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Error estimates for partition of unity finite element solutions of the Helmholtz equation

L. Hervella-Nieto[a], P. M. López-Pérez[b], A. Prieto[c]

Departamento de Matemáticas, Universidade da Coruña. E-mails:
[a]`luis.hervella@udc.es`,
[b]`paula.lopez@udc.es`, [c]`andres.prieto@udc.es`.

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Abstract

The Helmholtz equation is widely used as the reference model in time-harmonic acoustic propagation problems. At middle and high frequency regime, its numerical approximation, computed by a nodal Finite Element Method (FEM), differs significantly from the exact solution due to the so-called "pollution" effect [1]. So, the accuracy and reliability of Helmholtz numerical approximations are based on pollution-free discrete methods, which should have a robust behavior with respect to the wavenumber. The Partition of Unity Finite Element Method (PUFEM) [4] has been considered among these pollution-free methods. Computational advantages and implementation drawbacks of the PUFEM discretization have been studied numerically in [5]. Error estimates for Finite Element solutions of the one-dimensional Helmholtz equation have been already studied by Babǔska and Ihlenburg in [3]. In this work, *a priori* error estimates are derived for PUFEM, where plane waves are used to enrich the discretization space. The approximability of the exact solution in such discrete space is deduced from some interpolation estimates involving only exponential-type basis functions. Error-estimates for PUFEM are obtained in terms of the wavenumber on the Helmholtz equation, the mesh size and an additional perturbation parameter introduced in the wavenumbers of the basis functions of the discrete PUFEM space.

1. Introduction

Boundary-value problems for the Helmholtz equation $\Delta u + k^2 u = 0$ arise in a number of physical applications, in particular in problems of wave scattering in Acoustics, Optics and Electromagnetism. It is well known (see, for instance, [2]) that the mesh size h for finite element and finite difference computations should depend on the wavenumber k, usually following a "rule of the thumb" which ensures a minimum number of nodes per wavelength. In problems where the typical size of the computational domain has the same order of magnitude as the wavelength of the harmonic motion, this criterion leads to accurate results. However, the quality of the numerical approximation deteriorates if the computational domain or the wavenumber are large enough. Under certain assumptions on the magnitude hk, it has been shown in [3] that the H^1-relative error of the FEM solution e_{fe} can be bounded by

$$e_{\text{fe}} \leq C_1 kh + C_2 k^3 h^2, \tag{1}$$

where the second term on the right-hand side is the so-called numerical pollution error. In [4], the Partition of the Unity Finite Element Method was proposed with the aim of mitigating the pollution effect of standard FEM approximations. The computational advantages of this method have been illustrated by a variety of numerical results (see e.g. [5]). In this work, a PUFEM discretization based on a plane wave enrichment is applied to the one-dimensional Helmholtz equation. Once the stability of the discrete problem is assumed, an non-optimal error estimate is shown without any restriction on the mesh size and the wavenumber values. The one-dimensional Helmholtz boundary-value problem with Dirichlet and Robin boundary conditions is described in Section 2. Then, its variational formulation and the associated *inf-sup* condition are stated in Section 3. The remainder of this paper in organized as follows: FEM and PUFEM discretizations are discussed in detail in Section 4, where an approximation result of the weak solution is shown for the PUFEM discrete spaces. Then, assuming a discrete *inf-sup* condition, the existence and uniqueness of the discrete solution and its stability with respect to the boundary data are obtained. A non-optimal error estimate for the PUFEM is also included in this section. Finally, some numerical results are presented in Section 5, showing that the order of accuracy of the PUFEM is not limited to any restricted range of the mesh size. In particular, it is illustrated that the pollution effect suffered by the FEM discretization is avoided with the PUFEM approximation.

Notation: Through the rest of the paper, standard notations about functional Sobolev spaces are used without explicit definition. For instance, $\|\cdot\|_0$ and $|\cdot|_1$ denote, respectively, the L^2-norm and the H^1-seminorm.

2. Model Problem

Let the boundary-value problem be

$$\left\{\begin{array}{rcl} u'' + k^2 u &=& 0 \quad \text{in } (0,1), \\ u(0) &=& u_0, \\ u'(1) - iku(1) &=& u_1, \end{array}\right. \tag{2}$$

being $u_0, u_1 \in \mathbb{C}$ and $k > 0$ constant. From an acoustic point of view, u could be understood as the harmonic vibration of the pressure field in a compressible fluid at a fixed wavenumber k. At $x = 0$, a Dirichlet boundary condition (prescribed pressure) is given, and a Robin condition is imposed at $x = 1$ (with an analogous left hand side to the one-dimensional Sommerfeld radiation condition at $x \to +\infty$).

