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A FAMILY OF EXACT BOUNDED PML FOR THE HELMHOLTZ PROBLEM

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Abstract. *We study the Helmholtz equation with a Sommerfeld radiation condition in an unbounded domain. We introduce an exact bounded perfectly matched layer (PML) for this problem, in the sense that we recover the exact solution in the physical domain by choosing a singular PML function in a bounded domain. We compute the solution for the PML problem using a standard finite element method and assess its performance through numerical tests.*

1 INTRODUCTION

This work deals with the numerical resolution of a Helmholtz problem in an unbounded domain. For this reason, the use of finite element or finite difference discretization is not straightforward. Thus, as a first step, we use the PML (Perfectly Matched Layer) method to reduce the original problem to a problem in a bounded domain. This method was introduced by Bérenger in [2] for the electromagnetic equations. In this reference, Bérenger shows how to construct a perfectly matched absorbing boundary layer in rectilinear coordinates. This layer absorbs waves of any wavelength and any frequency without reflections. The idea for its construction is based on a non-physical decomposition of the unknowns of the problem. Each of these non-physical unknowns is damped separately using a damping function σ, which must be positive and increasing. In [2] σ is chosen linear or parabolic. The original equations are kept in a bounded domain containing Ω, and they are modified in some PML zone surrounding this domain.

Mathematically, the PML method can be understood as a complex change of variables, as it has been shown in [5]. In this reference, the method is analyzed for the case of a radial layer.

In the present work, we propose a particular PML method. Instead of the above mentioned classical choices for the damping function σ, we take an unbounded function with unbounded integral as, for example, $\sigma(r) = \frac{1}{R^*-r}$ for $r \in (R, R^*)$, where the PML zone is the set $\{x \in \mathbb{R}^2 : \ R < |x| < R^*\}$. In this case, we prove that the solution of the PML problem restricted to $\{x \in \mathbb{R}^2 \setminus \Omega : \ 0 < |x| \leq R\}$, is exactly the one of the original unbounded problem.

We report numerical results for a finite element approximation of this PML problem in the more interesting case of a rectangular physical domain. We compare them with those obtained for classical PML showing the good behaviour of our method.

2 The Helmholtz problem in an unbounded domain

We consider the following Helmholtz problem which models the propagation of a wave of frequency $\omega > 0$ and velocity of propagation $c > 0$ in an unbounded homogeneous medium:

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \Omega_{\mathrm{E}}, \\ u = u_{\mathrm{D}} & \text{on } \Gamma, \\ \displaystyle\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u}{\partial r} - iku\right) = 0, & \end{cases} \tag{1}$$

where $k := \omega/c$ is the wave number, $\Omega_{\mathrm{E}} := \mathbb{R}^2 \setminus \overline{\Omega}_{\mathrm{I}}$, with Ω_{I} being a bounded domain in $\mathbb{R}^2$ with regular boundary Γ, and $u_{\mathrm{D}} \in \mathrm{H}^{\frac{1}{2}}(\Gamma)$ a given function. Throughout this work, Sommerfeld-like conditions as the third equation in (1) are assumed to hold uniformly in all directions.

Problem (1) is a classical scattering problem, whose existence and uniqueness of solution is well known (see for instance [6]).

We assume that the origin is in Ω_{I} and we choose $R > 0$ such that $\Omega_{\mathrm{I}} \subset B_R$, where B_R is the ball of radius R centered at the origin and whose boundary is the circumference S_R (see Figure 1).

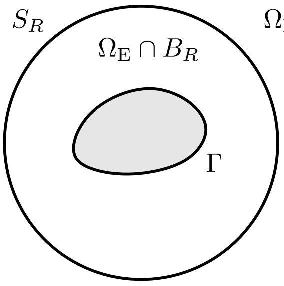

Figure 1: Scatterer and artificial circular boundary.

