Additional results for model-based nonparametric variance estimation for systematic sampling in a forestry survey *

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Abstract

Systematic sampling is frequently used in natural resource and other surveys, because of its ease of implementation and its design efficiency. An important drawback of systematic sampling, however, is that no direct estimator of the design variance is available. We describe a new estimator of the model-based expectation of the design variance, under a nonparametric model for the population. The nonparametric model is sufficiently flexible that it can be expected to hold at least approximately for many practical situations. We prove the consistency of the estimator for both the anticipated variance and the design variance. We compare the nonparametric variance estimators with several design-based estimators on data from a forestry survey.

Key Words: local polynomial regression, two-per-stratum variance approximation, smoothing.

*Short title: Nonparametric variance estimation.

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1 Introduction

The *Forest Inventory and Analysis* (FIA) is a program within the US Department of Agriculture Forest Service that conducts nationwide forest surveys (see e.g. Frayer and Furnival 1999). Sampling for the FIA surveys has traditionally followed a stratified systematic design. In these surveys, the population quantities of interest are, for example, total tree volume, growth and mortality, and area by forest type. Design-based estimates of such quantities are produced on a regular basis. In this article, we are considering survey data collected during the 1990's by the Forest Service within a 2.5 million ha ecological province that includes the Wasatch and Uinta Mountain Ranges of northern Utah (Bailey et al. 1994). Forest resource data are collected through field visits on sample plots located on a regular spatial grid. These field-level data are supplemented by remotely sensed data available on a much finer spatial grid. Figure 1 displays the study region and sample locations for the survey data and additional remote sensing data. The latter can be used as auxiliary information to improve the precision of survey estimators, as previously done by Opsomer et al. (2007) who explored nonparametric model-assisted estimation methods. In the current article, we will use the auxiliary information to construct an estimator for the variance of survey estimators.

[Figure 1 about here.]

A well-known issue in surveys that follow a systematic sampling design is the lack of a theoretically justified, generally applicable design-based variance estimator. This issue was also encountered in Opsomer et al. (2007). These authors used a synthetic approach, in which a model was fit to the sample data and extrapolated to the population, and the exact design-based variance of the proposed estimators was computed at the population level based on the model-generated data. The resulting variance "estimator" was used to evaluate the efficiency of a proposed nonparametric model-assisted estimator for the FIA data. While this approach was effective in providing a way to gauge the relative efficiency of different survey estimates, the resulting variance "estimates" are difficult to interpret from a statistical perspective, because they clearly depend on the model specification, estimation and extrapolation but these factors are not explicitly accounted for. We therefore revisit the issue of estimating the variance for systematic sampling estimators. Unlike in Opsomer et al. (2007), the proposed estimator will be explicitly model-based, which will allow us to obtain its statistical properties.

Variance estimation for systematic sampling is a long-standing issue in statistics. A whole chapter of the recently reissued classic monograph by Wolter (2007) is devoted to this issue, and a number of possible estimation approaches are evaluated there. In particular, it considers a set of eight "model-free" estimators, some of which we will discuss further below, and outlines a model-based estimation approach. For the set of eight estimators, their statistical properties are evaluated for several model scenarios and through simulation experiments. None of these estimators is best overall, and there was a clear interaction between the behavior of the estimators and the underlying data model. Despite this implicit model dependence, the two estimators based on averages of pairwise differences (see next section) are found to be the best compromise between good performance and general applicability among this set of eight estimators. They are also widely used in practice.

In the model-based estimation approach described in Wolter (2007), the model dependence is explicitly recognized and a Rao-Blackwell type estimator is proposed, which minimizes the model mean squared error in estimating the sampling variance. The models considered in Wolter (2007) are parametric, and the Rao-Blackwell estimator therefore depends on unknown parameters that must be estimated from the sample data. A recent example of this approach is Bartolucci and Montanari (2006), who proposed an unbiased model-based variance estimator when the population follows a linear regression model.

In practice, despite its potential efficiency, wide applicability of the model-based method is viewed as being hampered by lack of robustness. Wolter (2007, p.305) noted that:

> "Since [the model] is never known exactly, the practicing statistician must make a professional judgment about the form of the model and then derive [the variance estimator] based on the chosen form. The 'practical' variance estimator [with estimated parameters] is then subject not only to errors of estimation [...] but also to errors of model misspecification."

However, this lack of robustness can be at least partly offset by the use of a nonparametric model specification. Compared to parametric models, this class of models makes much less restrictive assumptions on the shape of the relationship between variables, typically only requiring that the relationship be continuous and smooth, i.e. possessing a pre-specified number of derivatives. Hence, the risk of model misspecification is significantly reduced. This is particularly important in the survey context, because the same variance estimation method often needs to be applied to many survey variables collected in the same survey, and a single parametric model is much less likely to be correct for all these variables.

Bartolucci and Montanari (2006) discussed the use of nonparametric estimation as a way to "robustify" the model-based approach. They evaluate the bias properties of the resulting estimator under the linear population model, and then consider the behavior under nonlinear population models through simulation. The latter results suggest that the nonparametric approach remains effective in estimating the variance under a range of population model specifications.

In the current article, we will consider a broadly applicable model for the data, in which both the mean and the variance are left unspecified subject only to smoothness assumptions. We propose a model-based nonparametric variance estimator, in which both the mean and the variance functions of the data are estimated nonparametrically. The smoothing method we will use is local polynomial regression (see Wand and Jones (1995) for an overview). We will show that the proposed estimator is consistent for the design variance of the survey estimator, subject only to the population smoothness assumptions. The theoretical portion of the article will focus on the case of estimating the finite population mean using the sample mean for a systematic sample, but there is no inherent difficulty in extending the method to estimate the (approximate) variance of more complicated estimators such as model-assisted estimators.

The rest of the article is organized as follows. In Section 2, we describe the systematic sampling estimation context and the main variance estimators in use today. In Section 3, we introduce the nonparametric variance estimator and describe its statistical properties. Section 4 evaluates the practical properties of the estimator in a simulation study. Section 5 returns to the analysis for the northern Utah forestry data.

2 Systematic sampling and design-based variance estimation

We will be sampling from a finite population U of size is N. For now, we consider a single study variable $Y_j \in \mathbb{R}$, $j = 1, 2, \cdots, N$ with population mean

$$\bar{Y}_N = \frac{1}{N} \sum_{j=1}^{N} Y_j.$$

Let n denote the sample size and $k = N/n$ denote the *sampling interval*. For simplicity, we assume throughout this article that N is an integral multiple of n, i.e. k is an integer. The variable Y will only be observed on the sampled elements only.

Let $\mathbf{x}_j \in \mathbb{R}^p$ $(j = 1, 2, \cdots, N)$ be vectors of auxiliary variables available for all the elements in the population. To draw a systematic sample, the population is first sorted by some appropriate criterion. For example, we can sort by one or several of the auxiliary variables in $\mathbf{x}_j$. If the study variable Y and auxiliary variables $\mathbf{x}$ are related to each other, sorting by $\mathbf{x}$ and then drawing a systematic sample has been long known to reduce the variance of the sample mean. Conversely, if the population is sorted by a criterion that is not related to Y, for instance, by a variable in $\mathbf{x}$ which is independent of Y, then we will have a random permutation of the population. In this case, systematic sampling is equivalent to simple random sampling without replacement (SRS hereafter). After sorting the population, drawing a systematic sample is done by randomly choosing an element among the first k with equal probability, say the bth one, after which the systematic sample, denoted by S_b, consists of the observations with labels $\{b,\ b+k,\ \ldots,\ b+(n-1)k\}$. The random sample S can therefore only take on k values on the set of possible samples $\{S_1, \ldots, S_k\}$.

The sample mean,

$$\bar{Y}_S = \frac{1}{n} \sum_{j \in S} Y_j,$$

is the Horvitz-Thompson estimator for the finite population mean. Its design-based variance was first derived by Madow and Madow (1944) and is equal to

$$\mathrm{Var}_p(\bar{Y}_S) = \frac{1}{k} \sum_{b=1}^{k} (\bar{Y}_{S_b} - \bar{Y}_N)^2. \tag{1}$$

It should be clear that, if only a single systematic sample is drawn and hence only one of the $\bar{Y}_{S_b}$ is observed, no unbiased design-based estimator of $\mathrm{Var}_p(\bar{Y}_S)$ exists for general variable Y. A more formal way to state this is that the systematic sampling design is *not measurable* (Särndal et al. 1992, p.33).

We describe the three main methods used in practice to estimate $\mathrm{Var}_p(\bar{Y}_S)$, all of which are part of the eight estimators evaluated by Wolter (2007) and mentioned in Section 1. The simplest estimator is to treat the systematic sample as if it had been obtained by SRS. This estimator is defined as

$$\hat{V}_{SRS} = \frac{1-f}{n}\frac{1}{n-1}\sum_{j\in S}(Y_j - \bar{Y}_S)^2, \tag{2}$$

where $f = n/N$. The two remaining estimators are based on pairwise differences and are recommended in Wolter (2007) as being the best general-purpose estimators of $\mathrm{Var}_p(\bar{Y}_S)$). They are defined as

$$\hat{V}_{OL} = \frac{1-f}{n}\frac{1}{2(n-1)}\sum_{j=2}^{n}(Y_j - Y_{j-1})^2, \tag{3}$$

which uses all successive pairwise differences (and hence uses overlapping differences, OL), and

$$\hat{V}_{NO} = \frac{1-f}{n}\frac{1}{n}\sum_{j=1}^{n/2}(Y_{2j} - Y_{2j-1})^2. \tag{4}$$

which takes successive non-overlapping differences (NO). Additional estimators based on higher-order differences are described in Wolter (2007) but will not be further considered here.

All three estimators just described are design biased for $\mathrm{Var}_p(\bar{Y}_S)$ in general. The estimator $\hat{V}_{SRS}$ is viewed as suitable when the ordering of the population is thought to have no effect on $\bar{Y}_S$, or is considered as a conservative estimator when the ordering is related to the variable Y. However, as discussed in Opsomer et al. (2007), the unbiasedness of $\hat{V}_{SRS}$ for uninformative ordering only holds if one averages over samples *and* over orderings of the population (see Cochran 1977, Thm 8.5), so this is not, strictly speaking, design unbiasedness. The design bias of $\hat{V}_{SRS}$ for a fixed ordering of the population can be large and either positive or negative, so that relying on its conservativeness can be potentially misleading. This was clearly seen in the synthetic estimator approach used in Opsomer et al. (2007), for instance. The remaining two estimators tended to have smaller bias in the simulation experiments reported in Wolter (2007), but their statistical properties as estimators of $\mathrm{Var}_p(\bar{Y}_S)$ are not generally available.

3 Variance estimation under a nonparametric model

In the model-based context, the finite population is regarded as a random realization from a superpopulation model. A simple approach consists of assuming a parametric model for this superpopulation model. Under the assumption of linearity for the model, Bartolucci and Montanari (2006) proposed an unbiased estimator for the *anticipated variance* $\mathrm{E}[\mathrm{Var}_p(\bar{Y}_S)]$, denoted by $\hat{V}_L$, using a least squares estimator for the regression parameters and a model unbiased estimator for the variance of the errors. In this section, we propose a model consistent variance estimator under a nonparametric model. The nonparametric superpopulation model is

$$Y_j = m(x_j) + v(x_j)^{1/2}\, e_j \quad 1 \leq j \leq N, \tag{5}$$

where $m(\cdot)$ and $v(\cdot)$ are continuous and bounded functions. The errors e_j, $1 \leq j \leq N$, are independent random variables with model mean 0 and variance 1. Define $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_N)^T$, $\mathbf{m} = (m(x_1), \ldots, m(x_N))^T$ and $\mathbf{\Sigma} = \mathrm{diag}\{v(x_1), v(x_2), \cdots, v(x_N)\}$.

The design variance of $\bar{Y}_S$ can be written as

$$\mathrm{Var}_p(\bar{Y}_S) = \frac{1}{k}\sum_{b=1}^{k}(\bar{Y}_{S_b} - \bar{Y}_N)^2 = \frac{1}{kn^2}\mathbf{Y}^T\mathbf{D}\mathbf{Y}, \tag{6}$$

where $\mathbf{D} = \mathbf{M}^T\mathbf{H}\mathbf{M}$, with $\mathbf{M} = \mathbf{1}_n^T \otimes \mathbf{I}_k$ and $\mathbf{H} = \mathbf{I}_k - \frac{1}{k}\mathbf{1}_k\mathbf{1}_k^T$, with $\otimes$ denoting Kronecker product and $\mathbf{1}_r$ a vector of 1's of length r. Stated more explicitly, $\mathbf{H}$ is a $k \times k$ matrix with diagonal elements being $1 - \frac{1}{k}$ and off-diagonal element being $-\frac{1}{k}$, and $\mathbf{D}$ is a $N \times N$ matrix composed of $n \times n$ $\mathbf{H}$s. Then, the model anticipated variance of $\bar{Y}_S$ under model (5) is

$$\mathrm{E}[\mathrm{Var}_p(\bar{Y}_S)] = \frac{1}{kn^2}\mathbf{m}^T\mathbf{D}\mathbf{m} + \frac{1}{kn^2}\mathrm{tr}(\mathbf{D}\mathbf{\Sigma}). \tag{7}$$

To estimate $\mathrm{E}[\mathrm{Var}_p(\bar{Y}_S)]$, we propose the following estimator

$$\hat{V}_{NP} = \frac{1}{kn^2}(\hat{\mathbf{m}}^T\mathbf{D}\hat{\mathbf{m}}) + \frac{1}{kn^2}\mathrm{tr}(\mathbf{D}\hat{\mathbf{\Sigma}}), \tag{8}$$

where $\hat{\mathbf{m}} = (\hat{m}(x_1), \cdots, \hat{m}(x_N))^T$, with $\hat{m}(x_j)$ the local polynomial regression (LPR) estimator of $m(x_j)$ computed on the observations in the sample S and $\hat{\mathbf{\Sigma}} = \text{diag}\{\hat{v}(x_1), \hat{v}(x_2), \cdots, \hat{v}(x_N)\}$, with $\hat{v}(x_j)$ the LPR estimator of $v(x_j)$. We briefly describe the two LPR estimators, and for simplicity we will assume that the x_j are univariate and that the degree of the two local polynomials is equal to p. Both assumptions are readily relaxed, but lead to more complicated notation and derivations. Note that there is no restriction that the x_j should or should not be related to the sorting variable used to draw the systematic sample.