It is easy to check that problem (2) has a unique analytic solution given by

$$u(x) = u_0 e^{ikx} + \left(\frac{i}{2k} u_1 e^{ik}\right)\left(e^{-ikx} - e^{ikx}\right), \tag{3}$$

satisfying the estimate

$$\|u\|_0 \leq |u_0| + k^{-1}|u_1|. \tag{4}$$

3. Variational formulation and weak solution

Considering the subspace $V = \Big\{v \in H^1(0,1)\,;\ v(0) = 0\Big\}$, the variational formulation of problem (2) is described as follows:

$$\begin{cases} \text{Find } u \in H^1(0,1);\ u(0) = u_0, \\ B(u,v) = u_1 \bar{v}(1) \quad \forall v \in V, \end{cases} \tag{5}$$

where $B(u,v) = \displaystyle\int_0^1 \Big(u'(x)\bar{v}'(x) - k^2 u(x)\bar{v}(x)\Big)\,dx - iku(1)\bar{v}(1) \quad \forall u,v \in H^1(0,1)$.
Following [3], the *inf-sup* condition of the sesquilinear form B is obtained.

Theorem 3.1 *Let $B : V \times V \to \mathbb{C}$ be as defined above. Then the Ladyzhenskaya-Babŭska-Brezzi (LBB) constant,*

$$\gamma = \inf_{u \in V} \sup_{v \in V} \frac{|B(u,v)|}{|u|_1 |v|_1} > 0,$$

is of order k^{-1}. More precisely, there exist positive constants C_1, C_2 not depending on k such that

$$\frac{C_1}{k} \leq \gamma \leq \frac{C_2}{k}. \tag{6}$$

Applying the LBB estimate (6), a trace inequality and a translation of the solution in (5), the stability of the weak problem is derived, $|u|_1 \leq Ck(|u_0| + |u_1|)$. Straightforwardly from the estimate stated above, problem (5) has a unique solution in $H^1(0,1)$ and it coincides with (3).

4. Discrete problem and error estimates

Consider $\{x_j = h(j-1);\ j = 1, \ldots, n+1\}$ a uniform mesh of $n+1$ nodes in $(0,1)$, where the mesh size is $h = 1/n$.

4.1. Finite Element Method

Let $\widetilde{X}_h$ be the standard Lagrange $\mathbb{P}_1$ (piecewise linear) finite element space,

$$\widetilde{X}_h = \{v \in C^0(0,1)\,;\ v|_{[x_j, x_{j+1}]} \in \mathbb{P}_1\,,\ j = 1, ..., n\} \subset H^1(0,1),$$

and let $\widetilde{V}_h$ be the finite space of functions in $\widetilde{X}_h$ satisfying a homogeneous Dirichlet condition at $x = 0$, $\widetilde{V}_h = \{v \in \widetilde{X}_h\,;\ v(0) = 0\} \subset V$. Since $\widetilde{X}_h$ and $\widetilde{V}_h$ are finite-dimensional spaces, it holds $\widetilde{X}_h = \langle\{\varphi_1, \ldots, \varphi_{n+1}\}\rangle$ and $\widetilde{V}_h = \langle\{\varphi_2, \ldots, \varphi_{n+1}\}\rangle$, where the piecewise linear basis functions φ_j satisfy $\varphi_j(x_l) = \delta_{jl}$ for $j, l = 1, \ldots, n+1$, being δ_{jl} the Kronecker delta. Hence, the FEM approximation u_{fe} of the exact solution of problem (5) is defined as the solution of the following linear problem: Find $u_{\mathrm{fe}} \in \widetilde{X}_h$ such that $u_{\mathrm{fe}}(0) = u_0$ and

$$B(u_{\mathrm{fe}}, v) = u_1\bar{v}(1) \quad \forall v \in \widetilde{V}_h.$$

To avoid the pollution effect derived from the numerical FEM solution u_{fe}, a PUFEM discretization is applied by enriching the FEM space with plane wave functions.