We define the exterior Dirichlet-to-Neumann (DtN) operator, $\widetilde{G}$, that maps any function $g \in \mathrm{H}^{\frac{1}{2}}(S_R)$ to $\widetilde{G}(g) := (\partial\widetilde{u}/\partial r)|_{S_R} \in \mathrm{H}^{-\frac{1}{2}}(S_R)$, with $\widetilde{u}$ being the solution of

$$\begin{cases} \Delta\widetilde{u} + k^2\widetilde{u} = 0 & \text{in } \mathbb{R}^2 \setminus \overline{B}_R, \\ \widetilde{u} = g & \text{on } S_R, \\ \displaystyle\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial\widetilde{u}}{\partial r} - ik\widetilde{u}\right) = 0. & \end{cases} \tag{2}$$

Then, the solution of problem (1) satisfies

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \Omega_{\mathrm{E}} \cap B_R, \\ u = u_{\mathrm{D}} & \text{on } \Gamma, \\ \dfrac{\partial u}{\partial r} = \widetilde{G}(u|_{S_R}) & \text{on } S_R. \end{cases} \tag{3}$$

It is well known (see for instance [7]) that, if we write $g = \sum_{n=-\infty}^{\infty} g_n \,\mathrm{e}^{in\theta} \in \mathrm{H}^{\frac{1}{2}}(S_R)$, then the DtN mapping verifies that

$$\widetilde{G}(g) = \sum_{n=-\infty}^{\infty} k g_n \frac{[\mathrm{H}_n^{(1)}]'(kR)}{\mathrm{H}_n^{(1)}(kR)} \,\mathrm{e}^{in\theta},$$

where $\mathrm{H}_n^{(1)}$ is the n-th Hankel function of first class.

3 PML problem in a bounded domain

The numerical solution of problem (1) by finite elements or finite differences is not straightforward because the domain is unbounded. Thus, as a first step, we reduce it to a bounded domain by using a PML (*Perfectly Matched Layer*) method introduced by Bérenger in [2] in the context of electromagnetic waves. To this aim, we consider a ball B_{R^*} centered at the origin and with radius $R^* > R$ (see Figure 2).

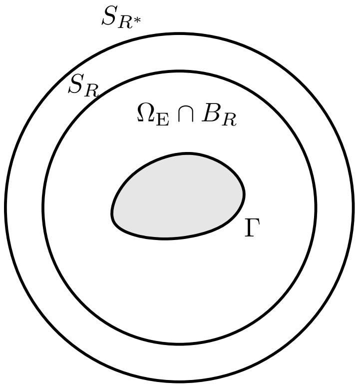

Figure 2: Domains for the PML problem.

The PML problem associated with (1) can be written in polar coordinates as follows (see [5]):

$$\left\{\begin{array}{ll} \dfrac{1}{r}\left\{\dfrac{\partial}{\partial r}\left[\dfrac{\widehat{\gamma}(r)r}{\gamma(r)}\dfrac{\partial v}{\partial r}\right]+\dfrac{\gamma(r)}{\widehat{\gamma}(r)r}\dfrac{\partial^2 v}{\partial \theta^2}\right\}+k^2\gamma(r)\widehat{\gamma}(r)v=0 & \text{in } \Omega_{\rm E}\cap B_{R^*}, \\ v=u_{\rm D} & \text{on } \Gamma, \\ \displaystyle\lim_{r\to R^*}\sqrt{r\widehat{\gamma}(r)}\left[\dfrac{1}{\gamma(r)}\dfrac{\partial v}{\partial r}-ikv\right]=0, & \end{array}\right. \tag{4}$$

where

$$\gamma(r) := \frac{\omega + i\sigma(r)}{\omega}$$

and

$$\widehat{\gamma}(r) := 1 + \frac{i}{r\omega}\int_R^r \sigma(s)\,ds,$$

with σ being an increasing function defined in $[0, R^*)$ and vanishing in $[0, R)$.

The PML equations (4) can be formally obtained by performing in the Helmholtz equations the complex change of variables given by (see [5])

$$\widehat{r} = \widehat{r}(r) := r + \frac{i}{\omega}\int_0^r \sigma(s)\,ds \qquad \forall r \in [0, R^*)\,. \tag{5}$$

Notice that $\widehat{r} = r$ for $r \in [0, R)$, because σ vanishes in this interval.

The PML method depends on the choice of σ in $[R, R^*)$. The classical choice is a linear or quadratic function taking a finite value σ^* in R^* (see [3, 5]). Also, the boundary condition on S_{R^*} in (4) is typically replaced by a homogeneous Dirichlet or Neumann condition. According to the literature, the value σ^* should be large enough as to minimize the reflections due to the fictitious boundary S_{R^*}, but not too large in order to avoid numerical errors arising from the discretization.