For the estimator of the mean function,

$$\hat{m}(x_j) = \mathbf{e}_1^T(\mathbf{X}_{Sj}^T\mathbf{W}_{Sj}\mathbf{X}_{Sj})^{-1}\mathbf{X}_{Sj}^T\mathbf{W}_{Sj}\mathbf{Y}_S,$$

with $\mathbf{e}_1$ a vector of length $(p+1)$ having 1 in the first entry and all other entries 0, $\mathbf{Y}_S$ a vector containing the $Y_j \in S$, $\mathbf{X}_{Sj}$ a matrix with ith row equal to $(1, (x_i - x_j), \ldots, (x_i - x_j)^p)$, $i \in S$, and

$$\mathbf{W}_{Sj} = \text{diag}\left\{K\left(\frac{x_i - x_j}{h_m}\right), \quad i \in S\right\},$$

where h_m is the bandwidth and K is a kernel function. For the estimator of the variance function, the expression is completely analogous, except that $\mathbf{Y}_S$ is replaced by the vector of squared residuals $\hat{\mathbf{r}}_S$ with elements $\hat{r}_j = (Y_j - \hat{m}(x_j))^2, j \in S$, and a different bandwidth h_v is used instead of h_m in the weight matrix $\mathbf{W}_{Sj}$. This variance estimator was previously used in Fan and Yao (1998) in a different context and does not include a "degrees of freedom" correction term as in Ruppert et al. (1997). While the latter estimator could certainly be used here, we found little difference between both in this setting, so that we chose the simpler estimator.

Under suitable regularity conditions on the population and the nonparametric estimator, which are stated in the Appendix, we obtain the following results on the asymptotic properties of $\hat{V}_{NP}$. An outline of the proof is given in the Appendix. The theorem shows that $\hat{V}_{NP}$ is a model consistent estimator for $\text{E}[\text{Var}_p(\bar{Y}_S)]$ and a model consistent predictor for $\text{Var}_p(\bar{Y}_S)$.

Theorem 3.1 *Assume that the degree p of the local polynomial is odd. Using superpopulation model* (5) *and under assumptions A.1–A.6 in the Appendix, the design variance is* *model consistent for the anticipated variance, in the sense that*

$$Var_p(\bar{Y}_S) - E[Var_p(\bar{Y}_S)] = O_p\left(\frac{1}{\sqrt{N}}\right), \tag{9}$$

and the nonparametric variance estimator is model consistent for the anticipated variance and for the design variance, in the sense that

$$\hat{V}_{NP} - E[Var_p(\bar{Y}_S)] = O_p(h_m^{p+1}) + O_p\left(\frac{1}{\sqrt{nh_m}}\right) \tag{10}$$

and

$$\hat{V}_{NP} - Var_p(\bar{Y}_S) = O_p(h_m^{p+1}) + O_p\left(\frac{1}{\sqrt{nh_m}}\right). \tag{11}$$

The best bandwidth h_m should satisfy the condition $h_m^{p+1} = O\left(\frac{1}{\sqrt{nh_m}}\right)$, which leads to $h_m = cn^{-1/(2p+3)}$, the usual optimal rate for local polynomial regression (see e.g. Fan and Gijbels 1996, p.67). Hence, it is expected that the usual bandwidth selection methods such as GCV or a plug-in method could be applied in this context as well. We do not explicitly address bandwidth selection in this article, however.

In these results, the effect of estimating the variance function is asymptotically negligible, because of assumption A.4 on the relationship between h_m and h_v. Without that assumption, model consistency of $\hat{V}_{NP}$ would continue to hold but a more complicated expression for the convergence rates would apply. Similarly, the restriction that p be odd simplifies the expressions for the rates but does not affect the overall consistency.

In Li (2006), a simpler nonparametric estimator is defined as

$$\hat{V}_{NP}^{ho} = \frac{1}{kn^2}(\hat{\mathbf{m}}_S^T \mathbf{D} \hat{\mathbf{m}}_S) + \frac{1}{kn^2}\mathrm{tr}(\mathbf{D})\hat{\sigma}_S^2 \tag{12}$$

with

$$\hat{\sigma}_S^2 = \frac{1}{n}\sum_{j\in S}(Y_j - \hat{m}(x_j))^2, \tag{13}$$

and its properties were studied under the special case of superpopulation model (5) with homoscedastic errors, i.e. when $v(x_j) \equiv \sigma^2, j = 1\ldots, N$. Under this model, Li (2006) obtained the same results for $\hat{V}_{NP}^{ho}$ as in Theorem 3.1. Because this estimator does not require the additional smoothing step on the residuals, it is easier to compute and, as we will discuss in the next two sections, it performs very similarly to the more complicated estimator $\hat{V}_{NP}$, even in the presence of heteroscedasticity.

4 Simulation Study

The practical behavior of the proposed nonparametric estimator is evaluated in a simulation study. The covariate x_j is uniformly distributed in the interval [0,1], and the errors e_j are generated as an independent and identically distributed (*iid*) sample from a standard normal distribution. Superpopulations of size $N = 2,000$ are generated according to model (5) with two different mean functions

$$\begin{aligned} \text{“linear”:} \quad m(x_j) &= 5 + 2x_j \\ \text{“quadratic”:} \quad m(x_j) &= 5 + 2x_j - 2x_j^2 \end{aligned}$$

and three different shapes for the variance functions

$$\begin{aligned} \text{“constant”:} \quad v(x_j) &= \beta \\ \text{“linear”:} \quad v(x_j) &= \beta x_j \\ \text{“quadratic”:} \quad v(x_j) &= \beta(1 - 4\,(x_j - 0.5)^2). \end{aligned}$$

The values of β for the three variance functions are selected to achieve two levels for the population coefficient of determination (R^2), equal to $R^2 = 0.75$ (the “precise model”) and $R^2 = 0.25$ (the “diffuse model”).

Several of the estimators are sensitive to the relationship between the modeling covariate and the sorting variable used in generating the systematic samples. We therefore investigate three sorting scenarios, based on the strength of the association between the sorting variable z_j and the x_j. We construct the z_j as $z_j = x_j + \sigma_z \eta_j$ with the η_j *iid* standard normal, and we select the value of σ_z to achieve $R^2 = 1$ (i.e. sorting by x), $R^2 = 0.75$ (“z strongly associated with x”) and $R^2 = 0.25$ (“z weakly associated with x”). We consider samples of sizes $n = 500$ and $n = 100$.

For each simulation scenario, we compare the performance of the estimators $\hat{V}_{NP}$, $\hat{V}_{NP}^{ho}$, $\hat{V}_L$, $\hat{V}_{OL}$, $\hat{V}_{NO}$ and $\hat{V}_{SRS}$ described in Sections 2 and 3. Additionally, we also compute an estimator similar to $\hat{V}_L$, but assuming a linear model for the variance of the errors and using the least squares estimator to estimate this function from the parametric residuals, this estimator is denoted by $\hat{V}_L^{lin}$. This will allow us to evaluate the performance of the nonparametric approach relative to the parametric approach and the commonly used design-based estimators. For the nonparametric estimators, we used local linear ($p = 1$) regression and the Epanechnikov kernel equal to $K(t) = (1-t^2)$ if $|t| \leq 1$ and 0 otherwise. We consider three different values for the bandwidths, $h_m = 0.10, 0.25, 0.50$ and the same values for h_v in the case of $\hat{V}_{NP}$. The stratification-based estimators $\hat{V}_{OL}$, $\hat{V}_{NO}$ construct pairs of observations based on the sorting variable z_j.

It is important to note that the anticipated variance in (7) contains two components, with the first related to the difference in the sample means and the second to the model variance. It can readily be shown that both components are of the same order of magnitude in general. In addition, while the first component can in principle be decreased by choosing an appropriate sorting variable (this is a major theme in the systematic sampling literature), the second component is independent of the sorting. Because the proposed model-based variance estimator targets the anticipated variance, it would therefore appear critical to capture the model variance correctly in order to obtain a good variance estimator. This was the main reason for the nonparametric specification of the function $v(\cdot)$.

In summary, we study 72 scenarios (twelve superpopulation models, three sorting criteria and two sample sizes) for $\hat{V}_L$, $\hat{V}_{OL}$, $\hat{V}_{NO}$ and $\hat{V}_{SRS}$, 216 scenarios (three bandwidth values in addition) for $\hat{V}_{NP}^{ho}$ and 648 scenarios (three additional bandwidths to estimate the variance function) for $\hat{V}_{NP}$.

In each simulation run, we keep the population x_j and the z_j fixed but generate new population errors e_j, and draw a systematic sample according to the sorted z values (corresponding to the model-based setting we are considering in this article). Each simulation setting is repeated $B = 10000$ times and the results are obtained by averaging over the B replicates. We consider both $\mathrm{E}(\mathrm{Var}_p(\bar{Y}_S))$ and $\mathrm{Var}_p(\bar{Y}_S)$ as targets for the estimators, with $\mathrm{Var}_p(\bar{Y}_S)$ computed exactly for each replicate. Letting $\hat{V}$ denote one of the estimators above, we calculate the relative bias (RB) and mean squared error (MSE) and mean squared prediction error (MSPE), where

$$\begin{aligned}
\mathrm{RB} &= \frac{\mathrm{E}^*(\hat{V}) - \mathrm{E}^*[\mathrm{Var}_p(\bar{Y}_S)]}{\mathrm{E}^*[\mathrm{Var}_p(\bar{Y}_S)]}, \\
\mathrm{MSE} &= \mathrm{E}^*(\hat{V} - \mathrm{E}^*[\mathrm{Var}_p(\bar{Y}_S)])^2, \\
\mathrm{MSPE} &= \mathrm{E}^*(\hat{V} - \mathrm{Var}_p(\bar{Y}_S))^2,
\end{aligned}$$

with E^* indicating which expectations are obtained by averaging across the replicates.

Tables 1 and 2 report the relative bias (in percent) of $\hat{V}_L$, $\hat{V}_L^{lin}$, $\hat{V}_{NP}^{ho}$, $\hat{V}_{NP}$, $\hat{V}_{OL}$, $\hat{V}_{NO}$ and $\hat{V}_{SRS}$, when populations are sorted by the covariate x, for the sample size $n = 500$ and the precise ($R^2 = 0.75$) regression model, and $n = 500$ and the diffuse ($R^2 = 0.25$) regression model, respectively. Tables 3 and 4 show the same information but with $n = 100$.

[Table 1 about here.]

[Table 2 about here.]

[Table 3 about here.]

[Table 4 about here.]

Similar conclusions can be deduced from these four tables. Linear estimators $\hat{V}_L^{ho}$ and $\hat{V}_L^{lin}$ perform similarly very well when the superpopulation model is linear, they are almost unbiased. However, when the superpopulation model is not linear, their biases increases dramatically due to the misspecification. The nonparametric estimators $\hat{V}_{NP}^{ho}$ and $\hat{V}_{NP}$ perform well under all superpopulation models if proper bandwidths h_m and h_v values are chosen. Specifically, when the superpopulation model is linear, $\hat{V}_{NP}^{ho}$ or $\hat{V}_{NP}$ tend to favour bigger bandwidth h_m, due to the use of the local linear regression in the calculation of $\hat{V}_{NP}^{ho}$ and $\hat{V}_{NP}$. When the superpopulation is quadratic, they tend to favour smaller bandwidth of h_m. For example, the bias for $\hat{V}_{NP}^{ho}$ for $h_m = 0.5$, $R_2^2 = 0.75$ and $n = 500$ is more than 54%. It appears that some care is needed in selecting the bandwidths. In this paper we will not further discuss the bandwidth selection problem. We choose these three bandwidth values for h_m and h_v to illustrate the bandwidth effect.

The results also show that the variance function specification generally has only a modest effect on the bias of the estimators. An interesting result is that the estimator $\hat{V}_{NP}^{ho}$, which uses the mean squared residuals, appears to perform better than many of the more complicated $\hat{V}_{NP}$ even when the errors were heteroscedastic. We conjecture that this is due to the fact that under heteroscedasticity, the variance component of the anticipated variance is of the form $c\sum_{j=1}^{N} v(x_j)/N$, and the mean of the squared residuals is a very good estimator for the mean of the variance over the population for approximately balanced samples, such as those obtained by systematic sampling.