4.2. Partiton of the Unity Finite Element Method

Babŭska and Melenk [4] proposed a new Galerkin method which mitigates the FEM pollution effects in the discretization of the Helmholtz equation. In that work, the standard finite element spaces were enriched with plane waves of the form $e^{\lambda x}$, being λ a root of the characteristic polynomial of the Helmholtz equation, i.e., $\lambda = \pm ik$. In the present work, an additional perturbation parameter $\delta > 0$ is introduced on these roots, and so, the PUFEM space involves plane waves $e^{\lambda x}$, where $\lambda = \pm i(k+\delta)$. Consequently, the *a priori* estimates derived in this section will depend also on δ. The trial and test PUFEM spaces are given, respectively, by $X_h = \langle\psi_1, \ldots, \psi_{3n+3}\rangle$ and $V_h = \{v \in X_h\,;\ v(0) = 0\}$, where $\psi_{3j-2}(x) = \varphi_j(x)$, $\psi_{3j-1}(x) = \varphi_j(x)e^{-i(k+\delta)x}$ and $\psi_{3j}(x) = \varphi_j(x)e^{i(k+\delta)x}$ for $j = 1, \ldots, n+1$.
Then, the discrete PUFEM solution u_{pu} is defined as the solution of the following linear problem: Find $u_{\mathrm{pu}} \in X_h$ such that $u_{\mathrm{pu}}(0) = u_0$ and

$$B(u_{\mathrm{pu}}, v) = u_1\bar{v}(1) \quad \forall v \in V_h. \tag{7}$$

The first step to analyze the error in the PUFEM discretization consists in the derivation of an approximability result in X_h.

Lemma 4.1 *Let $u \in H^1(0,1)$ the solution of the variational problem* (5) *and k_0 a fixed positive value. Then, if $k \geq k_0 > 2$, there exists $u_I \in X_h$ such that*

$$\inf_{v_h \in X_h} \|u - v_h\|_0 \leq \|u - u_I\|_0 \leq C_3\delta^2 h^2 \|u\|_0, \tag{8}$$

$$\inf_{v_h \in X_h} |u - v_h|_1 \leq |u - u_I|_1 \leq \left(C_4\delta^2 h + C_5\delta^2 h^2 k\right) \|u\|_0, \tag{9}$$

where $C_3, C_4, C_5 > 0$ do not depend on h, δ and k.

Proof. To ensure $u_I \in X_h$, any restriction of u_I to the interval $[x_j, x_{j+1}]$ should be written as a linear combination of basis functions in X_h, i.e., for $j = 1, \ldots, n$,

$$u_I(x) = \alpha_1^j\varphi_j(x)e^{i(k+\delta)x} + \alpha_2^j\varphi_{j+1}(x)e^{i(k+\delta)x} + \alpha_3^j\varphi_j(x)e^{-i(k+\delta)x} + \alpha_4^j\varphi_{j+1}(x)e^{-i(k+\delta)x} + \alpha_5^j\varphi_j(x) + \alpha_6^j\varphi_{j+1}(x), \quad x \in [x_j, x_{j+1}].$$

Since the exact solution for the homogeneous Helmholtz equation is given by $u(x) = Ae^{ikx} + Be^{-ikx}$, its expression can be split in two terms, $u = u^+ + u^-$, with $u^+(x) = Ae^{ikx}$ and $u^-(x) = Be^{-ikx}$, being A and B complex-valued constants. Using the same splitting procedure and neglecting the terms in u_I coming from the original FEM basis, that is, $\alpha_5^j=\alpha_6^j=0$, it holds $u_I=u_I^+ + u_I^-$, where $u_I^+(x)=\alpha_1^j\varphi_j(x)e^{i(k+\delta)x}+\alpha_2^j\varphi_{j+1}(x)e^{i(k+\delta)x}$ and $u_I^-(x)=\alpha_3^j\varphi_j(x)e^{-i(k+\delta)x}+\alpha_4^j\varphi_{j+1}(x)e^{-i(k+\delta)x}$ for $x\in[x_j,x_{j+1}]$. If the other coefficients are computed by interpolation, $u_I^+(x_j)=u^+(x_j)$, $u_I^+(x_{j+1})=u^+(x_{j+1})$, $u_I^-(x_j)=u^-(x_j)$ and $u_I^-(x_{j+1})=u^-(x_{j+1})$, it is easy to check that $\alpha_1^j=Ae^{-i\delta x_j}$, $\alpha_2^j=Ae^{-i\delta x_{j+1}}$, $\alpha_3^j=Be^{i\delta x_j}$ and $\alpha_4^j=Be^{i\delta x_{j+1}}$ for $j=1,\dots,n$, and it holds