Instead, we propose to choose a function σ such that $\int_R^{R^*} \sigma(s)\, ds = +\infty$ as, for instance,

$$\sigma(r) := \begin{cases} 0 & r \in [0, R)\,, \\ \dfrac{1}{R^* - r} & r \in [R, R^*)\,. \end{cases}$$

Let us remark that although σ is discontinuous on the interface, this does not produce any spurious reflection (see [1] for further discussions on this topic).

We will show that, in this case, the solution of the PML problem (4) coincides in $\Omega_{\mathrm{E}} \cap B_R$ with the solution of the original problem (1).

If we restrict the first equation in problem (4) to $\Omega_{\mathrm{E}} \cap B_R$, since σ vanishes in $[0, R)$, we recover the Helmholtz equation. More precisely, any solution of problem (4) satisfies

$$\begin{cases} \Delta v + k^2 v = 0 & \text{in } \Omega_{\mathrm{E}} \cap B_R, \\ v = u_{\mathrm{D}} & \text{on } \Gamma, \\ \dfrac{\partial v}{\partial r} = \widehat{G}(v|_{S_R}) & \text{on } S_R, \end{cases} \tag{6}$$

where $\widehat{G}$ is an operator mapping any function $g \in \mathrm{H}^{\frac{1}{2}}(S_R)$ to

$$\widehat{G}(g) := \left. \frac{1}{\gamma} \frac{\partial \widehat{v}}{\partial r} \right|_{S_R} \in \mathrm{H}^{-\frac{1}{2}}(S_R),$$

with $\widehat{v}$ being the solution of

$$\begin{cases} \dfrac{1}{r} \left\{ \dfrac{\partial}{\partial r} \left[\dfrac{\widehat{\gamma}(r) r}{\gamma(r)} \dfrac{\partial \widehat{v}}{\partial r} \right] + \dfrac{\gamma(r)}{\widehat{\gamma}(r) r} \dfrac{\partial^2 \widehat{v}}{\partial \theta^2} \right\} + k^2 \gamma(r) \widehat{\gamma}(r) \widehat{v} = 0 & \text{in } B_{R^*} \setminus \overline{B}_R, \\ \widehat{v} = g & \text{on } S_R, \\ \displaystyle\lim_{r \to R^*} \sqrt{r \widehat{\gamma}(r)} \left[\dfrac{1}{\gamma(r)} \dfrac{\partial \widehat{v}}{\partial r} - ik\widehat{v} \right] = 0. & \end{cases} \tag{7}$$

As a consequence of the following theorem, this operator $\widehat{G}$ is well defined.

Let us denote by $\mathcal{V}$ the space of measurable functions u defined in $B_{R^*} \setminus \overline{B}_R$ and such that

$$\int_0^{2\pi} \int_R^{R^*} \left[\left| \frac{\widehat{\gamma}(r) r}{\gamma(r)} \right| \left| \frac{\partial u}{\partial r}(r, \theta) \right|^2 + \left| \frac{\gamma(r)}{\widehat{\gamma}(r) r} \right| \left| \frac{\partial u}{\partial \theta}(r, \theta) \right|^2 + |r \gamma(r) \widehat{\gamma}(r)|\, |u(r, \theta)|^2 \right] dr\, d\theta < \infty.$$

Then the following results hold (see [4]):

Theorem 3.1 *Let σ be an increasing function defined in $[0, R^*)$, vanishing in $[0, R)$, smooth in (R, R^*), and such that $\int_R^{R^*} \sigma(s)\, ds = +\infty$. Then, problem (7) has a unique solution in $\mathcal{V}$ given by*

$$\widehat{v}(r,\theta) = \sum_{n=-\infty}^{\infty} \frac{g_n}{\mathrm{H}_n^{(1)}(kR)} \mathrm{H}_n^{(1)}(k\widehat{r}(r))\, \mathrm{e}^{in\theta} \qquad \forall r \in (R, R^*)\,. \tag{8}$$

Corollary 3.2 *The operator $\widehat{G}$ is well defined and $\widehat{G}(g) = \widetilde{G}(g)$ $\forall g \in \mathrm{H}^{\frac{1}{2}}(S_R)$.*

