational domain as the thickness of the layer goes to infinity. This result was generalized by Hohage et al.[21] using techniques based on the pole condition. Similarly, Bécache et al.[6] proved an analogous result for the convected Helmholtz equation.

Once the problem is discretized, the approximation error typically becomes larger. Increasing the thickness of the PML may be a remedy, but not always available because of computational cost. An alternative usual choice to achieve low error levels is to take larger values for the absorption coefficients in the layer. However, Collino and Monk[14] showed that this methodology may produce an increasing error in the discretized problem. Consequently, an optimization problem arises: given a data set and a mesh, to choose an optimal absorbing function (i.e., a variable absorption coefficient) to minimize the error. In this framework, Asvadurov et al.[3] proposed a pure imaginary stretching to optimize the error of the PML method. They recovered exponential error estimates using finite-difference grid optimization. However, to the best of the authors' knowledge, the optimization problem is still open in that there is no optimal criterion to choose the absorbing function independently of data and meshes.

We have proposed[11] an alternative procedure to avoid this drawback: to use an absorbing function with unbounded integral on the PML. We have shown that this leads to a theoretically exact bounded PML. More precisely, this kind of absorbing functions on a circular annular layer allows recovering the exact solution of the time-harmonic scattering problem in the domain of interest, up to discretization errors, even though the thickness of the layer is finite.

In this paper, we consider Cartesian perfectly matched layers. We choose a particularly convenient non-integrable absorbing function[12], which only depends on the sound speed of the fluid. We have shown that this choice leads to a robust PML method, easy to implement in a finite-element code and significantly more efficient than standard PML techniques based on classical bounded absorbing functions.

The outline of this paper is as follows. In Section 2 we recall the classical two-dimensional scattering problem with Cartesian perfectly matched layers. A finite-element method to solve this problem is introduced in Section 3. We also compare in this section the proposed strategy with other classical PML methods. Finally, in Section 4, we report the numerical results obtained with our PML technique applied to a realistic wave propagation problem.

2 A PERFECTLY MATCHED LAYER METHOD

We deal with the time-harmonic acoustic scattering problem in an unbounded exterior domain. Let Ω be a bounded domain of $\mathbb{R}^2$ occupied by an obstacle to the propagation of acoustic waves as shown in Fig. 1.

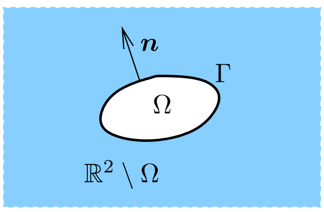

Figure 1: Two-dimensional unbounded domain.

Our goal is to solve the following exterior Helmholtz problem:

$$\begin{cases} \Delta p + k^2 p = 0 & \text{in } \mathbb{R}^2 \setminus \Omega, \\ \dfrac{\partial p}{\partial \boldsymbol{n}} = g & \text{on } \Gamma, \\ \lim\limits_{r\to\infty} \sqrt{r}\left(\dfrac{\partial p}{\partial r} - ikp\right) = 0. \end{cases} \tag{1}$$

In these equations, p is the unknown amplitude of the pressure wave and $k = \omega/c$ is the wave number, with ω being the angular frequency of the waves and c the sound speed of the fluid in the exterior domain.

We introduce perfectly matched layers (PML) on the x and y directions to truncate the unbounded domain, as shown in Fig. 2. The inner rectangle contains the obstacle Ω as well as the physical domain Ω_{F}, i.e., the subdomain occupied by the fluid surrounding the obstacle where we are interested in computing the solution of problem (1).

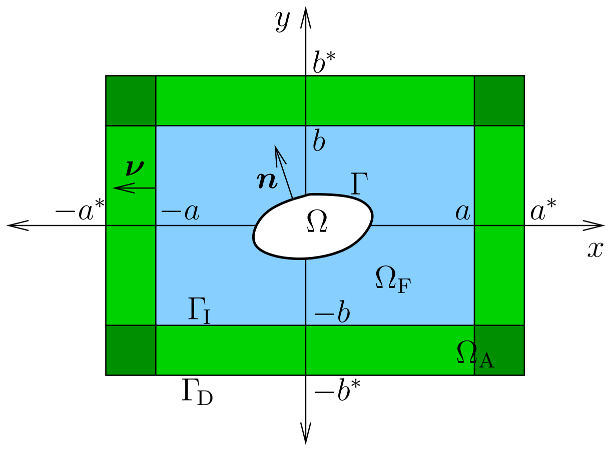

Figure 2: Cartesian PML on a two-dimensional domain.