Estimators $\hat{\mathrm{V}}_{OL}$ and $\hat{\mathrm{V}}_{NO}$ have small biases under the models presented in those Tables. This is because when the populations are sorted by x before drawing systematic samples, $\hat{\mathrm{V}}_{OL}$ and $\hat{\mathrm{V}}_{NO}$ can capture the population trend very well and thus very efficient. The most inefficient variance estimator in this case is $\hat{\mathrm{V}}_{SRS}$. It always overestimates the true variance.

Tables 5 and 6 report the relative bias (in percent) of $\hat{V}_L$, $\hat{V}_L^{lin}$, $\hat{V}_{NP}^{ho}$, $\hat{V}_{NP}$, $\hat{V}_{OL}$, $\hat{V}_{NO}$ and $\hat{V}_{SRS}$, when populations are sorted by a variable z strongly associated with x (case $R^2 = 0.75$), for the sample size $n = 500$ and the precise ($R^2 = 0.75$) regression model, and $n = 500$ and the diffuse ($R^2 = 0.25$) regression model, respectively. Tables 7 and 8 show the same information but with $n = 100$.

[Table 5 about here.]

[Table 6 about here.]

[Table 7 about here.]

[Table 8 about here.]

For $\hat{\mathrm{V}}_L^{ho}$, $\hat{\mathrm{V}}_L^{lin}$, $\hat{\mathrm{V}}_{NP}^{ho}$ and $\hat{\mathrm{V}}_{NP}$, we can draw similar conclusions to those in Tables 1, 2, 3 and 4. The linear estimators $\hat{\mathrm{V}}_L^{ho}$ and $\hat{\mathrm{V}}_L^{lin}$ do well when the superpopulation model is linear, but have large biases when the superpopulation model is quadratic. However, under the quadratic superpopulation model, their relative biases are smaller than those in Tables 1, 2, 3 and 4. This is because when populations are sorted by a variable strongly associated with covariate x, the systematic sample is closer to a simple random sample and does not capture the quadratic trend as well as the case where populations are sorted by x. So misspecification has less effect because the sample is less quadratic. The nonparametric estimators $\hat{\mathrm{V}}_{NP}^{ho}$ and $\hat{\mathrm{V}}_{NP}$ again perform well if we choose proper bandwidths. The most important difference with respect to the case of populations sorted by x is that $\hat{\mathrm{V}}_{OL}$ and $\hat{\mathrm{V}}_{NO}$ have larger bias values than those in Tables 1, 2, 3 and 4. When the populations is sorted by a variable strongly associated with x, the overlapping and nonoverlapping difference estimators $\hat{\mathrm{V}}_{OL}$ and $\hat{\mathrm{V}}_{NO}$ cannot capture the population trend as well as when the population is sorted by x. In contrast, the nonparametric estimator is able to use the correct modeling variable x_j in all cases and is able to capture any unknown but smooth trend.

Tables 9 and 10 report the relative bias (in percent) of $\hat{V}_L$, $\hat{V}_L^{lin}$, $\hat{V}_{NP}^{ho}$, $\hat{V}_{NP}$, $\hat{V}_{OL}$, $\hat{V}_{NO}$ and $\hat{V}_{SRS}$, when populations are sorted by a variable weakly associated with covariate x (case $R^2 = 0.25$), for the sample size $n = 500$ and the precise ($R^2 = 0.75$) regression model, and $n = 500$ and the diffuse ($R^2 = 0.25$) regression model, respectively. Tables 11 and 12 show the same information but with $n = 100$.

[Table 9 about here.]

[Table 10 about here.]

[Table 11 about here.]

[Table 12 about here.]

We can see a similar trend as in previous Tables. When the superpopulation model is linear, the linear estimators $\hat{\mathrm{V}}_L^{ho}$ and $\hat{\mathrm{V}}_L^{lin}$ perform well. When the superpopulation model deviates from linearity, they become more biased. Given a proper bandwidth h_m, the nonparametric estimator $\hat{\mathrm{V}}_{NP}^{ho}$ has small bias for all superpopulation models and both sample sizes. We can also see that the overlapping difference estimator $\hat{\mathrm{V}}_{OL}$ and nonoverlapping difference estimator $\hat{\mathrm{V}}_{NO}$ tend to have more biases than those in Tables 5, 6, 7 and 8. Note that this is the case where the populations are sorted by a variable weakly associated with x, which results in almost random permutations of populations. So, the systematic samples are close to SRS samples. Thus, it is not surprising to see that $\hat{\mathrm{V}}_{OL}$ and $\hat{\mathrm{V}}_{NO}$ have similar bias to $\hat{\mathrm{V}}_{SRS}$, especially when the superpopulation model is quadratic. The performance of the stratification-based estimators decreases substantially when the relationship between the sorting variable and the model covariate becomes weaker, resulting in substantial bias and large MSE values (as we will see later). The interpretation of this result is that the stratification-based variance estimators work well only when the "implicit model" under which it is constructed is correct. This implicit model assumes that the relationship between z_j and y_j is well approximated by a piecewise constant function. This is true when $R^2 = 1$, but not in the remaining cases. On the other hand, the nonparametric estimator have the advantage of using auxiliary information to improve its precision under this circumstance.

Tables 13 and 14 report the MSE of $\hat{V}_L$, $\hat{V}_L^{lin}$, $\hat{V}_{NP}^{ho}$, $\hat{V}_{NP}$, $\hat{V}_{OL}$, $\hat{V}_{NO}$ and $\hat{V}_{SRS}$ when populations are sorted by variable x, for the sample size $n = 500$ and the precise ($R^2 = 0.75$) regression model, and $n = 500$ and the diffuse ($R^2 = 0.25$) regression model, respectively. Tables 15 and 16 show the same information but with $n = 100$. The MSE results for the different estimators are normalized by dividing by the MSE of $\hat{V}_{NP}^{ho}$ with bandwdith $h_m = 0.10$, to facilitate comparison.

[Table 13 about here.]

[Table 14 about here.]

[Table 15 about here.]

[Table 16 about here.]

We can see that when the superpopulation model is linear, the linear estimators $\hat{\mathrm{V}}_L^{ho}$ and $\hat{\mathrm{V}}_L^{lin}$ and the nonparametric estimators $\hat{\mathrm{V}}_{NP}^{ho}$ and $\hat{\mathrm{V}}_{NP}$ with larger bandwidth h_m values tend to be favourable choices (except $\hat{\mathrm{V}}_{NP} : h_m = 0.5, h_v = 0.5$). When the superpopulation model is quadratic, the nonparametric estimators $\hat{\mathrm{V}}_{NP}^{ho}$ and $\hat{\mathrm{V}}_{NP}$ perform the best if given a proper bandwidth h_m. This can be seen by noting that in the last three columns of these Tables, only a few values are less than one, and they are the ratios of the nonparametric estimators at different bandwidth h_m values. In all cases, although never the best, $\hat{\mathrm{V}}_{OL}$ and $\hat{\mathrm{V}}_{NO}$ do not perform too badly either. They are close to the best choice in each case. The linear estimators $\hat{\mathrm{V}}_L^{ho}$ and $\hat{\mathrm{V}}_L^{lin}$ drastically fails when the superpopulation model is quadratic. The SRS variance estimator $\hat{\mathrm{V}}_{SRS}$ is almost always a bad choice.

Tables 17 and 18 report the MSE of $\hat{V}_L$, $\hat{V}_L^{lin}$, $\hat{V}_{NP}^{ho}$, $\hat{V}_{NP}$, $\hat{V}_{OL}$, $\hat{V}_{NO}$ and $\hat{V}_{SRS}$ when populations are sorted by a variable strongly associated with covariate x, for the sample size $n = 500$ and the precise ($R^2 = 0.75$) regression model, and $n = 500$ and the diffuse ($R^2 = 0.25$) regression model, respectively. Tables 19 and 20 show the same information but with $n = 100$. Tables 21, 22, 23 and 24 present the results in analogous settings as in

Tables 17, 18, 19 and 20, but when populations are sorted by a variable weakly associated with covariate x. Similarly as previously, the different estimators are normalized by dividing by the MSE of $\hat{V}_{NP}^{ho}$ with bandwdith $h_m = 0.10$.

[Table 17 about here.]

[Table 18 about here.]

[Table 19 about here.]

[Table 20 about here.]

[Table 21 about here.]

[Table 22 about here.]

[Table 23 about here.]

[Table 24 about here.]

We see a similar trend as when sorting by x, although differently, now, the sorting variable deviates further from the model variable x, and the overlapping difference estimator $\hat{\text{V}}_{OL}$ and nonoverlapping difference estimator $\hat{\text{V}}_{NO}$ tend to be worse.

Finally, Tables 25 and 26 present the MSPE of $\hat{V}_L$, $\hat{V}_L^{lin}$, $\hat{V}_{NP}^{ho}$, $\hat{V}_{NP}$, $\hat{V}_{OL}$, $\hat{V}_{NO}$ and $\hat{V}_{SRS}$ when populations are sorted by the variable x, for the sample size $n = 500$ and the precise ($R^2 = 0.75$) regression model, and $n = 500$ and the diffuse ($R^2 = 0.25$) regression model, respectively. Tables 27 and 28 show the same information but with $n = 100$. Tables 29, 30, 31 and 32 present the results in analogous settings as in Tables 25, 26, 27 and 28, but when populations are sorted by a variable strongly associated with x (case $R^2 = 0.75$), while Tables 33, 34, 35 and 36 present the results when populations are sorted by a variable weakly associated with x (case $R^2 = 0.25$). Similarly as in the case of the MSE, the different estimators are normalized by dividing by the MSPE of $\hat{V}_{NP}^{ho}$ with bandwdith $h_m = 0.10$.

[Table 25 about here.]

[Table 26 about here.]

[Table 27 about here.]

[Table 28 about here.]

[Table 29 about here.]

[Table 30 about here.]

[Table 31 about here.]

[Table 32 about here.]

[Table 33 about here.]

[Table 34 about here.]

[Table 35 about here.]

[Table 36 about here.]

Similarly, when MSPE is computed instead of MSE (and hence $\mathrm{Var}_p(\bar{Y}_S)$ is targeted instead of the anticipated variance), the conclusions just stated continue to hold. The main difference is that because of the randomness of $\mathrm{Var}_p(\bar{Y}_S)$, the mean squared differences between the estimators and the target are larger than the differences between the estimators and $\mathrm{E}(\mathrm{Var}_p(\bar{Y}_S))$, and hence the normalized MSPEs are all closer to 1.

5 Application in Forest Inventory and Analysis

We now return to the FIA data collected in Northern Utah. As illustrated in Figure 1, the forest survey data are collected using a two-phase systematic sampling design. In phase one, remote sensing data and geographical information system (GIS) coverage information are gathered on an intensive sample grid. In phase two, a field-visited subset of the phase one grid is taken. Several hundred variables are collected during these field visits, ranging from individual tree characteristics and size measurements to complex ecological health ratings. There are 24,980 phase one sample points and 968 phase two sample points. It should be noted that the phase one data are available at essentially any desirable resolution, so this grid of points is somewhat arbitrary and can be used as an approximation for the underlying continuous population. We therefore treat the phase one plots as the population of interest and phase two plots as a systematic sample drawn from that population. At the "population" level, we have auxiliary information such as location (**LOC**, bivariate scaled longitude and latitude) and elevation (ELEV). At the sample level, information is available for the field-collected forestry variables in addition to the population-level variables.

We consider here the following representative forestry variables, which are a subset of all the variables collected in the survey:

- BIOMASS - total wood biomass per acre in tons
- CRCOV - percent crown cover
- BA - tree basal area per acre
- NVOLTOT - total cuft volume per acre
- FOREST - forest/nonforest indicator.

We are interested in estimating the population mean for these variables using the systematic sample mean $\bar{Y}_S$, and estimating its design-based variance $\mathrm{Var}_p(\bar{Y}_S)$. We will consider two traditional design-based variance estimators, $\hat{V}_{SRS}$ as in (2) and $\hat{V}_{ST}$ (see below), and the model-based nonparametric variance estimator $\hat{V}_{NP}$. The stratified sampling variance estimator $\hat{V}_{ST}$ is similar to the nonoverlapping differences estimator $\hat{V}_{NO}$ in (4), generalized to a spatial setting by considering an approximate 4-per-stratum design:

$$\hat{V}_{ST} = \frac{1-f}{n}\frac{1}{n}\sum_{h=1}^{H}\frac{n_h}{n_h-1}\sum_{j\in S_h}(Y_j - \bar{Y}_{S_h})^2,$$

where S_h denotes the sample in cell h and n_h the corresponding cell sample size. Note that for points near the edge of the map, there may be more or less than four points per cell, because we collapsed all cells that contained less than two point with their closest neighbor. Figure 2 displays this 4-per-stratum design prior to cell collapsing.

[Figure 2 about here.]