Corollary 3.3 *If u and v are the solutions of problems (1) and (4), respectively, then $u = v$ in $\Omega_{\mathrm{E}} \cap B_R$.*

Remark 1 *By using the Green's formula and the behaviour of the outgoing fundamental solution of the PML problem as $|x| \to R^*$, we deduce (see [4]) that the above solution $\widehat{v}$ also satisfies $\widehat{v} = 0$ on S_{R^*}. Thus, in practice, we can use this Dirichlet boundary condition rather than (7) when solving the PML problem.*

4 Numerical results

Although the previous analysis has been made for the Helmholtz problem in polar coordinates, in practice it is more interesting to solve numerically the problem with a rectangular PML layer as that shown in Figure 3.

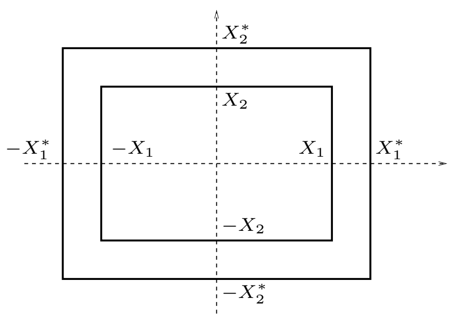

Figure 3: PML layers in Cartesian coordinates.

We denote $\Omega_{\mathrm{P}} := [-X_1, X_1] \times [-X_2, X_2]$ (physical domain) and $\Omega := [-X_1^*, X_1^*] \times [-X_2^*, X_2^*]$. Instead of a prescribed Dirichlet data, we consider a harmonic source with support contained in Ω_{P}. The equations of the corresponding PML problem are the following (see [5]):

$$\begin{cases} \dfrac{\partial}{\partial x_1}\left(\dfrac{\gamma_2}{\gamma_1}\dfrac{\partial u}{\partial x_1}\right) + \dfrac{\partial}{\partial x_2}\left(\dfrac{\gamma_1}{\gamma_2}\dfrac{\partial u}{\partial x_2}\right) + k^2\gamma_1\gamma_2 u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{9}$$

Here

$$\gamma_1 = \gamma_1(x_1) := \frac{\omega + i\sigma_1(x_1)}{\omega},$$

with σ_1 being the even function vanishing for $|x_1| < X_1$, and such that

$$\sigma_1(x_1) := \frac{c}{(X_1^* - |x_1|)}, \qquad x_1 \in [X_1, X_1^*)\,.$$

We recall that c is the velocity of propagation of the wave. The definition of $\gamma_2 = \gamma_2(x_2)$ is analogous.

We solve the above problem with a finite element method based on Q_1-Lagrange rectangular elements on uniform partitions of mesh-size h. We use exact integration to compute the element stiffness and mass matrices. We denote by u_h the discrete solution. Figure 4 shows one of the meshes we have used. The physical domain is drawn in dark blue, whereas the other colors represent the PML layers.

To test the method we take as f the Dirac delta measure supported at the origin, $k = \omega/c = 750/340$, $X_1 = X_2 = 0.5$, and $X_1^* = X_2^* = 0.75$. In this case, the exact solution for the unbounded domain is explicitly known:

$$u^{\mathrm{ex}}(x_1, x_2) := \frac{i}{4}\mathrm{H}_0^{(1)}(k\sqrt{x_1^2 + x_2^2}).$$

Figure 5 shows a plot of the relative error versus the mesh-size h. We measure the error in the L^2-norm in Ω_{P} excluding a small neighborhood of the origin: $\|u_h - u^{\mathrm{ex}}\|_{0,\widetilde{\Omega}_{\mathrm{P}}} / \|u^{\mathrm{ex}}\|_{0,\widetilde{\Omega}_{\mathrm{P}}}$, with $\widetilde{\Omega}_{\mathrm{P}} := \{x \in \Omega_{\mathrm{P}} :\ |x| > 0.05\}$. It can be clearly seen that a quadratic order of convergence is achieved.