We consider variable absorption coefficients in the PML, σ_x and σ_y, acting on the vertical and horizontal layers, respectively; both absorption coefficients act in the corner layers, as well. These coefficients, σ_x and σ_y, are allowed to be functions of x and y, respectively.

The PML approximate solution of problem (1) is the solution $\tilde{p}$ of the following equations[13]:

$$\begin{cases} \Delta \tilde{p} + k^2 \tilde{p} = 0 & \text{in } \Omega_{\mathrm{F}}, \\ \dfrac{1}{\gamma_x}\dfrac{\partial}{\partial x}\left(\dfrac{1}{\gamma_x}\dfrac{\partial \tilde{p}}{\partial x}\right) + \dfrac{1}{\gamma_y}\dfrac{\partial}{\partial y}\left(\dfrac{1}{\gamma_y}\dfrac{\partial \tilde{p}}{\partial y}\right) + k^2\tilde{p} = 0 & \text{in } \Omega_{\mathrm{A}}, \\ \dfrac{\partial \tilde{p}}{\partial \boldsymbol{n}} = g & \text{on } \Gamma, \\ \tilde{p} \quad \text{and} \quad \left(\dfrac{1}{\gamma_x}\dfrac{\partial \tilde{p}}{\partial \nu_x} + \dfrac{1}{\gamma_y}\dfrac{\partial \tilde{p}}{\partial \nu_y}\right) & \text{continuous on } \Gamma_{\mathrm{I}}, \\ \tilde{p} = 0 & \text{on } \Gamma_{\mathrm{D}}, \end{cases} \tag{2}$$

where

$$\gamma_x(x) = \begin{cases} 1, & \text{if } |x| < a, \\ 1 + \dfrac{i}{\omega}\sigma_x(|x|), & \text{if } a \leq |x| < a^*, \end{cases}$$

and

$$\gamma_y(y) = \begin{cases} 1, & \text{if } |y| < b, \\ 1 + \dfrac{i}{\omega}\sigma_y(|y|), & \text{if } b \leq |y| < b^*. \end{cases}$$

Although constant, linear or quadratic functions are the typical choices for σ_x and σ_y, we have shown[11] that a convenient choice consists of unbounded functions σ_x and σ_y such that

$$\int_a^{a^*} \sigma_x(s)\, ds = +\infty \qquad \text{and} \qquad \int_b^{b^*} \sigma_y(s)\, ds = +\infty.$$

In some cases, this choice allows recovering the exact solution in the physical domain.

In particular we are going to use the absorbing functions

$$\sigma_x(x) = \frac{c}{a^* - x}, \qquad \sigma_y(y) = \frac{c}{b^* - y}, \tag{3}$$

which, we have shown to be an optimal choice[12].

3 FINITE ELEMENT DISCRETIZATION

Next, we introduce a convenient finite-element method for the numerical solution of problem (2) such that the resulting discrete problem is well posed.

Consider a partition in triangles of the physical domain Ω_F and a partition in rectangles of the absorbing layer Ω_A, matching on the common interface Γ_I as shown in Fig. 3. The reason to use such hybrid meshes is that triangles are more adequate to fit the boundary of the obstacle, whereas rectangles allow us to compute explicitly the integrals involving the singular absorbing functions that appear in the elements in the layer.

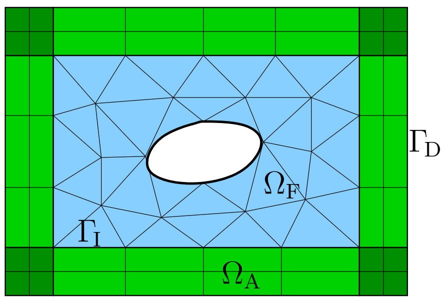

Figure 3: Hybrid mesh on PML and physical domain.