For the purpose of constructing the model-based nonparametric variance estimator $\hat{V}_{NP}$, we consider the following model with location (**LOC**) as bivariate auxiliary variables:

$$Y_j = m(\mathbf{LOC}_j) + \varepsilon_j. \tag{14}$$

We considered both homoscedastic and heteroscedastic versions of this model. Firstly, we assumed that the errors in (14) are independent with homogeneous variance. Under this assumption, we implemented the nonparametric variance estimator $\hat{V}_{NP}$ given in (12) and (13) with x_j replaced by $\mathbf{LOC}_j$. Here $m(\cdot)$ is estimated by bivariate local linear regression, and the estimator $\hat{m}(\cdot)$ is obtained using loess() in R. In loess(), the bandwidth parameter h is replaced by the *span*, the fraction of the sample observations that have non-zero weight in the computation of $\hat{m}(\mathbf{LOC}_j)$. Since the samples points are approximated equally spaced (5×5 km grid), using loess() will produce similar results to those obtained using a fixed bandwidth in the interior of the estimation region. At the boundaries of the region, it will tend to select larger bandwidths and hence reduce some of the increased variability often experienced close to boundaries in fixed-bandwidth smoothing. This results in improved overall stability of the fits. In order to evaluate the sensitivity of the results to the choice of the smoothing parameters, we choose three spans: 0.1, 0.2 and 0.5. After obtaining $\hat{m}(\cdot)$, we can calculate the nonparametric variance estimator $\hat{V}_{NP}$ for each response variable. Table 37 presents the sample means and the estimated variances using $\hat{V}_{SRS}$, $\hat{V}_{ST}$ and $\hat{V}_{NP}$.

[Table 37 about here.]

While we do not know the true variance, the estimator $\hat{V}_{ST}$ is likely to be a reasonable approximation as long as the Y_j can be modeled as a spatial trend plus random errors. The naive estimator $\hat{V}_{SRS}$ produces the largest values among the five variance estimators for all response variables and so is likely to be biased upwards for this survey. In contrast, the nonparametric variance estimator $\hat{V}_{NP}$ results in estimates that are close to those of $\hat{V}_{ST}$, with smaller spans leading to slightly smaller estimates.

As already discussed in Section 4, an important advantage of the model-based nonparametric method is that one is not restricted to using only the sampling variables (**LOC** in this case) in the construction of the estimator, if other variables are thought to be good predictors of the survey variables. We illustrate this here by considering more sophisticated models that also includes elevation (ELEV) in additive to **LOC**:

$$Y_j = m_1(\mathbf{LOC}_j) + m_2(ELEV_j) + \varepsilon_j. \tag{15}$$

Assuming that the errors in (15) are independent with homogeneous variance, we fit model (15) in R using the Generalized Additive Models (gam) package. We use the same span for both **LOC** and ELEV, as well as span = 0.1 for location and span = 0.3 for elevation.

[Table 38 about here.]

Table 38 shows that, relative to the simpler model without elevation, the estimated variances all decreased, by 8-14%. This decrease is due primarily to a reduction in the $\hat{\sigma}_S^2$ component in (12), which accounts for the fact that the extended mean model in (15) captures more of the observed behavior of these forestry variables. This can be observed by comparing Tables 37 and 38 with Tables 39 and 40, which show the $\hat{\sigma}_S^2$ component in (12) under homoscedastic models (14) and (15), respectively.

[Table 39 about here.]

[Table 40 about here.]

To complete the study with the homoscedastic models, we computed the model assisted mean and variance estimates for five response under models (14) and (15). The results are shown in Tables 41 and 42, respectively.

[Table 41 about here.]

[Table 42 about here.]

The next step in our study was considering heteroscedastic versions of models (14) and (15). First, we considered model (14), but assuming that the errors have a variance function, $v(\cdot)$, depending on the variable **LOC**, that is,

$$Y_j = m(\mathbf{LOC}_j) + \varepsilon_j = m(\mathbf{LOC}_j) + v^{1/2}(\mathbf{LOC}_j)\, e_j, \tag{16}$$

where $e_j's$ are *iid* random variables with model mean 0 and variance 1.

Under this heteroscedastic model, we computed the nonparametric estimator (8). For this, estimators of $m(\cdot)$ and $v(\cdot)$ using loess() with different spans were used. In particular, we choose three spans to estimate the mean function, $h_m = 0.1, 0.2$ and 0.5, and three spans, $h_v = 0.2, 0.4$ and 0.6, to estimate the variance function. It is important to note that the nonparametric estimator of $v(\cdot)$ could give some negative values. In those cases, we assigned value 0 to the estimator of $v(\cdot)$. The results obtained are shown in Table 43. As expected from the simulations study in the previous Section, no important differences with respect to the simpler homoscedastic case are observed.

[Table 43 about here.]

Finally, we considered heteroscedastic versions of the regression model (15) that also includes elevation (ELEV) in additive to **LOC**. We studied a model with a variance function, $v(\cdot)$, depending on the variables **LOC** and ELEV,

$$Y_j = m_1(\mathbf{LOC}_j) + m_2(ELEV_j) + v^{1/2}(\mathbf{LOC}_j, ELEV_j)\, e_j, \tag{17}$$

where $e_j's$ are *iid* random variables with model mean 0 and variance 1.

There are several possibilities to model the variance function $v(\mathbf{LOC}_j, ELEV_j)$. After some tests, we selected the following two alternatives:

$$v(\mathbf{LOC}_j, ELEV_j) = v_1(\mathbf{LOC}_j) + v_2(ELEV_j). \tag{18}$$

$$v(\mathbf{LOC}_j, ELEV_j) = v_3(ELEV_j) \tag{19}$$

To estimate nonparametrically these variance functions, we used loess() and gam package (in the case of (18)) with different spans. We used the same spans as before for the regression function (h_m), and spans $h_v = 0.2, 0.4$ and 0.6 for the variance functions estimators. As it was stated before, if the estimation of $v(\cdot)$ is negative (which happens very rarely), we set value 0 for those estimations. The results obtained for model (18) are given in Table 44 and for model (19) in Table 45.

[Table 44 about here.]

[Table 45 about here.]

Comparing Tables 44 and 45, it can be observed that the results are almost identical and they are practically the same as those corresponding to the model with homoscedastic errors (Table 38). We can also see that the degree of smoothing in the estimator of the variance function does not play an important role in the final results.

In general, we observed that the nonparametric variance estimator $\hat{\mathrm{V}}_{NP}$ produced very good estimates for the variance in this FIA example. The results were close to $\hat{\mathrm{V}}_{ST}$, which we believe to be a good estimator because the stratification should capture the spatial trend very well. Both $\hat{\mathrm{V}}_{NP}$ and $\hat{\mathrm{V}}_{ST}$ were better than $\hat{\mathrm{V}}_{SRS}$. An advantage of using $\hat{\mathrm{V}}_{NP}$ is the flexibility. We can include more auxiliary variables, change the bandwidth (span) of nonparametric fitting or model the variance of the errors, and further improve the results. Moreover, the homoscedastic version of the nonparametric estimator appeared to behave at least as well as the more complicated estimator that captures heteroscedasticity.

Acknowledgments

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A Assumptions

A.1 *The x_j's are treated as fixed with respect to superpopulation model (5). The x_j's are independent and identically distributed with $F(x) = \int_{-\infty}^{x} f(t)dt$, where $f(\cdot)$ is a density function with compact support $[a_x, b_x]$ and $f(x) > 0$ for all $x \in [a_x, b_x]$. The first derivative of f exists for all $x \in [a_x, b_x]$.*

A.2 *The third and fourth moments of e_j exist and are bounded.*

A.3 *The sample size n and sampling interval k are positive integers with $nk{=}N$. We assume that $n, N \to \infty$ and allow $k = O(1)$ or $k \to \infty$.*

A.4 *As $n \to \infty$, we assume $h^* \to 0$ and $nh^* \to \infty$, where h^* is h_m or h_v. Additionally, $\left\{h_m^{2(p+1)} + (nh_m)^{-1}\right\} = o\left(h_v^{p+1}\right)$, $h_v^{p+1} = o(nh_m^{p+1})$ and $\frac{1}{n^2 h_v^{1/2}} = o\left(\frac{1}{n^{1/2} h_m^{1/2}}\right)$.*

A.5 *The kernel function $K(\cdot)$ is a compactly supported, bounded, symmetric kernel with $\int u^{q+1} K(u)du = \mu_{q+1}(K)$. Assume that $\mu_{p+1}(K) \neq 0$.*

A.6 *The $(p+1)$th derivatives of the mean function $m(\cdot)$ and the variance function $v(\cdot)$ exist and are bounded on $[a_x, b_x]$.*

B Outline of Proof of Theorem 3.1

Statement (9) is obtained by showing that $\text{Var}[\text{Var}_p(\bar{Y}_S)] = O(1/N)$. This is done by computing the variance of the quadratic form in (6) under model (5), and then bounding each of the terms using assumptions A.1-A.3 and A.6. See Theorem 1.1 in Li (2006) for details.

In order to prove (10), we write

$$\begin{aligned} \hat{V}_{NP} - \text{E}[\text{Var}_p(\bar{Y}_S)] &= \frac{1}{Nn}(\hat{\mathbf{m}}^T\mathbf{D}\hat{\mathbf{m}} - \mathbf{m}^T\mathbf{D}\mathbf{m}) + \left(1 - \frac{n}{N}\right)\frac{1}{Nn}\sum_{j \in U}(\hat{v}(x_j) - v(x_j)) \\ &= A + B. \end{aligned} \quad (20)$$

The term A in (20) can be broken down into components that are functions of $\frac{1}{n}\sum_{j\in S_b}(\hat{m}(x_j)-m(x_j))^l$ for $b=1,\ldots,k$ and $\frac{1}{N}\sum_{j\in U}(\hat{m}(x_j)-m(x_j))^l$ with $l=1,2$. Using the same approach as in the proof of Theorem 4.1 in Ruppert and Wand (1994) except that we are treating the x_j as fixed, and applying assumptions A.1-A.6, we approximate the required moments of these quantities. Bounding arguments for the expectation and variance of each of the components of A show that $A=O_p(h_m^{p+1}+(nh_m)^{-1/2})$. Theorem 1.2 in Li (2006) provides a complete description.

For the term B in (20), the squared residuals are decomposed into $\hat{r}_j = v(x_j)e_j^2 + (\hat{m}(x_j)-m(x_j))^2 - 2\sqrt{v(x_j)}e_j(\hat{m}(x_j)-m(x_j)) = r_j + b_{1j} + b_{2j}$, with corresponding sample vectors $\hat{\mathbf{r}}_S = \mathbf{r}_S + \mathbf{b}_{1S} + \mathbf{b}_{2S}$. Hence, $\mathbf{r}_S$ contains the true model errors for the sample observations and does not depend on the first nonparametric regression. Letting $\tilde{v}$ denote the local polynomial regression fit using $\mathbf{r}_S$ instead of $\hat{\mathbf{r}}_S$, straightforward moment approximations and bounding arguments show that

$$\left(1-\frac{n}{N}\right)\frac{1}{Nn}\sum_{j\in U}(\tilde{v}(x_j)-v(x_j)) = O_p\left(\frac{h_v^{p+1}}{n}+\frac{1}{n^2h_v^{1/2}}\right) = o_p\left(h_m^{p+1}+\frac{1}{\sqrt{nh_m}}\right)$$

by assumption A.4. Using the fact that $\mathrm{E}(b_{1j}) = O(h_m^{2p+2}+(nh_m)^{-1})$ and A.4 again, we can show that the local polynomial regression for $\mathbf{b}_{1S}$ is $o_p(h_v^{p+1})$. Similarly, the local polynomial regression for $\mathbf{b}_{2S}$ leads to terms that are of the same or smaller order. Hence, we conclude that $B = o_p\left(h_m^{p+1}+\frac{1}{\sqrt{nh_m}}\right)$.

Finally, statement (11) follows directly from (9) and (10).

List of Figures

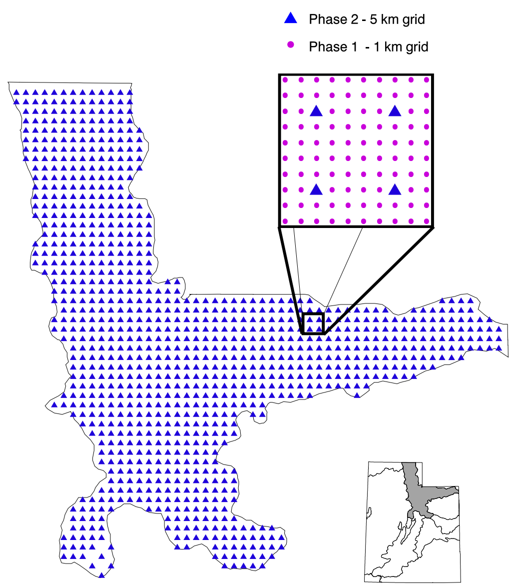

Figure 1: Map of the study region in northern Utah. Each triangle represents a field-visited phase two sample point. Each dot in the magnified section represents a phase one sample point.

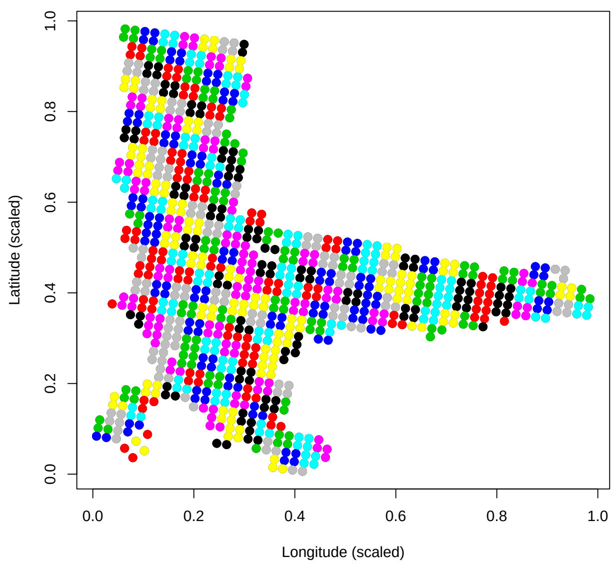

Figure 2: Four-per-stratum partition of sample points for computing stratified variance estimator for systematic sampling.