Table 1 shows a comparison between the performance of our PML method with singular σ_1 and σ_2, versus the classical PML method with σ_i being the quadratic function defined by

$$\sigma_i(x_i) = c\sigma^* \frac{(|x_i| - X_i)^2}{(X_i^* - X_i)^2}, \quad x_i \in [X_i, X_i^*)\,; \qquad i = 1, 2,$$

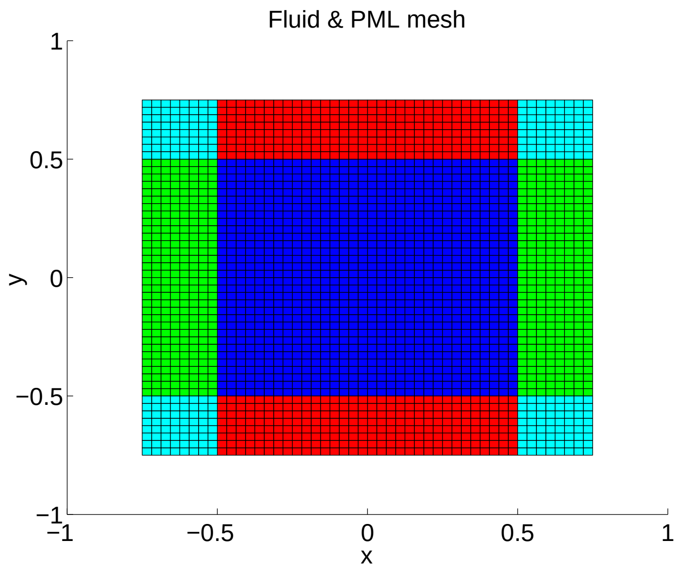

Figure 4: Mesh with 2401 d.o.f.

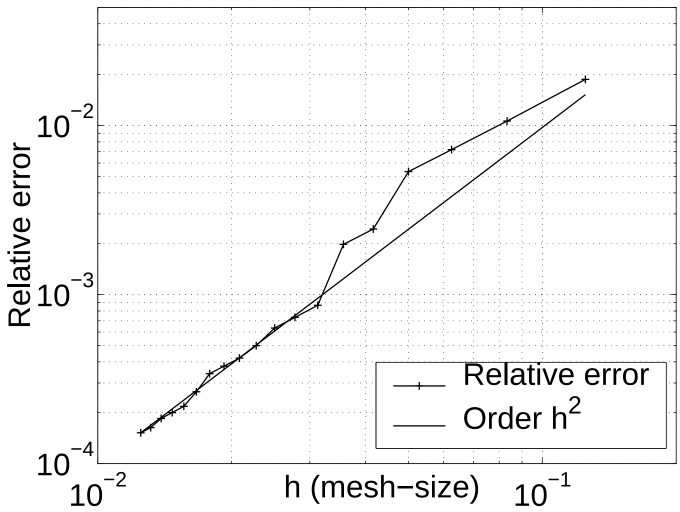

Figure 5: Error curve for the exact PML method.

where the value of σ^* is chosen as to keep the error as small as possible.

Table 1 shows that our PML method has essentially the same computational cost than the classical one, but it is much more efficient, even though an optimal value of σ^* has been used for the latter. We remark that for the classical PML method, the reported CPU time is somehow fictitious, in that the time necessary to find the optimal value of σ^* is not included. Moreover, such optimal value of σ^* can only be found when the exact solution of the problem is known.

Number of d.o.f.	σ type	Relative error (%)	CPU time (s)
625	singular	0.720	11.54
625	quadratic ($\sigma^\star = 31.00$)	3.543	11.11
2401	singular	0.086	46.16
2401	quadratic ($\sigma^* = 39.36$)	1.047	44.35
5329	singular	0.042	106.99
5329	quadratic ($\sigma^* = 44.37$)	0.506	102.45
9409	singular	0.022	195.40
9409	quadratic ($\sigma^\star = 47.84$)	0.300	187.98

Table 1: Performance comparison: PML with singular σ versus PML with quadratic σ.

Figure 6 shows the real part of the solution of the PML problem with σ singular. We have used the mesh shown in Figure 4.