We compute an approximation $\tilde{p}_h$ of the pressure amplitude by using linear triangular finite elements in the physical domain and bilinear rectangular finite elements in the absorbing layer. The degrees of freedom defining the finite-element solution in both cases are the values of $\tilde{p}_h$ at the vertices of the elements.

Moreover, we impose the vanishing Dirichlet boundary condition on the finite-element solution. Hence, $\tilde{p}_h$ does not have degrees of freedom on the outer boundary. This fact is essential for the resulting discrete problem to be well posed.

Standard arguments in this finite-element framework lead to the following discrete problem from the weak formulation of problem (2):

$$\begin{aligned}
&\int_{\Omega_{\mathrm{F}}} \left(\nabla \tilde{p}_h \cdot \nabla \bar{q}_h \, dx\, dy - k^2 \tilde{p}_h \bar{q}_h \, dx\, dy\right) \\
&+ \int_{\Omega_{\mathrm{A}}} \left(\frac{\gamma_y}{\gamma_x} \frac{\partial \tilde{p}_h}{\partial x} \frac{\partial \bar{q}_h}{\partial x} \, dx\, dy + \frac{\gamma_x}{\gamma_y} \frac{\partial \tilde{p}_h}{\partial y} \frac{\partial \bar{q}_h}{\partial y} \, dx\, dy - k^2 \gamma_x \gamma_y \tilde{p}_h \bar{q}_h \, dx\, dy\right) \ = \int_{\Gamma} g \bar{q}_h \, ds,
\end{aligned}$$

for all functions q_h in this finite-element space.

Once the discrete problem is written in matrix form, it yields a system of linear equations whose unknowns are the nodal values of $\tilde{p}_h$. The entries of the system matrix are computed by assembling the element matrices; in particular, the following ones involve the unbounded absorbing functions:

$$\int_K \frac{\gamma_y}{\gamma_x} \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} \, dx\, dy, \qquad \int_K \frac{\gamma_x}{\gamma_y} \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} \, dx\, dy, \quad \text{and} \quad \int_K k^2 \gamma_x \gamma_y N_i N_j \, dx\, dy,$$

with K being a rectangular element in Ω_{A} and N_i the nodal finite-element basis functions of this element.

For the discrete problem to be well posed, it is necessary that all the integrals above be finite, what is not trivial since they involve singular functions whenever K is a rectangle with an edge lying on the outer boundary Γ_{D}. However, it has been shown that all these integrals are finite for the absorbing functions defined by (3)[12].

On the other hand, the performance of these absorbing functions is significantly better than that of one of the most competitive classical alternatives: quadratic absorbing functions of the form

$$\sigma_x(x) = \sigma^*(x-a)^2 \qquad \text{and} \qquad \sigma_y(y) = \sigma^*(y-b)^2, \tag{4}$$

where σ^* is a parameter to be fixed.

When these quadratic absorbing functions are used, the standard procedure to minimize the spurious reflections produced at the outer boundary of the PML consists of taking large values for σ^*. However, larger values of σ^* lead to larger discretization errors. Therefore, σ^* cannot be chosen arbitrarily large because, otherwise, the discretization errors would be dominant, deteriorating the overall accuracy of the method.

For a given problem and a given mesh there is an optimal value of σ^* leading to minimal errors[14]. Unfortunately, such optimal value depends strongly on the problem data as well as on the particular mesh. Thus, in practice, it is necessary to find in advance a reasonable value of σ^*. No theoretical procedure to tune this parameter is known to date. Some efforts have been done[31], but the dependency of σ^* with respect to the mesh has not been avoided.