Mean function	Linear			Quadr.		
Variance function	Const.	Linear	Quadr.	Const.	Linear	Quadr.
$\hat{V}_L$	-0.3127	-0.3511	-0.2291	297.74	300.83	296.88
$\hat{V}_L^{lin}$	-0.3128	-0.3508	-0.2287	297.74	300.83	296.88
$V_{NP}^{ho} : h_m = 0.10$	-1.9668	-2.0256	-1.7116	-1.7942	-1.8517	-1.5383
$V_{NP}^{ho} : h_m = 0.25$	-0.9096	-0.9593	-0.6956	4.9906	5.0035	5.1873
$V_{NP}^{ho} : h_m = 0.50$	-0.5563	0.5986	-0.3943	54.257	54.790	54.248
$V_{NP} : h_m = 0.10, h_v = 0.10$	-2.0708	-2.1382	-2.7455	-1.9034	-1.9694	-2.5848
$V_{NP} : h_m = 0.10, h_v = 0.25$	-2.0751	-2.1526	-6.9501	-1.9134	-1.9897	-6.8289
$V_{NP} : h_m = 0.10, h_v = 0.50$	-2.1174	-2.2070	-14.536	-1.9576	-2.0460	-14.478
$V_{NP} : h_m = 0.25, h_v = 0.10$	-0.9640	-1.0195	-1.7267	4.9498	4.9578	4.1621
$V_{NP} : h_m = 0.25, h_v = 0.25$	-0.9955	-1.0603	-5.9868	4.4660	4.4597	-0.5829
$V_{NP} : h_m = 0.25, h_v = 0.50$	-1.0479	-1.1235	-13.661	3.9995	3.9783	-8.7312
$V_{NP} : h_m = 0.50, h_v = 0.10$	-0.5792	-0.6243	-1.4250	54.966	55.507	53.939
$V_{NP} : h_m = 0.50, h_v = 0.25$	-0.6070	-0.6614	-5.6844	53.188	53.700	47.901
$V_{NP} : h_m = 0.50, h_v = 0.50$	-0.6734	-0.7384	-13.393	41.711	42.094	28.754
$\hat{V}_{OL}$	-0.6699	-0.7169	-0.4740	0.1673	0.1287	0.3621
$\hat{V}_{NO}$	-0.7085	-0.7540	-0.6797	0.1274	0.0895	0.1554
$\hat{V}_{SRS}$	328.30	331.63	327.32	298.93	302.03	297.99

Table 1: Simulated relative bias (in percent) with sorting variable $R^2 = 1$, $n = 500$ and regression model $R^2 = 0.75$.

Mean function	Linear			Quadr.		
Variance function	Const.	Linear	Quadr.	Const.	Linear	Quadr.
$\hat{V}_L$	-0.3148	-0.3535	-0.2306	29.510	29.787	29.497
$\hat{V}_L^{lin}$	-0.3149	-0.3531	-0.2302	29.509	29.787	29.497
$V_{NP}^{ho} : h_m = 0.10$	-1.9804	-2.0400	-1.7239	-1.9630	-2.0224	-1.7066
$V_{NP}^{ho} : h_m = 0.25$	-0.9160	-0.9662	-0.7006	-0.3259	-0.3696	-0.1124
$V_{NP}^{ho} : h_m = 0.50$	-0.5604	-0.6029	-0.3973	4.9252	4.9408	5.0708
$V_{NP} : h_m = 0.10, h_v = 0.10$	-2.0852	-2.1534	-2.7653	-2.0683	-2.1363	-2.7492
$V_{NP} : h_m = 0.10, h_v = 0.25$	-2.0895	-2.1679	-7.0005	-2.0732	-2.1515	-6.9884
$V_{NP} : h_m = 0.10, h_v = 0.50$	-2.1322	-2.2227	-14.642	-2.1160	-2.2064	-14.636
$V_{NP} : h_m = 0.25, h_v = 0.10$	-0.9708	-1.0268	-1.7392	-0.3793	-0.4285	-1.1504
$V_{NP} : h_m = 0.25, h_v = 0.25$	-1.0025	-1.0679	-6.0302	-0.4565	-0.5156	-5.4901
$V_{NP} : h_m = 0.25, h_v = 0.50$	-1.0553	-1.1316	-13.760	-0.5509	-0.6213	-13.267
$V_{NP} : h_m = 0.50, h_v = 0.10$	-0.5834	-0.6288	-1.4355	4.9752	4.9899	4.1048
$V_{NP} : h_m = 0.50, h_v = 0.25$	-0.6114	-0.6662	-5.7258	4.7717	4.7748	-0.3639
$V_{NP} : h_m = 0.50, h_v = 0.50$	-0.6782	-0.7438	-13.491	3.5610	3.5414	-9.2759
$\hat{V}_{OL}$	0.0455	0.0057	0.2406	0.1299	0.0910	0.3249
$\hat{V}_{NO}$	0.0062	-0.0321	0.0333	0.0903	0.0526	0.1176
$\hat{V}_{SRS}$	33.120	33.416	33.030	29.985	30.266	29.893

Table 2: Simulated relative bias (in percent) with sorting variable $R^2 = 1$, $n = 500$ and regression model $R^2 = 0.25$.

Mean function	Linear			Quadr.		
Variance function	Const.	Linear	Quadr.	Const.	Linear	Quadr.
$\hat{V}_L$	-1.9309	-1.9296	-1.6173	296.15	299.08	295.25
$\hat{V}_L^{lin}$	-1.9291	-1.9257	-1.6059	296.16	299.09	295.27
$V_{NP}^{ho} : h_m = 0.10$	-10.000	-10.058	-8.9370	-10.158	-10.205	-9.0553
$V_{NP}^{ho} : h_m = 0.25$	-4.8354	-4.8525	-3.9188	0.9053	0.9458	1.8255
$V_{NP}^{ho} : h_m = 0.50$	-3.1101	-3.1068	-2.4236	51.644	52.189	52.131
$V_{NP} : h_m = 0.10, h_v = 0.10$	-10.530	-10.625	-9.8869	-10.710	-10.796	-10.042
$V_{NP} : h_m = 0.10, h_v = 0.25$	-10.587	-10.663	-13.690	-10.774	-10.841	-13.981
$V_{NP} : h_m = 0.10, h_v = 0.50$	-10.617	-10.691	-20.491	-10.807	-10.870	-21.016
$V_{NP} : h_m = 0.25, h_v = 0.10$	-5.1193	-5.1666	-4.8513	0.6280	0.6365	0.8751
$V_{NP} : h_m = 0.25, h_v = 0.25$	-5.3172	-5.3389	-8.9307	-0.0215	0.0099	-3.7877
$V_{NP} : h_m = 0.25, h_v = 0.50$	-5.3986	-5.4124	-16.166	-0.5266	-0.4939	-11.692
$V_{NP} : h_m = 0.50, h_v = 0.10$	-3.2404	-3.2555	-3.3541	52.269	52.798	51.910
$V_{NP} : h_m = 0.50, h_v = 0.25$	-3.4191	-3.4068	-7.4285	50.332	50.874	45.956
$V_{NP} : h_m = 0.50, h_v = 0.50$	-3.5718	-3.5516	-14.840	38.725	39.151	26.876
$\hat{V}_{OL}$	-3.3266	-3.3881	-2.4816	0.7037	0.6806	1.5563
$\hat{V}_{NO}$	-3.2996	-3.3637	-3.3716	0.7358	0.6964	0.6481
$\hat{V}_{SRS}$	320.01	322.91	318.88	301.50	304.46	300.19

Table 3: Simulated relative bias (in percent) with sorting variable $R^2 = 1$, $n = 100$ and regression model $R^2 = 0.75$.

Mean function	Linear			Quadr.		
Variance function	Const.	Linear	Quadr.	Const.	Linear	Quadr.
$\hat{\mathrm{V}}_L$	-1.9908	-1.9876	-1.6666	27.889	28.145	28.029
$\hat{\mathrm{V}}_L^{lin}$	-1.9889	-1.9836	-1.6548	27.892	28.151	28.042
$\mathrm{V}_{NP}^{ho} : h_m = 0.10$	-10.312	-10.363	-9.2115	-10.329	-10.376	-9.2240
$\mathrm{V}_{NP}^{ho} : h_m = 0.25$	-4.9880	-5.0026	-4.0408	-4.4144	-4.4236	-3.4689
$\mathrm{V}_{NP}^{ho} : h_m = 0.50$	-3.2075	-3.2020	-2.4984	2.2814	2.3347	2.9594
$\mathrm{V}_{NP} : h_m = 0.10, h_v = 0.10$	-10.857	-10.948	-10.191	-10.877	-10.964	-10.207
$\mathrm{V}_{NP} : h_m = 0.10, h_v = 0.25$	-10.916	-10.988	-14.111	-10.936	-11.004	-14.141
$\mathrm{V}_{NP} : h_m = 0.10, h_v = 0.50$	-10.948	-11.016	-21.121	-10.968	-11.032	-21.175
$\mathrm{V}_{NP} : h_m = 0.25, h_v = 0.10$	-5.2806	-5.3265	-5.0020	-4.7060	-4.7468	-4.4321
$\mathrm{V}_{NP} : h_m = 0.25, h_v = 0.25$	-5.4847	-5.5042	-9.2070	-4.9536	-4.9681	-8.6942
$\mathrm{V}_{NP} : h_m = 0.25, h_v = 0.50$	.5.5686	-5.5799	-16.665	-5.0817	-5.0892	-16.222
$\mathrm{V}_{NP} : h_m = 0.50, h_v = 0.10$	-3.3418	-3.3554	-3.4575	2.2266	2.2601	2.0717
$\mathrm{V}_{NP} : h_m = 0.50, h_v = 0.25$	-3.5261	-3.5114	-7.6574	1.8672	1.9281	-2.3146
$\mathrm{V}_{NP} : h_m = 0.50, h_v = 0.50$	-3.6835	-3.6607	-15.297	0.5548	0.6103	-11.131
$\hat{\mathrm{V}}_{OL}$	-0.5290	-0.5618	0.3336	-0.1140	-0.1431	0.7474
$\hat{\mathrm{V}}_{NO}$	-0.4941	-0.5304	-0.5720	-0.0810	-0.1178	-0.1606
$\hat{\mathrm{V}}_{SRS}$	32.972	33.217	32.770	30.201	30.468	29.946

Table 4: Simulated relative bias (in percent) with sorting variable $R^2 = 1$, $n = 100$ and regression model $R^2 = 0.25$.

Mean function	Linear			Quadr.		
Variance function	Const.	Linear	Quadr.	Const.	Linear	Quadr.
$\hat{V}_L$	-0.2528	-0.2846	-0.2061	147.28	148.21	147.05
$\hat{V}_L^{lin}$	-0.2534	-0.2838	-0.1940	147.30	148.23	147.08
$V_{NP}^{ho} : h_m = 0.10$	-0.7034	-0.8418	-0.6627	-1.1463	-1.2843	-1.1094
$V_{NP}^{ho} : h_m = 0.25$	-0.4834	-0.5552	-0.4101	-2.4125	-2.5012	-2.3362
$V_{NP}^{ho} : h_m = 0.50$	-0.3708	-0.4039	-0.2832	13.257	13.309	13.317
$V_{NP} : h_m = 0.10, h_v = 0.10$	-0.7926	-0.9308	-1.3498	-1.2329	-1.3709	-1.7665
$V_{NP} : h_m = 0.10, h_v = 0.25$	-0.7892	-0.9236	-4.1138	-1.2339	-1.3682	-4.4072
$V_{NP} : h_m = 0.10, h_v = 0.50$	-0.8048	-0.9461	-9.0962	-1.2497	-1.3906	-9.1605
$V_{NP} : h_m = 0.25, h_v = 0.10$	-0.5276	-0.6026	-1.0860	-2.4015	-2.4932	-2.9285
$V_{NP} : h_m = 0.25, h_v = 0.25$	-0.5512	-0.6204	-3.8940	-2.7369	-2.8247	-5.9191
$V_{NP} : h_m = 0.25, h_v = 0.50$	-0.5754	-0.6504	-8.9362	-3.0248	-3.1202	-10.992
$V_{NP} : h_m = 0.50, h_v = 0.10$	-0.3917	-0.4276	-0.9577	13.812	13.863	13.245
$V_{NP} : h_m = 0.50, h_v = 0.25$	-0.4145	-0.4443	-3.7658	12.666	12.717	9.4456
$V_{NP} : h_m = 0.50, h_v = 0.50$	-0.4484	-0.4838	-8.8311	5.4618	5.4592	-2.5421
$\hat{V}_{OL}$	16.815	16.892	16.912	75.122	75.597	75.064
$\hat{V}_{NO}$	17.264	17.359	17.240	75.859	76.344	75.691
$\hat{V}_{SRS}$	181.96	183.19	181.59	148.29	149.23	148.02

Table 5: Simulated relative bias (in percent) with sorting variable $R^2 = 0.75$, $n = 500$ and regression model $R^2 = 0.75$.