$$\begin{aligned}\|u-u_I\|_0^2 \le{}& |A|^2\sum_{j=1}^n\int_{x_j}^{x_{j+1}}|e^{ikx}-e^{-i\delta x_j}\varphi_j(x)e^{i(k+\delta)x}-e^{-i\delta x_{j+1}}\varphi_{j+1}(x)e^{i(k+\delta)x}|^2dx\\ &+|B|^2\sum_{j=1}^n\int_{x_j}^{x_{j+1}}|e^{-ikx}-e^{i\delta x_j}\varphi_j(x)e^{-i(k+\delta)x}-e^{i\delta x_{j+1}}\varphi_{j+1}(x)e^{-i(k+\delta)x}|^2dx.\end{aligned}$$

If the integrals are computed analytically and a Taylor expansion is applied for the exponential expressions, inequality (8) is obtained as follows:

$$\|u-u_I\|_0^2\le |A|^2\sum_{j=1}^n\widetilde{C}_1\delta^4h^5+|B|^2\sum_{j=1}^n\widetilde{C}_2\delta^4h^5\le C_3^2\delta^4h^4\|u\|_0^2,$$

where $C_3>0$ is a constant independent of h, δ and k.
Now, using an analogous procedure to estimate the H^1-seminorm of $u-u_I$, it holds

$$\begin{aligned}|u-u_I|_1^2 \;\le\;& |A|^2\sum_{j=1}^n(\widetilde{C}_3\delta^4h^3+\widetilde{C}_4\delta^4h^5k^2+\widetilde{C}_5\delta^5h^5k)\\ +\;& |B|^2\sum_{j=1}^n(\widetilde{C}_6\delta^4h^3+\widetilde{C}_7\delta^4h^5k^2+\widetilde{C}_8\delta^5h^5k)\le (C_4^2\delta^4h^2+C_5^2\delta^4h^4k^2)\|u\|_0^2,\end{aligned}$$

and so, estimate (9) is obtained, with positive constants C_4 and C_5 independent of h, δ and k. Notice that in the estimates stated above, it has been used $(|A|^2+|B|^2)\le C_0k_0/(k_0-2)\|u\|_0^2$, with $C_0>0$ independent of k. □

Combining (4) and (9), the H^1-approximability of the PUFEM discrete space for the Helmholtz problem is immediately obtained.

Theorem 4.2 *Let $u\in H^1(0,1)$ be the solution of problem (5), k_0 a fixed positive value and $u_I\in X_h$ its interpolated function as it has been defined on Lemma 1. If $k\ge k_0>2$ then, there exist positive constants C_4 and C_5 independent of h, δ and k such that*

$$|u-u_I|_1\le (C_4\delta^2h+C_5\delta^2h^2k)(|u_0|+k^{-1}|u_1|).$$

Assumption 1 *If $\delta>0$ is small enough, it is assumed that there exists a positive discrete inf-sup constant for the PUFEM discretization. More precisely, the LBB discrete condition, with*

$$\gamma_h=\inf_{u\in V_h}\sup_{v\in V_h}\frac{|B(u,v)|}{|u|_1|v|_1}>0,$$

holds, and there exist positive constants C_6 and C_7 not depending on k, δ and h such that

$$\frac{C_6}{k}\le\gamma_h\le\frac{C_7}{k}. \tag{10}$$

Applying the LBB discrete estimate (10), a trace inequality and a translation of the solution in (7), the stability of the discrete problem is obtained,

$$|u_{\mathrm{pu}}|_1 \leq \widetilde{C}k((\delta^2 + 2k\delta + \delta)|u_0| + |u_1|), \tag{11}$$

where $\widetilde{C} > 0$ is independent of k, δ and h. Straightforwardly from (11), the discrete problem has a unique solution.

Theorem 4.3 *Let $u \in H^1(0,1)$ be the exact solution of the variational problem* (5)*, k_0 a fixed positive value and let u_{pu} be the solution of the PUFEM discrete problem defined in* (7)*. If $k \geq k_0 > 2$ then, it holds*

$$\frac{|u - u_{\mathrm{pu}}|_1}{|u|_1} \leq \widehat{C}_1\delta^2 hk^{-1} + \widehat{C}_2\delta^2 h^2 + \widehat{C}_3\delta^2 h^2 k^2, \tag{12}$$

where $\widehat{C}_1, \widehat{C}_2$ and $\widehat{C}_3$ are positive constants not depending on h, δ and k.