To appreciate the advantageous performance of the PML strategy we propose, we have solved problem (1) with the obstacle Ω being the unit circle centered at the origin and the data g such that the exact solution of the problem is

$$p(x,y) = \frac{i}{4} \mathrm{H}_0^{(1)} \left(k\sqrt{(x-0.5)^2 + y^2}\right).$$

In the expression above, $\mathrm{H}_0^{(1)}$ is the Hankel function of order zero and first kind, and $k = \omega/c$, with $c = 340\,\mathrm{m/s}$ and different values of the frequency ω.

In Table 1 we compare the errors of the PML method with the unbounded absorbing functions (3) and with the quadratic absorbing functions (4). For the latter, we have used the optimal value of σ^*, which is also reported in the table. We also include in the table the condition number κ of the system matrix for each discrete problem.

		Unbounded (3)		Quadratic (4)		
ω(rad/s)	d.o.f.	Error(%)	κ	σ^*	Error(%)	κ
	464	0.763	6.7e+02	$22.28\,c$	11.644	4.7e+02
250	1720	0.131	5.1e+03	$29.57\,c$	3.675	5.0e+03
	6768	0.029	4.1e+04	$38.37\,c$	1.134	4.6e+04
	464	1.700	1.1e+02	$27.67\,c$	7.602	1.1e+02
750	1720	0.447	7.0e+02	$35.52\,c$	2.291	9.4e+02
	6768	0.109	5.6e+03	$43.49\,c$	0.698	8.2e+03

Table 1: Comparison of PML methods with unbounded and quadratic absorbing functions.

A clear advantage of the PML method based on the proposed unbounded absorbing functions (3) can be clearly appreciated from this table. The errors with the quadratic absorbing functions are much larger in all cases, even though the optimal value of σ^* has been used. On the other hand, in spite of the singular character of the unbounded functions, the condition numbers of the resulting system matrices are essentially of the same order as those of the quadratic functions.

Let us emphasize that an additional benefit of choosing the absorbing functions (3) is that they are free of parameters to be fitted.

4 NUMERICAL TESTS

In this section we solve a *real-life* scattering problem by using the proposed finite-element/PML method with the non integrable absorbing function (3).

The obstacle is the diapason shown in Figure 4. Its thickness is 0.2 m, its interior aperture 1 m, and its length 4.1 m.

First, we have computed the waves scattered by the diapason generated by a plane wave advancing in the positive x-direction. We have used two meshes, which are refinements of that shown in Figure 4. The coarser mesh has 9140 triangles in the fluid domain and 3120 rectangles in the PML, whereas the finer mesh has 36610 triangles and 12480 rectangles.

In Figures 5 and 6 we show the results obtained with the coarser mesh for a wave number $k = 2\pi\,\mathrm{m}^{-1}$.

Figures 7 and 8 show the results obtained for a higher wave number: $k = 10\pi\,\mathrm{m}^{-1}$. In this case we have used the finer mesh.

In the second numerical experiment we have used the same geometry and meshes as above, but with a monopole (i.e., a Dirac's delta) acting inside the arc of the diapason as source term. In Figures 9 to 12 we show the reflected pressure waves generated by the monopole for the wave numbers $k = 2\pi\,\mathrm{m}^{-1}$ (with the coarser mesh) and $k = 10\pi\,\mathrm{m}^{-1}$ (with the finer mesh).

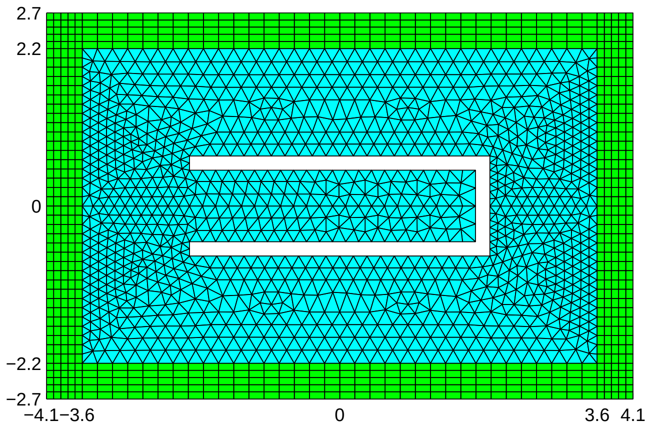

Figure 4: Mesh of the fluid domain and PML surrounding the diapason.