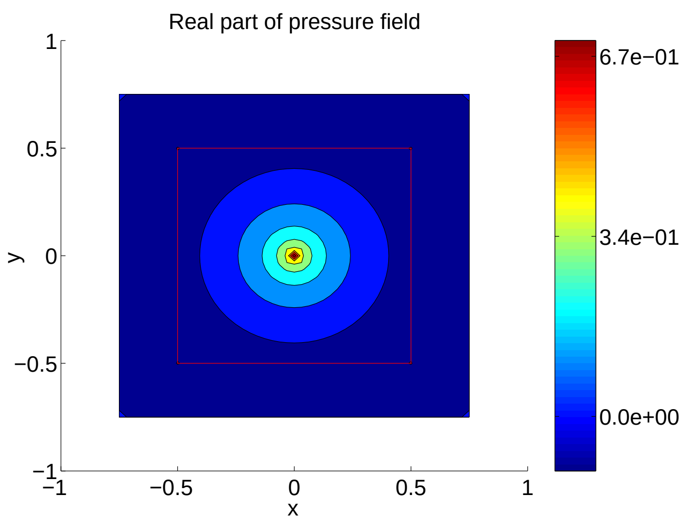

Figure 6: Real part of the solution for the PML problem (singular function).

Finally, we have tested our PML method in a more challenging case. We have simulated a Dirac delta measure supported near an edge and near a corner of the physical domain. Figures 7 and 8 show the real part of the respective solutions. It can be seen that the behaviour of the method is very good in these cases, too. This shows that, in practice, the PML layer could be situated very close to the obstacle.

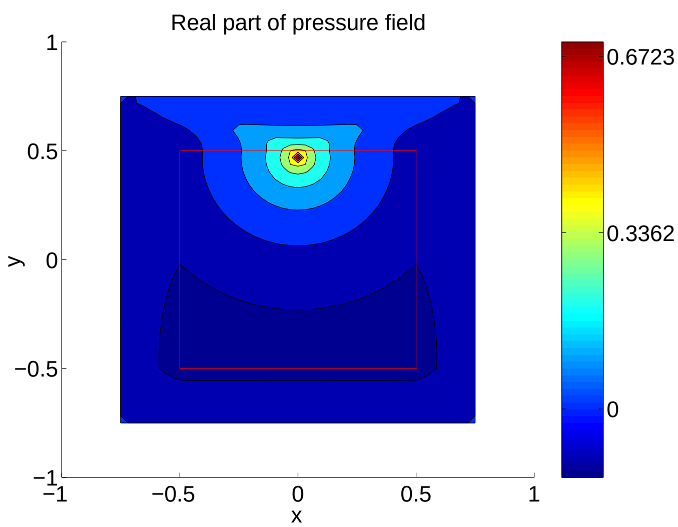

Figure 7: Dirac delta supported near an edge.

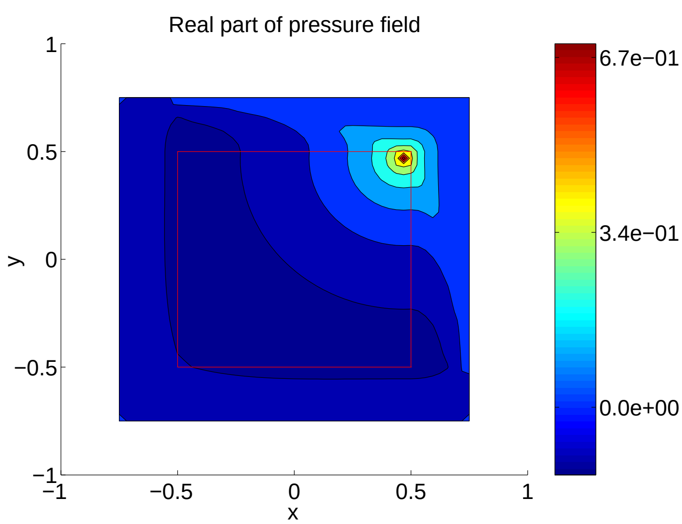

Figure 8: Dirac delta supported near a corner.

5 Concluding remarks

We have solved the Helmholtz problem in an unbounded domain with a PML method, based on a singular PML function. We have shown that the exact solution is perfectly recovered in the physical domain.

According to the numerical experiments that we have reported, the method is very robust, even in challenging cases as, for instance, when the sources are located near a corner or an edge of the PML.

In all cases, the method with a singular PML function turns out to be more effective than when a quadratic or linear function is used: smaller errors versus the same CPU time, and no parameters to fit.

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