Proof. Let $u_I \in X_h$ be the interpolant of u defined in Lemma 1. Following analogous arguments to those presented in [3, Theorem 5], define $z = u_{\mathrm{pu}} - u_I \in V_h$. Due to the B-orthogonality of the error in V_h and the linearity of the form B, it holds $B(u - u_I, v) = B(z, v)$ for all $v \in V_h$.
It is easy to check by partial integration that $\left((u - u_I)', v'\right) = 0$ if $v \in V_h$, and therefore,

$$B(u - u_I, v) = k^2(u - u_I, v) \quad \forall v \in V_h.$$

Hence, z is the solution of $B(z, v) = k^2(u - u_I, v)$ for all $v \in V_h$. Using the assumed LBB discrete condition, it is satisfied $|z|_1 \leq \widehat{C}k^3\|u - u_I\|_{-1}$. Then, using a triangular inequality,

$$|u - u_{\mathrm{pu}}|_1 \leq |u - u_I|_1 + |z|_1 \leq |u - u_I|_1 + \widehat{C}k^3\|u - u_I\|_{-1},$$

taking into account the estimates (8) and (9) in Lemma 1 and considering the continuous embedding of $L^2(0,1)$ in $H^{-1}(0,1)$, it holds

$$\begin{aligned} |u - u_{\mathrm{pu}}|_1 &\leq (C_4\delta^2 h + C_5\delta^2 h^2 k)\|u\|_0 + \widehat{C}C_3\delta^2 h^2 k^3\|u\|_0 \\ &\leq (\widehat{C}_1\delta^2 hk^{-1} + \widehat{C}_2\delta^2 h^2 + \widehat{C}_3\delta^2 h^2 k^2)|u|_1, \end{aligned}$$

where it has been used that for the plane wave solution, $\|u\|_0 \leq \widehat{C}_4 k_0/(k_0 - 2)|u|_1$ for $k \geq k_0$, being $\widehat{C}_4 > 0$ independent of k. □

5. Numerical Results

In this section, some numerical results are shown for FEM and PUFEM discrete approximations of the solution of the boundary-value problem (2) with non homogeneous boundary conditions. The boundary data is chosen to obtain $u(x) = e^{ikx} + 2e^{-ikx}$ as the exact solution.
Let e_{fe} be the relative error for the FEM discretization and e_{pu} the relative error for the PUFEM discretization, defined as follows: $e_{\mathrm{fe}} = |u - u_{\mathrm{fe}}|_{1,h}/|u|_{1,h}$ and $e_{\mathrm{pu}} = |u - u_{\mathrm{pu}}|_{1,h}/|u|_{1,h}$, where

$$|v|_{1,h}^2 = \sum_{j=1}^{n} |v'(y_j)|^2,$$

being y_j the midpoint of $[x_j, x_{j+1}]$ for $j = 1, \ldots, n$. The behavior of $e_{\rm fe}$ in the left plot of Figure 1 illustrates the second order of accuracy of the FEM approximation with respect to the mesh size, when the term $\mathcal{O}(hk)$ is negligible with respect to the pollution term in (1). In the right plot of Figure 1, second-order accuracy of the PUFEM approximation is checked and it confirms the error estimate (12) for high wavenumbers, i.e., once k is several orders of magnitude larger than $1/h$. Notice that $10^{-12} \leq e_{\rm pu} \leq 10^{-5}$ even with high wavenumbers, while the lowest error for FEM is of order 10^{-2}, achieved only for low wavenumber k.

In Figure 2, the dependence of FEM and PUFEM relative errors on the wavenumber k is observed. In the left plot, the behavior of $e_{\rm fe}$ is goberned by the pollution term in (1), whereas in the right plot it can be checked that the PUFEM relative error does not depend on the wavenumber values. The convergent second-order behavior of the perturbation parameter δ that holds in (12) for the PUFEM discretization can be checked in Figure 3.

The independent behavior of the PUFEM relative error of the wavenumber k, illustrated on the right plot of Figure 2, has not been obtained in the error estimate of Theorem 3 (see last term in (12)). Consequently, further work has to be done to obtain optimal error estimates.