5 Conclusions

We have introduced a PML method based on a non-integrable absorbing function for the numerical solution of time-harmonic problems in unbounded domains. The proposed method leads to significantly smaller errors than the classical ones based on bounded absorbing functions. To assess the efficiency of our approach, we have applied it to solve some realistic problems, obtaining very good results even with thin absorbing layers close to the obstacles.

Many subjects of further research remain open. In particular, a detailed analysis of the proposed PML technique including error estimates for its numerical solution and its application to time domain problems.

References

[1] S. Abarbanel, D. Gottlieb and J.S. Hesthaven, Well-posed perfectly matched layers for advective acoustics, *J. Comput. Phys.*, **154**, 266-283 (1999).

[2] R.J. Astley, Infinite elements for wave problems: A review of current formulations and an assessment of accuracy, *Int. J. Numer. Methods Eng.*, **49**, 951-976 (2000).

[3] S. Asvadurov, V. Druskin, M.N. Guddati and L. Knizhnerman, On optimal finite-difference approximation of PML, *SIAM J. Numer. Anal.*, **41**, 287-305 (2003).

[4] U. Basu and A.K. Chopra, Perfectly matched layers for time-harmonic elastodynamics of unbounded domains: theory and finite-element implementation, *Comput. Methods Appl. Mech. Eng.*, **192**, 1337-1375 (2003).

[5] A. Bayliss and E. Turkel, Radiation boundary conditions for wave-like equations, *Comm. Pure Appl. Math.*, **33**, 707-725 (1980).

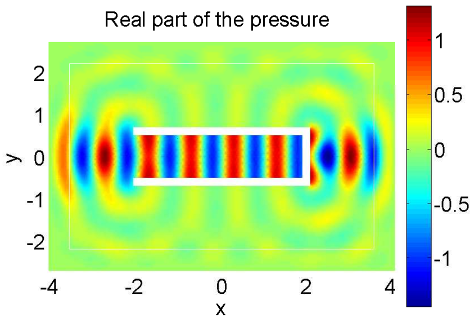

Figure 5: Real part of the pressure field generated by an incident plane wave, $k = 2\pi\,\mathrm{m}^{-1}$. Coarse mesh.

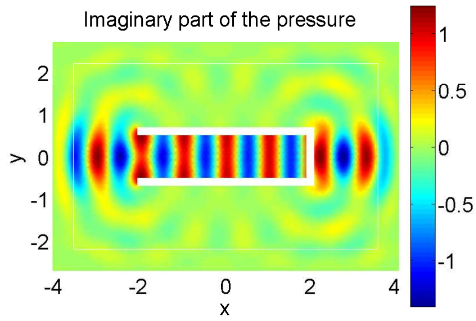

Figure 6: Imaginary part of the pressure field generated by an incident plane wave, $k = 2\pi\,\mathrm{m}^{-1}$. Coarse mesh.

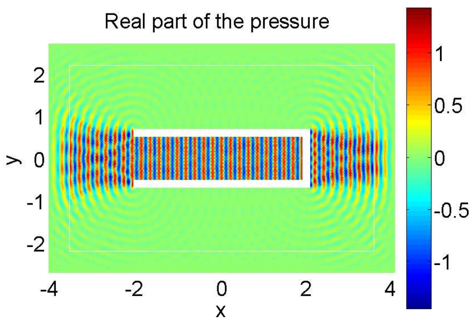

Figure 7: Real part of the pressure field generated by an incident plane wave, $k = 10\pi\,\mathrm{m}^{-1}$. Fine mesh.

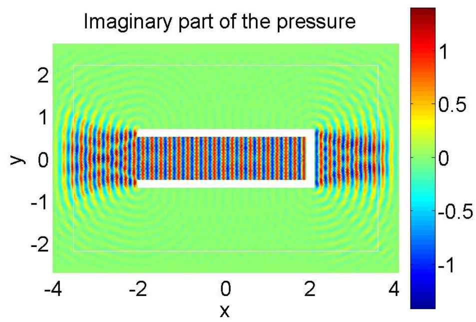

Figure 8: Imaginary part of the pressure field generated by an incident plane wave, $k = 10\pi\,\mathrm{m}^{-1}$. Fine mesh.