Mean function	Linear			Quadr.		
Variance function	Const.	Linear	Quadr.	Const.	Linear	Quadr.
$\hat{V}_L$	-0.3541	-0.4045	-0.2850	21.931	22.100	21.940
$\hat{V}_L^{lin}$	-0.3549	-0.4033	-0.2673	21.933	22.105	21.960
$V_{NP}^{ho} : h_m = 0.10$	-0.9891	-1.1934	-0.9353	-1.0304	-1.2353	-0.9784
$V_{NP}^{ho} : h_m = 0.25$	-0.7033	-0.8184	-0.5941	-0.9991	-1.1201	-0.8886
$V_{NP}^{ho} : h_m = 0.50$	-0.5368	-0.5943	-0.4084	1.5158	1.4765	1.6387
$V_{NP} : h_m = 0.10, h_v = 0.10$	-1.1187	-1.3232	-1.9332	-1.1593	-1.3644	-1.9694
$V_{NP} : h_m = 0.10, h_v = 0.25$	-1.1137	-1.3126	-5.9471	-1.1550	-1.3546	-5.9561
$V_{NP} : h_m = 0.10, h_v = 0.50$	-1.1363	-1.3455	-13.183	-1.1777	-1.3874	-13.141
$V_{NP} : h_m = 0.25, h_v = 0.10$	-0.7675	-0.8874	-1.5756	-1.0544	-1.1802	-1.8551
$V_{NP} : h_m = 0.25, h_v = 0.25$	-0.8018	-0.9135	-5.6536	-1.1356	-1.2535	-5.9515
$V_{NP} : h_m = 0.25, h_v = 0.50$	-0.8370	-0.9571	-12.976	-1.2109	-1.3377	-13.263
$V_{NP} : h_m = 0.50, h_v = 0.10$	-0.5671	-0.6289	-1.3880	1.5739	1.5305	0.7526
$V_{NP} : h_m = 0.50, h_v = 0.25$	-0.6002	-0.6531	-5.4660	1.3713	1.3357	-3.4653
$V_{NP} : h_m = 0.50, h_v = 0.50$	-0.6495	-0.7107	-12.822	0.2365	0.1810	-11.852
$\hat{V}_{OL}$	2.4182	2.3833	2.5779	11.323	11.387	11.437
$\hat{V}_{NO}$	2.5000	2.4899	2.4834	11.453	11.538	11.395
$\hat{V}_{SRS}$	26.424	26.644	26.350	22.397	22.568	22.342

Table 6: Simulated relative bias (in percent) with sorting variable $R^2 = 0.75$, $n = 500$ and regression model $R^2 = 0.25$.

Mean function	Linear			Quadr.		
Variance function	Const.	Linear	Quadr.	Const.	Linear	Quadr.
$\hat{V}_L$	-1.3784	-1.3505	-1.1660	128.70	129.53	128.66
$\hat{V}_L^{lin}$	-1.3832	-1.3538	-1.1435	128.77	129.60	128.75
$V_{NP}^{ho}: h_m = 0.10$	-3.6505	-3.7267	-3.4001	-4.0916	-4.1751	-3.8778
$V_{NP}^{ho}: h_m = 0.25$	-2.6141	-2.6340	-2.1813	-5.6521	-5.6866	-5.2541
$V_{NP}^{ho}: h_m = 0.50$	-2.0237	-1.9985	-1.5952	7.8444	7.9283	8.2108
$V_{NP}: h_m = 0.10, h_v = 0.10$	-4.4362	-4.4585	-4.3331	-4.7921	-4.8267	-4.7102
$V_{NP}: h_m = 0.10, h_v = 0.25$	-4.1650	-4.2227	-6.6840	-4.5566	-4.6235	-6.8217
$V_{NP}: h_m = 0.10, h_v = 0.50$	-4.1075	-4.1775	-11.195	-4.5057	-4.5836	-10.860
$V_{NP}: h_m = 0.25, h_v = 0.10$	-2.8574	-2.8655	-2.7806	-5.6476	-5.6706	-5.5686
$V_{NP}: h_m = 0.25, h_v = 0.25$	-2.9764	-2.9825	-5.5598	-6.1666	-6.1898	-8.4686
$V_{NP}: h_m = 0.25, h_v = 0.50$	-3.0311	-3.0337	-10.437	-6.4992	-6.5220	-13.116
$V_{NP}: h_m = 0.50, h_v = 0.10$	-2.1011	-2.0621	-2.1603	8.9888	9.0864	8.9140
$V_{NP}: h_m = 0.50, h_v = 0.25$	-2.2424	-2.1997	-4.9453	7.6463	7.7432	5.2087
$V_{NP}: h_m = 0.50, h_v = 0.50$	-2.3738	-2.3274	-9.9636	0.4276	0.4818	-6.3686
$\hat{V}_{OL}$	17.632	17.739	17.959	64.660	64.913	64.971
$\hat{V}_{NO}$	15.707	15.748	15.544	62.460	62.680	62.353
$\hat{V}_{SRS}$	181.94	183.21	181.57	132.75	133.59	132.52

Table 7: Simulated relative bias (in percent) with sorting variable $R^2 = 0.75$, $n = 100$ and regression model $R^2 = 0.75$.

Mean function	Linear			Quadr.		
Variance function	Const.	Linear	Quadr.	Const.	Linear	Quadr.
$\hat{V}_L$	-1.9822	-1.9483	-1.6699	18.800	19.044	19.049
$\hat{V}_L^{lin}$	-1.9891	-1.9532	-1.6372	18.803	19.048	19.092
$V_{NP}^{ho} : h_m = 0.10$	-5.2695	-5.3991	-4.8979	-5.2867	-5.4208	-4.9269
$V_{NP}^{ho} : h_m = 0.25$	-3.7963	-3.8362	-3.1596	-4.2749	-4.3179	-3.6449
$V_{NP}^{ho} : h_m = 0.50$	-2.9304	-2.9068	-2.3021	-1.3523	-1.3128	-0.7387
$V_{NP} : h_m = 0.10, h_v = 0.10$	-6.4156	-6.4700	-6.2575	-6.4128	-6.4726	-6.2630
$V_{NP} : h_m = 0.10, h_v = 0.25$	-6.0200	-6.1250	-9.6836	-6.0251	-6.1349	-9.6329
$V_{NP} : h_m = 0.10, h_v = 0.50$	-5.9361	-6.0588	-16.257	-5.9427	-6.0699	-16.096
$V_{NP} : h_m = 0.25, h_v = 0.10$	-4.1512	-4.1750	-4.0330	-4.5852	-4.6122	-4.4651
$V_{NP} : h_m = 0.25, h_v = 0.25$	-4.3248	-4.3461	-8.0834	-4.8235	-4.8486	-8.5153
$V_{NP} : h_m = 0.25, h_v = 0.50$	-4.4046	-4.4211	-15.191	-4.9478	-4.9690	-15.549
$V_{NP} : h_m = 0.50, h_v = 0.10$	-3.0433	-2.9998	-3.1257	-1.2573	-1.1994	-1.3443
$V_{NP} : h_m = 0.50, h_v = 0.25$	-3.2495	-3.2012	-7.1844	-1.6605	-1.5982	-5.5342
$V_{NP} : h_m = 0.50, h_v = 0.50$	-3.4411	-3.3880	-14.498	-2.9899	-2.9355	-13.861
$\hat{V}_{OL}$	2.3395	2.3290	2.8992	10.096	10.096	10.677
$\hat{V}_{NO}$	1.9519	1.8583	1.8061	9.6582	9.5902	9.5549
$\hat{V}_{SRS}$	26.263	26.570	26.190	20.971	21.224	20.917

Table 8: Simulated relative bias (in percent) with sorting variable $R^2 = 0.75$, $n = 100$ and regression model $R^2 = 0.25$.

Mean function	Linear			Quadr.		
Variance function	Const.	Linear	Quadr.	Const.	Linear	Quadr.
$\hat{V}_L$	-0.2064	-0.1801	-0.1899	-61.523	-61.577	-61.499
$\hat{V}_L^{lin}$	-0.2063	-0.1766	-0.1839	-61.510	-61.563	-61.485
$V_{NP}^{ho} : h_m = 0.10$	-0.2661	-0.3652	-0.3429	-1.4390	-1.4457	-1.4439
$V_{NP}^{ho} : h_m = 0.25$	-0.1614	-0.2269	-0.2169	-12.872	-12.887	-12.871
$V_{NP}^{ho} : h_m = 0.50$	-0.2080	-0.2391	-0.1828	-41.996	-42.037	-41.975
$V_{NP} : h_m = 0.10, h_v = 0.10$	-0.3565	-0.4474	-0.9467	-1.4546	-1.4599	-1.5470
$V_{NP} : h_m = 0.10, h_v = 0.25$	-0.3477	-0.4325	-3.3561	-1.4538	-1.4580	-1.9582
$V_{NP} : h_m = 0.10, h_v = 0.50$	-0.3474	-0.4356	-7.6904	-1.4539	-1.4588	-2.6969
$V_{NP} : h_m = 0.25, h_v = 0.10$	-0.2071	-0.2659	-0.8090	-12.872	-12.886	-12.964
$V_{NP} : h_m = 0.25, h_v = 0.25$	-0.2210	-0.2720	-3.2551	-12.921	-12.934	-13.427
$V_{NP} : h_m = 0.25, h_v = 0.50$	-0.2302	-0.2840	-7.6430	-12.965	-12.979	-14.218
$V_{NP} : h_m = 0.50, h_v = 0.10$	-0.2315	-0.2549	-0.7723	-41.865	-41.905	-41.941
$V_{NP} : h_m = 0.50, h_v = 0.25$	-0.2444	-0.2597	-3.2184	-42.051	-42.090	-42.542
$V_{NP} : h_m = 0.50, h_v = 0.50$	-0.2640	-0.2822	-7.6283	-43.191	-43.232	-44.430
$\hat{V}_{OL}$	98.785	99.368	98.638	-63.386	-63.448	-63.363
$\hat{V}_{NO}$	97.473	98.117	97.310	-61.386	-61.437	-61.368
$\hat{V}_{SRS}$	146.59	147.51	146.31	-61.380	-61.434	-61.362

Table 9: Simulated relative bias (in percent) with sorting variable $R^2 = 0.25$, $n = 500$ and regression model $R^2 = 0.75$.

Mean function	Linear			Quadr.		
Variance function	Const.	Linear	Quadr.	Const.	Linear	Quadr.
$\hat{V}_L$	-0.3154	-0.2796	-0.2918	-32.985	-33.136	-32.915
$\hat{V}_L^{lin}$	-0.3153	-0.2738	-0.2819	-32.978	-33.126	-32.902
$V_{NP}^{ho} : h_m = 0.10$	-0.3252	-0.5208	-0.4717	-0.9691	-1.0621	-1.0366
$V_{NP}^{ho} : h_m = 0.25$	-0.1607	-0.2944	-0.2700	-6.9808	-7.0778	-7.0199
$V_{NP}^{ho} : h_m = 0.50$	-0.2735	-0.3479	-0.2455	-22.554	-22.706	-22.497
$V_{NP} : h_m = 0.10, h_v = 0.10$	-0.4725	-0.6550	-1.4539	-1.0513	-1.1367	-1.5848
$V_{NP} : h_m = 0.10, h_v = 0.25$	-0.4582	-0.6307	-5.3727	-1.0436	-1.1236	-3.7721
$V_{NP} : h_m = 0.10, h_v = 0.50$	-0.4577	-0.6357	-12.423	-1.0435	-1.1265	-7.7063
$V_{NP} : h_m = 0.25, h_v = 0.10$	-0.2352	-0.3580	-1.2331	-7.0182	-7.1089	-7.5532
$V_{NP} : h_m = 0.25, h_v = 0.25$	-0.2577	-0.3682	-5.2117	-7.0553	-7.1393	-9.7980
$V_{NP} : h_m = 0.25, h_v = 0.50$	-0.2727	-0.3877	-12.349	-7.0868	-7.1734	-13.804
$V_{NP} : h_m = 0.50, h_v = 0.10$	-0.3116	-0.3738	-1.2044	-22.505	-22.649	-22.961
$V_{NP} : h_m = 0.50, h_v = 0.25$	-0.3327	-0.3815	-5.1830	-22.614	-22.751	-25.278
$V_{NP} : h_m = 0.50, h_v = 0.50$	-0.3647	-0.4183	-12.356	-23.238	-23.382	-29.887
$\hat{V}_{OL}$	16.115	16.274	16.090	-33.811	-33.987	-33.747
$\hat{V}_{NO}$	15.910	16.151	15.867	-32.728	-32.857	-32.678
$\hat{V}_{SRS}$	23.862	24.144	23.764	-32.743	-32.894	-32.705

Table 10: Simulated relative bias (in percent) with sorting variable $R^2 = 0.25$, $n = 500$ and regression model $R^2 = 0.25$.