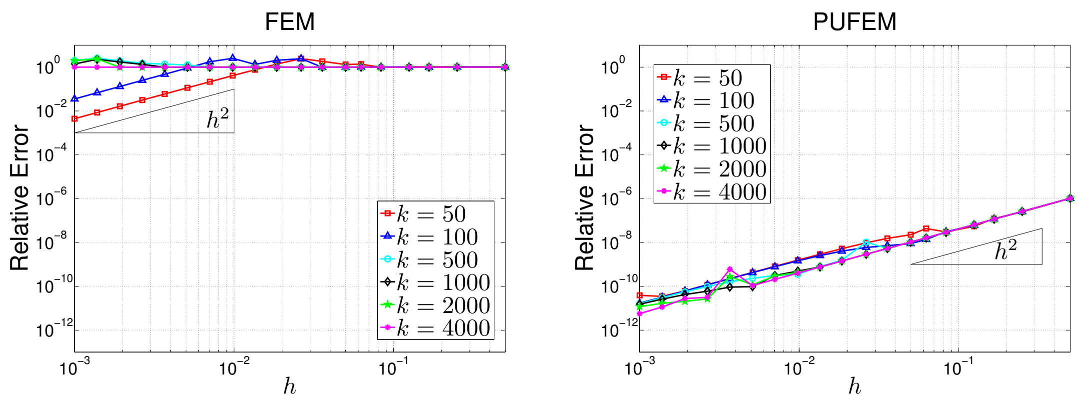

Figure 1: Relative error of the FEM (left) and PUFEM (right) discretization, plotted (using $\delta = 10^{-2}$) with respect to the mesh size

6. Conclusions

In this work, a non-optimal error estimate for a PUFEM discretization of a one-dimensional Helmholtz problem is deduced. For middle and high frequency regimes, such estimate ensures that the PUFEM H^1-relative error is bounded by a term of order $\mathcal{O}(\delta^2 h^2 k^2)$. Despite the independence of the relative error with respect to the wavenumber has not been obtained theoretically, the error estimate (12) involves only a term of order k^2, showing the potential advantages of the PUFEM approximation with respect to the FEM pollution term, given by $\mathcal{O}(k^3)$. In addition, the order of accuracy of the PUFEM approximation is not limited to any restricted range of the mesh size h, the perturbation parameter δ or the wavenumber k.

Índice

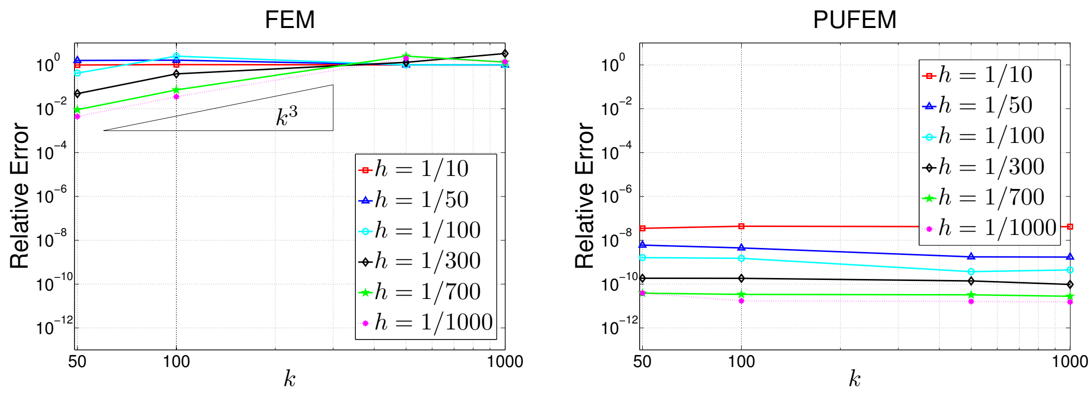

Figure 2: Relative error of the FEM (left) and PUFEM (right) discretization, plotted (using $\delta = 10^{-2}$) with respect to the wavenumber

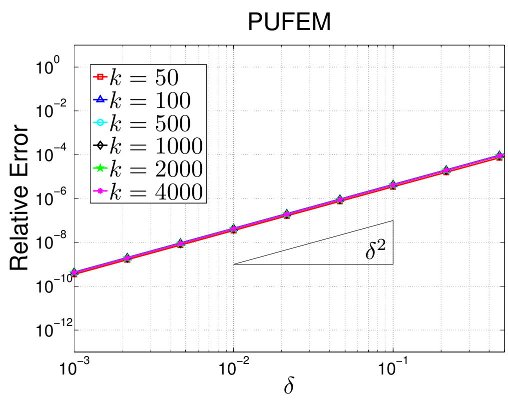

Figure 3: Relative error of the FEM (left) and PUFEM (right) discretization, plotted (using $h = 1/10$) with respect to the additional perturbation parameter

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