[6] E. Bécache, A.-S. Bonnet-Benn Dhia and G. Legendre, Perfectly matched layer for the convected Helmholtz equation, *SIAM J. Numer. Anal.*, **42**, 409-433 (2004)

[7] E. Bécache and P. Joly, On the analysis of Bérenger's perfectly matched layers for Maxwell's equations, *M2AN, Math. Model. Numer. Anal.*, **36**, 87-119 (2002).

[8] J.P. Berenger, A perfectly matched layer for the absorption of electromagnetic waves, *J. Comput. Phys.*, **114**, 185-200 (1994).

[9] J.P. Berenger, Perfectly matched layer for the FDTD solution of wave-structure interaction problems, *IEEE Trans. Antennas Propag.*, **44**, 110-117 (1996).

[10] J.P. Berenger, Three-dimensional perfectly matched layer for the absorption of electromagnetic waves, *J. Comput. Phys.*, **114**, 363-379 (1996).

[11] A. Bermúdez, L.M. Hervella-Nieto, A. Prieto and R. Rodríguez, An exact

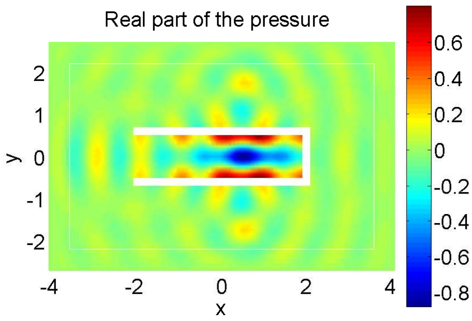

Figure 9: Real part of pressure field generated by a monopole, $k = 2\pi\,\mathrm{m}^{-1}$. Coarse mesh.

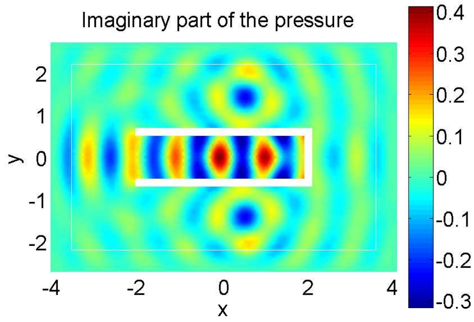

Figure 10: Imaginary part of the pressure field generated by a monopole, $k = 2\pi\,\mathrm{m}^{-1}$. Coarse mesh.

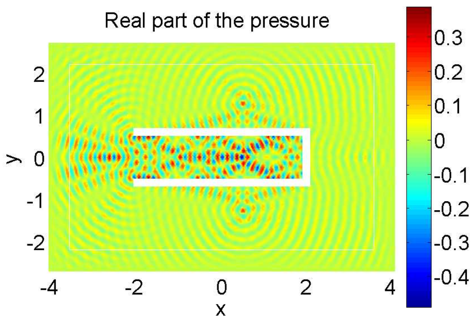

Figure 11: Real part of the pressure field generated by a monopole, $k = 10\pi\,\mathrm{m}^{-1}$. Fine mesh.

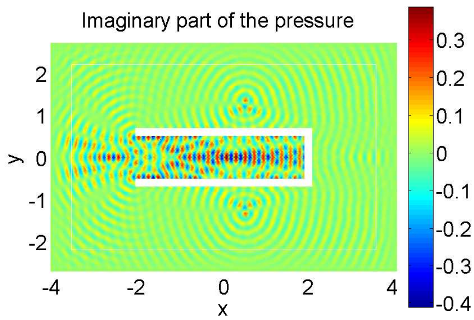

Figure 12: Imaginary part of the pressure field generated by a monopole, $k = 10\pi\,\mathrm{m}^{-1}$. Fine mesh.

bounded PML for the Helmholtz equation, *C. R. Acad. Sci. Paris, Ser. I, Math.*, **339**, 803-808 (2004).