Mean function	Linear			Quadr.		
Variance function	Const.	Linear	Quadr.	Const.	Linear	Quadr.
$\hat{V}_L$	-0.6368	-0.6108	-0.5631	-22.671	-22.725	-22.626
$\hat{V}_L^{lin}$	-0.6388	-0.6022	-0.5159	-22.342	-22.388	-22.275
$V_{NP}^{ho} : h_m = 0.10$	-1.4003	-1.7870	-1.5716	-1.5914	-1.8646	-1.7554
$V_{NP}^{ho} : h_m = 0.25$	-0.9841	-1.0562	-0.9385	-10.508	-10.597	-10.510
$V_{NP}^{ho} : h_m = 0.50$	-0.8339	-0.8294	-0.6742	-32.190	-32.267	-32.101
$V_{NP} : h_m = 0.10, h_v = 0.10$	-2.0578	-2.4309	-2.2522	-1.9072	-2.1732	-2.0826
$V_{NP} : h_m = 0.10, h_v = 0.25$	-1.8413	-2.2327	-3.6647	-1.8055	-2.0806	-2.7665
$V_{NP} : h_m = 0.10, h_v = 0.50$	-1.7714	-2.1569	-6.4827	-1.7723	-2.0446	-4.1259
$V_{NP} : h_m = 0.25, h_v = 0.10$	-1.2092	-1.2642	-1.3330	-10.536	-10.616	-10.620
$V_{NP} : h_m = 0.25, h_v = 0.25$	-1.2794	-1.3483	-3.0519	-10.706	-10.793	-11.585
$V_{NP} : h_m = 0.25, h_v = 0.50$	-1.3046	-1.3661	-6.1250	-10.821	-10.905	-13.169
$V_{NP} : h_m = 0.50, h_v = 0.10$	-0.9206	-0.9034	-1.0329	-31.641	-31.712	-31.688
$V_{NP} : h_m = 0.50, h_v = 0.25$	-1.0114	-1.0009	-2.7599	-32.131	-32.204	-32.965
$V_{NP} : h_m = 0.50, h_v = 0.50$	-1.0937	-1.0741	-5.9242	-34.614	-34.689	-36.931
$\hat{V}_{OL}$	44.076	44.348	44.063	-23.118	-23.215	-23.070
$\hat{V}_{NO}$	39.375	39.643	39.271	-22.757	-22.837	-22.766
$\hat{V}_{SRS}$	77.575	77.932	77.426	-21.502	-21.557	-21.503

Table 11: Simulated relative bias (in percent) with sorting variable $R^2 = 0.25$, $n = 100$ and regression model $R^2 = 0.75$.

Mean function	Linear			Quadr.		
Variance function	Const.	Linear	Quadr.	Const.	Linear	Quadr.
$\hat{V}_L$	-1.3670	-1.3170	-1.1865	-9.1613	-9.2040	-9.0025
$\hat{V}_L^{lin}$	-1.3712	-1.2985	-1.0859	-9.0470	-9.0678	-8.8044
$V_{NP}^{ho} : h_m = 0.10$	-2.9013	-3.7375	-3.2654	-2.5725	-3.3486	-2.9622
$V_{NP}^{ho} : h_m = 0.25$	-2.0850	-2.2513	-1.9732	-5.2698	-5.4684	-5.2053
$V_{NP}^{ho} : h_m = 0.50$	-1.7834	-1.7828	-1.4208	-12.898	-12.996	-12.586
$V_{NP} : h_m = 0.10, h_v = 0.10$	-4.3043	-5.1182	-4.7158	-3.7131	-4.4691	-4.1421
$V_{NP} : h_m = 0.10, h_v = 0.25$	-3.8423	-4.6933	-7.7256	-3.3384	-4.1253	-6.5926
$V_{NP} : h_m = 0.10, h_v = 0.50$	-3.6933	-4.5307	-13.730	-3.2173	-3.9934	-11.480
$V_{NP} : h_m = 0.25, h_v = 0.10$	-2.5653	-2.6973	-2.8137	-5.6311	-5.8010	-5.8610
$V_{NP} : h_m = 0.25, h_v = 0.25$	-2.7153	-2.8777	-6.4766	-5.8022	-5.9969	-8.8906
$V_{NP} : h_m = 0.25, h_v = 0.50$	-2.7690	-2.9159	-13.025	-5.8820	-6.0646	-14.256
$V_{NP} : h_m = 0.50, h_v = 0.10$	-1.9685	-1.9415	-2.1852	-12.829	-12.907	-12.995
$V_{NP} : h_m = 0.50, h_v = 0.25$	-2.1622	-2.1505	-5.8653	-13.150	-13.239	-16.151
$V_{NP} : h_m = 0.50, h_v = 0.50$	-2.3379	-2.3074	-12.608	-14.171	-14.252	-22.513
$\hat{V}_{OL}$	9.3733	9.5301	9.5384	-8.4247	-8.5327	-8.2360
$\hat{V}_{NO}$	8.4007	8.5532	8.3179	-8.3273	-8.4210	-8.3533
$\hat{V}_{SRS}$	16.558	16.765	16.475	-7.7409	-7.7805	-7.7443

Table 12: Simulated relative bias (in percent) with sorting variable $R^2 = 0.25$, $n = 100$ and regression model $R^2 = 0.25$.

Mean function	Linear			Quadr.		
Variance function	Const.	Linear	Quadr.	Const.	Linear	Quadr.
$\hat{V}_L$	0.9304	0.9456	0.9601	2099.3	1637.5	1787.0
$\hat{V}_L^{lin}$	0.9304	0.9454	0.9601	2099.3	1637.5	1787.0
$V_{NP}^{ho} : h_m = 0.25$	0.9398	0.9524	0.9630	1.5273	1.4042	1.5105
$V_{NP}^{ho} : h_m = 0.50$	0.9326	0.9471	0.9605	70.627	55.243	60.610
$V_{NP} : h_m = 0.10, h_v = 0.10$	1.0221	1.0360	1.0843	1.0219	1.0360	1.0779
$V_{NP} : h_m = 0.10, h_v = 0.25$	1.0440	1.0950	1.8254	1.0449	1.0962	1.7983
$V_{NP} : h_m = 0.10, h_v = 0.50$	1.0967	1.2209	4.9348	1.0980	1.2235	4.8918
$V_{NP} : h_m = 0.25, h_v = 0.10$	0.9557	0.9837	1.0045	1.5318	1.4259	1.3072
$V_{NP} : h_m = 0.25, h_v = 0.25$	0.9782	1.0436	1.5836	1.4453	1.3997	0.8741
$V_{NP} : h_m = 0.25, h_v = 0.50$	1.0289	1.1685	4.4417	1.4039	1.4535	2.2054
$V_{NP} : h_m = 0.50, h_v = 0.10$	0.9477	0.9781	0.9894	72.480	56.711	59.916
$V_{NP} : h_m = 0.50, h_v = 0.25$	0.9695	1.0377	1.5160	67.951	53.203	47.391
$V_{NP} : h_m = 0.50, h_v = 0.50$	1.0198	1.1627	4.2980	42.240	33.241	17.4373
$\hat{V}_{OL}$	1.4072	1.4324	1.4571	1.4204	1.4425	1.4728
$\hat{V}_{NO}$	1.8900	1.9256	1.9686	1.9115	1.9430	1.9849
$\hat{V}_{SRS}$	2550.8	1994.8	2179.5	2116.1	1650.4	1800.4

Table 13: Comparisons between MSEs with sorting variable $R^2 = 1$, $n = 500$ and regression model $R^2 = 0.75$.

Mean function	Linear			Quadr.		
Variance function	Const.	Linear	Quadr.	Const.	Linear	Quadr.
$\hat{V}_L$	0.9303	0.9458	0.9600	21.723	17.188	18.697
$\hat{V}_L^{lin}$	0.9303	0.9456	0.9600	21.723	17.188	18.697
$V_{NP}^{ho} : h_m = 0.25$	0.9398	0.9526	0.9630	0.9244	0.9402	0.9547
$V_{NP}^{ho} : h_m = 0.50$	0.9326	0.9473	0.9606	1.5086	1.3918	1.4844
$V_{NP} : h_m = 0.10, h_v = 0.10$	1.0222	1.0360	1.0843	1.0221	1.0360	1.0836
$V_{NP} : h_m = 0.10, h_v = 0.25$	1.0441	1.0951	1.825	1.0443	1.0952	1.8227
$V_{NP} : h_m = 0.10, h_v = 0.50$	1.0967	1.2210	4.9346	1.0968	1.2212	4.9305
$V_{NP} : h_m = 0.25, h_v = 0.10$	0.9558	0.9839	1.0045	0.9389	0.9703	0.9717
$V_{NP} : h_m = 0.25, h_v = 0.25$	0.9782	1.0438	1.5835	0.9615	1.0302	1.4600
$V_{NP} : h_m = 0.25, h_v = 0.50$	1.0289	1.1687	4.4416	1.0119	1.1551	4.1751
$V_{NP} : h_m = 0.50, h_v = 0.10$	0.9478	0.9784	0.9894	1.5360	1.4325	1.2971
$V_{NP} : h_m = 0.50, h_v = 0.25$	0.9696	1.0379	1.5160	1.5101	1.4531	0.8747
$V_{NP} : h_m = 0.50, h_v = 0.50$	1.0198	1.1629	4.2980	1.3184	1.3902	2.3827
$\hat{V}_{OL}$	1.3965	1.4232	1.4536	1.3991	1.4253	1.4563
$\hat{V}_{NO}$	1.8781	1.9159	1.9594	1.8816	1.9189	1.9623
$\hat{V}_{SRS}$	27.049	21.330	23.108	22.390	17.699	19.174

Table 14: Comparisons between MSEs with sorting variable $R^2 = 1$, $n = 500$ and regression model $R^2 = 0.25$.

Mean function	Linear			Quadr.		
Variance function	Const.	Linear	Quadr.	Const.	Linear	Quadr.
$\hat{V}_L$	0.6929	0.7574	0.7833	316.90	270.34	299.58
$\hat{V}_L^{lin}$	0.6927	0.7569	0.7832	316.91	270.36	299.61
$V_{NP}^{ho} : h_m = 0.25$	0.7473	0.7958	0.8121	0.6750	0.7380	0.7781
$V_{NP}^{ho} : h_m = 0.50$	0.7064	0.7655	0.7899	10.307	8.9677	10.102
$V_{NP} : h_m = 0.10, h_v = 0.10$	1.0404	1.0426	1.0569	1.0407	1.0429	1.0566
$V_{NP} : h_m = 0.10, h_v = 0.25$	1.0584	1.0928	1.3107	1.0591	1.0942	1.3089
$V_{NP} : h_m = 0.10, h_v = 0.50$	1.1011	1.1878	2.0036	1.1029	1.1911	2.0017
$V_{NP} : h_m = 0.25, h_v = 0.10$	0.7615	0.8177	0.8342	0.6777	0.7494	0.7619
$V_{NP} : h_m = 0.25, h_v = 0.25$	0.7821	0.8734	0.9630	0.6885	0.7984	0.7334
$V_{NP} : h_m = 0.25, h_v = 0.50$	0.8259	0.9727	1.4635	0.7337	0.9007	0.9979
$V_{NP} : h_m = 0.50, h_v = 0.10$	0.7165	0.7855	0.8020	10.554	9.1836	10.009
$V_{NP} : h_m = 0.50, h_v = 0.25$	0.7333	0.8388	0.8862	9.8527	8.6355	7.9593
$V_{NP} : h_m = 0.50, h_v = 0.50$	0.7782	0.9407	1.3240	6.1807	5.5669	3.0334
$\hat{V}_{OL}$	1.1029	1.1962	1.2416	1.0809	1.1806	1.2417
$\hat{V}_{NO}$	1.4544	1.6030	1.6525	1.4610	1.6077	1.6458
$\hat{V}_{SRS}$	389.02	331.86	368.40	328.38	280.02	309.63

Table 15: Comparisons between MSEs with sorting variable $R^2 = 1$, $n = 100$ and regression model $R^2 = 0.75$.

Mean function	Linear			Quadr.		
Variance function	Const.	Linear	Quadr.	Const.	Linear	Quadr.
$\hat{V}_L$	0.6943	0.7595	0.7834	3.8196	3.4297	3.6903
$\hat{V}_L^{lin}$	0.6940	0.7590	0.7833	3.8200	3.4297	3.6927
$V_{NP}^{ho} : h_m = 0.25$	0.7481	0.7973	0.8120	0.7296	0.7824	0.7982
$V_{NP}^{ho} : h_m = 0.50$	0.7076	0.7676	0.7899	0.7030	0.7666	0.8081
$V_{NP} : h_m = 0.10, h_v = 0.10$	1.0405	1.0428	1.0569	1.0405	1.0428	1.0569
$V_{NP} : h_m = 0.10, h_v = 0.25$	1.0585	1.0932	1.3107	1.0585	1.0933	1.3104
$V_{NP} : h_m = 0.10, h_v = 0.50$	1.1015	1.1886	2.0037	1.1016	1.1891	2.0034
$V_{NP} : h_m = 0.25, h_v = 0.10$	0.7624	0.8194	0.8341	0.7427	0.8034	0.8164
$V_{NP} : h_m = 0.25, h_v = 0.25$	0.7829	0.8752	0.9630	0.7638	0.8599	0.9312
$V_{NP} : h_m = 0.25, h_v = 0.50$	0.8269	0.9749	1.4636	0.8093	0.9613	1.4101
$V_{NP} : h_m = 0.50, h_v = 0.10$	0.7177	0.7877	0.8020	0.7100	0.7839	0.7850
$V_{NP} : h_m = 0.50, h_v = 0.25$	0.7345	0.8411	0.8862	0.7165	0.8294	0.7153
$V_{NP} : h_m = 0.50, h_v = 0.50$	0.7796	0.9433	1.3241	0.7429	0.9158	0.9539
$\hat{V}_{OL}$	1.0646	1.1644	1.2197	1.0651	1.1657	1.2224
$\hat{V}_{NO}$	1.4122	1.5678	1.6076	1.4162	1.5705	1.6108
$\hat{V}_{SRS}$	4.9969	4.4203	4.6747	4.3246	3.8562	4.0902

Table 16: Comparisons between MSEs with sorting variable $R^2 = 1$, $n = 100$ and regression model $R^2 = 0.25$.