[12] A. Bermúdez, L.M. Hervella-Nieto, A. Prieto and R. Rodríguez, Optimal perfectly matched layers for time-harmonic acoustic scattering problems (submitted).

[13] F. Collino and P. Monk, The perfectly matched layer in curvilinear coordinates, *SIAM J. Sci. Comput.*, **19**, 2061-2090 (1998).

[14] F. Collino and P. Monk, Optimizing the perfectly matched layer, *Comput. Methods Appl. Mech. Eng.*, **164**, 157-171 (1998).

[15] F. Collino and C. Tsogka, Application of the PML absorbing layer model to the linear elastodynamic problem in anisotropic heterogeneous media, *Geophysics*, **66**, 294-307 (2001)

[16] L. Demkowicz and F. Ihlenburg, Analysis of a coupled finite-infinite element method for exterior Helmholtz problems, *Numer. Math.*, **88**, 43-73 (2001).

[17] B. Engquist and A. Majda, Absorbing boundary conditions for the numerical simulation of waves, *Math. Comp.*, **31**, 629-651 (1977).

[18] K. Feng, Finite element method and natural boundary reduction, *Proc. Int. Congress of Mathematicians*, Warsaw, 1439-1453 (1983).

[19] D. Givoli and B. Neta, High-order non-reflecting boundary scheme for time-dependent waves, *J. Comput. Phys.*, **186**, 24-46 (2004).

[20] I. Harari, M. Slavutin and E. Turkel, Analytical and numerical studies of a finite element PML for the Helmholtz equation, *J. Computat. Acoust.*, **8**, 121-137 (2000).

[21] T. Hohage, F. Schmidt and L. Zschiedrich, Solving time-harmonic scattering problems based on the pole condition II: convergence of the PML method, *SIAM J. Math. Anal.*, **35**, 547-560 (2003).

[22] F.Q. Hu, A stable, perfectly matched layer for linearized Euler equations in unsplit physical variables, *J. Comput. Phys.*, **173**, 455-480 (2001).

[23] F. Ihlenburg, *Finite Element of Analysis of Acoustical Scattering*, Springer-Verlag, New York, 1998.

[24] V.J. van Joolen, B. Neta and D. Givoli, High-order Higdon-like boundary conditions for exterior transient problems, *Int. J. Numer. Methods. Eng.*, **63**, 1041-1068 (2005).

[25] M. Lassas and E. Somersalo, On the existence and convergence of the solution of PML equations, *Computing*, **60**, 228-241 (1998).

[26] J.L. Lions, J. Métral and O. Vacus, Well-posed absorbing layer for hyperbolic problems, *Numer. Math.*, **92**, 535-562 (2002).

[27] I.M. Navon, B. Neta and M.Y. Hussaini, A perfectly matched layer approach to the linearized shallow water equations models, *Monthly Weather Review*, **132**, 1369-1378 (2004).

[28] J.-C.Nédélec, *Acoustic and Electromagnetic Equations. Integral Representations for Harmonic Problems*, Springer-Verlag, New York, 2000.

[29] Q. Qi and T.L. Geers, Evaluation of the perfectly matched layer for computational acoustics, *J. Comput. Phys.*, **139**, 166-183 (1998).

[30] J.J. Shirron and I. Babuška, A comparison of approximate boundary conditions and infinite element methods for exterior Helmholtz problems, *Comput. Methods Appl. Mech. Eng.*, **164**, 121-139 (1998).

[31] B. Sjogreen and N. A. Petersson, Perfectly matched layers for Maxwell's equations in second order formulation. *J. Comput. Phys.*, **209**, 19-46 (2005).

[32] E. Turkel and A. Yefet, Absorbing PML boundary layers for wave-like equations, *Appl. Numer. Math.*, **27**, 533-557 (1998).

[33] Y.Q. Zeng, J.Q. He and Q.H. Liu, The application of the perfectly matched layer in numerical modeling of wave propagation in poroelastic media, *Geophysics*, **66**, 1258-1266 (2001).