Mean function	Linear			Quadr.		
Variance function	Const.	Linear	Quadr.	Const.	Linear	Quadr.
$\hat{V}_L$	0.5348	0.5062	0.8135	368.36	370.94	331.91
$\hat{V}_L^{lin}$	0.5348	0.4984	0.8137	368.44	370.98	332.04
$V_{NP}^{ho} : h_m = 0.25$	0.6366	0.6084	0.8278	0.4952	0.5535	0.4874
$V_{NP}^{ho} : h_m = 0.50$	0.5611	0.5270	0.8103	3.3565	3.4276	3.0669
$V_{NP} : h_m = 0.10, h_v = 0.10$	1.0164	1.0158	1.0416	1.0082	1.0045	1.0182
$V_{NP} : h_m = 0.10, h_v = 0.25$	1.0205	1.0396	1.5259	1.0127	1.0255	1.2388
$V_{NP} : h_m = 0.10, h_v = 0.50$	1.0499	1.0970	3.8129	1.0292	1.0680	2.1676
$V_{NP} : h_m = 0.25, h_v = 0.10$	0.6503	0.6212	0.8538	0.5000	0.5555	0.5043
$V_{NP} : h_m = 0.25, h_v = 0.25$	0.6560	0.6471	1.2964	0.5309	0.6057	0.8804
$V_{NP} : h_m = 0.25, h_v = 0.50$	0.6854	0.7067	3.5405	0.5771	0.6786	2.1419
$V_{NP} : h_m = 0.50, h_v = 0.10$	0.5733	0.5380	0.8303	3.6300	3.6965	3.0460
$V_{NP} : h_m = 0.50, h_v = 0.25$	0.5801	0.5658	1.2470	3.1115	3.1987	1.6964
$V_{NP} : h_m = 0.50, h_v = 0.50$	0.6100	0.6268	3.4586	0.9094	1.0167	0.3456
$\hat{V}_{OL}$	10.163	7.5920	14.204	98.031	98.665	88.484
$\hat{V}_{NO}$	11.556	8.6492	15.982	100.64	101.29	90.536
$\hat{V}_{SRS}$	907.41	676.78	1224.8	373.39	375.96	336.25

Table 17: Comparisons between MSEs with sorting variable $R^2 = 0.75$, $n = 500$ and regression model $R^2 = 0.75$.

Mean function	Linear			Quadr.		
Variance function	Const.	Linear	Quadr.	Const.	Linear	Quadr.
$\hat{V}_L$	0.8862	0.8896	0.9575	11.601	9.7939	9.9758
$\hat{V}_L^{lin}$	0.8864	0.8739	0.9578	11.603	9.7709	9.9922
$V_{NP}^{ho} : h_m = 0.25$	0.9120	0.9139	0.9637	0.8021	0.8581	0.8180
$V_{NP}^{ho} : h_m = 0.50$	0.8925	0.8945	0.9574	0.8412	0.8882	0.8516
$V_{NP} : h_m = 0.10, h_v = 0.10$	1.0272	1.0274	1.0507	1.0218	1.0225	1.0416
$V_{NP} : h_m = 0.10, h_v = 0.25$	1.0391	1.0793	1.6573	1.0341	1.0737	1.5495
$V_{NP} : h_m = 0.10, h_v = 0.50$	1.0937	1.1937	4.5300	1.0808	1.1801	3.9312
$V_{NP} : h_m = 0.25, h_v = 0.10$	0.9357	0.9369	0.9969	0.8230	0.8784	0.8490
$V_{NP} : h_m = 0.25, h_v = 0.25$	0.9492	0.9906	1.5554	0.8374	0.9322	1.3635
$V_{NP} : h_m = 0.25, h_v = 0.50$	1.0037	1.1068	4.3817	0.8874	1.0435	3.8068
$V_{NP} : h_m = 0.50, h_v = 0.10$	0.9149	0.9160	0.9829	0.8677	0.9119	0.8045
$V_{NP} : h_m = 0.50, h_v = 0.25$	0.9291	0.9709	1.5078	0.8627	0.9512	0.937
$V_{NP} : h_m = 0.50, h_v = 0.50$	0.9837	1.0880	4.2918	0.8646	1.0254	3.1460
$\hat{V}_{OL}$	1.7004	1.5634	1.7740	4.3742	3.8501	3.9341
$\hat{V}_{NO}$	2.2256	2.0366	2.2609	4.9507	4.4007	4.3703
$\hat{V}_{SRS}$	18.563	14.363	16.612	12.054	10.158	10.310

Table 18: Comparisons between MSEs with sorting variable $R^2 = 0.75$, $n = 500$ and regression model $R^2 = 0.25$.

Mean function	Linear			Quadr.		
Variance function	Const.	Linear	Quadr.	Const.	Linear	Quadr.
$\hat{V}_L$	0.7732	0.7918	0.7727	121.48	107.87	148.55
$\hat{V}_L^{lin}$	0.7748	0.7796	0.7741	121.60	107.93	148.74
$V_{NP}^{ho} : h_m = 0.25$	0.8119	0.8270	0.8135	0.8774	0.8955	1.0510
$V_{NP}^{ho} : h_m = 0.50$	0.7850	0.8056	0.7727	1.1667	1.1828	1.5127
$V_{NP} : h_m = 0.10, h_v = 0.10$	1.0857	1.0712	1.0733	1.0705	1.0628	1.0829
$V_{NP} : h_m = 0.10, h_v = 0.25$	1.0538	1.0714	1.1596	1.0519	1.0676	1.2030
$V_{NP} : h_m = 0.10, h_v = 0.50$	1.0777	1.1463	1.5751	1.0673	1.1250	1.6800
$V_{NP} : h_m = 0.25, h_v = 0.10$	0.8786	0.8851	0.8640	0.9264	0.9399	1.1200
$V_{NP} : h_m = 0.25, h_v = 0.25$	0.8563	0.8910	0.9218	0.9401	0.9723	1.3577
$V_{NP} : h_m = 0.25, h_v = 0.50$	0.8821	0.9690	1.3083	0.9974	1.0658	2.0763
$V_{NP} : h_m = 0.50, h_v = 0.10$	0.8526	0.8676	0.8180	1.4207	1.4092	1.7300
$V_{NP} : h_m = 0.50, h_v = 0.25$	0.8260	0.8693	0.8516	1.1958	1.2322	1.1279
$V_{NP} : h_m = 0.50, h_v = 0.50$	0.8539	0.9490	1.2136	0.7534	0.8773	0.9889
$\hat{V}_{OL}$	6.0059	4.8903	5.5992	36.457	32.193	44.724
$\hat{V}_{NO}$	6.6147	5.3951	6.0676	36.084	31.864	43.691
$\hat{V}_{SRS}$	262.97	208.00	230.23	128.87	114.34	157.16

Table 19: Comparisons between MSEs with sorting variable $R^2 = 0.75$, $n = 100$ and regression model $R^2 = 0.75$.

Mean function	Linear			Quadr.		
Variance function	Const.	Linear	Quadr.	Const.	Linear	Quadr.
$\hat{V}_L$	0.8893	0.9209	0.9250	2.9748	2.5978	2.7833
$\hat{V}_L^{lin}$	0.8915	0.9054	0.9270	2.9792	2.5794	2.7925
$V_{NP}^{ho} : h_m = 0.25$	0.9159	0.9344	0.9395	0.9060	0.9277	0.9583
$V_{NP}^{ho} : h_m = 0.50$	0.9000	0.9253	0.9292	0.8510	0.8935	0.9277
$V_{NP} : h_m = 0.10, h_v = 0.10$	1.1092	1.0885	1.0902	1.1042	1.0854	1.0902
$V_{NP} : h_m = 0.10, h_v = 0.25$	1.0698	1.0920	1.2005	1.0700	1.0912	1.2045
$V_{NP} : h_m = 0.10, h_v = 0.50$	1.1039	1.1904	1.7336	1.1014	1.1845	1.7430
$V_{NP} : h_m = 0.25, h_v = 0.10$	1.0011	1.0075	1.0025	0.9873	0.9983	1.0240
$V_{NP} : h_m = 0.25, h_v = 0.25$	0.9744	1.0174	1.0770	0.9657	1.0107	1.1191
$V_{NP} : h_m = 0.25, h_v = 0.50$	1.0107	1.1196	1.5742	1.0054	1.1123	1.6570
$V_{NP} : h_m = 0.50, h_v = 0.10$	0.9871	1.0041	0.9872	0.9408	0.9748	0.9821
$V_{NP} : h_m = 0.50, h_v = 0.25$	0.9542	1.0082	1.0298	0.9022	0.9725	0.9685
$V_{NP} : h_m = 0.50, h_v = 0.50$	0.9914	1.1116	1.4933	0.9520	1.0813	1.4539
$\hat{V}_{OL}$	1.5454	1.5071	1.5698	2.2099	2.0115	2.2037
$\hat{V}_{NO}$	2.0277	1.9667	1.9863	2.6973	2.4803	2.5897
$\hat{V}_{SRS}$	4.7534	3.9486	4.0578	3.4147	2.9422	3.1309

Table 20: Comparisons between MSEs with sorting variable $R^2 = 0.75$, $n = 100$ and regression model $R^2 = 0.25$.

Mean function	Linear			Quadr.		
Variance function	Const.	Linear	Quadr.	Const.	Linear	Quadr.
$\hat{V}_L$	0.1748	0.2295	0.3573	114.05	121.56	214.80
$\hat{V}_L^{lin}$	0.1747	0.2266	0.3571	114.00	121.52	214.70
$V_{NP}^{ho} : h_m = 0.25$	0.7561	0.6627	0.8679	5.5855	5.8220	10.024
$V_{NP}^{ho} : h_m = 0.50$	0.4633	0.4013	0.7191	53.302	56.757	100.30
$V_{NP} : h_m = 0.10, h_v = 0.10$	1.0025	1.0045	0.9652	1.0011	0.9999	1.0036
$V_{NP} : h_m = 0.10, h_v = 0.25$	1.0067	1.0160	1.1152	1.0021	1.0010	1.0836
$V_{NP} : h_m = 0.10, h_v = 0.50$	1.0125	1.0374	1.8735	1.0025	1.0029	1.2768
$V_{NP} : h_m = 0.25, h_v = 0.10$	0.7585	0.6668	0.8296	5.5859	5.8241	10.167
$V_{NP} : h_m = 0.25, h_v = 0.25$	0.7621	0.6786	0.9716	5.6230	5.8630	10.856
$V_{NP} : h_m = 0.25, h_v = 0.50$	0.7677	0.7002	1.7280	5.6597	5.9041	12.085
$V_{NP} : h_m = 0.50, h_v = 0.10$	0.4658	0.4047	0.6774	52.978	56.411	100.15
$V_{NP} : h_m = 0.50, h_v = 0.25$	0.4692	0.4167	0.8166	53.446	56.907	103.02
$V_{NP} : h_m = 0.50, h_v = 0.50$	0.4751	0.4389	1.5743	56.361	60.021	112.30
$\hat{V}_{OL}$	99.235	103.98	165.77	121.31	129.32	228.46
$\hat{V}_{NO}$	97.720	102.51	162.94	114.22	121.72	215.07
$\hat{V}_{SRS}$	217.84	228.35	363.45	113.52	121.00	213.85

Table 21: Comparisons between MSEs with sorting variable $R^2 = 0.25$, $n = 500$ and regression model $R^2 = 0.75$.

Mean function	Linear			Quadr.		
Variance function	Const.	Linear	Quadr.	Const.	Linear	Quadr.
$\hat{V}_L$	0.6154	0.7029	0.8275	11.232	11.506	18.797
$\hat{V}_L^{lin}$	0.6153	0.6911	0.8272	11.228	11.508	18.783
$V_{NP}^{ho} : h_m = 0.25$	0.8801	0.8660	0.9610	1.1612	1.1068	1.5998
$V_{NP}^{ho} : h_m = 0.50$	0.7427	0.7638	0.9211	5.4779	5.5891	9.1398
$V_{NP} : h_m = 0.10, h_v = 0.10$	1.0140	1.0138	0.9074	1.0031	1.0031	0.9770
$V_{NP} : h_m = 0.10, h_v = 0.25$	1.0261	1.0562	1.2846	1.0068	1.0125	1.1568
$V_{NP} : h_m = 0.10, h_v = 0.50$	1.0550	1.1400	3.2426	1.0118	1.0302	1.8897
$V_{NP} : h_m = 0.25, h_v = 0.10$	0.8940	0.8797	0.8576	1.1686	1.1168	1.7009
$V_{NP} : h_m = 0.25, h_v = 0.25$	0.9054	0.9230	1.2153	1.1755	1.1294	2.3425
$V_{NP} : h_m = 0.25, h_v = 0.50$	0.9338	1.0070	3.1695	1.1865	1.1540	3.9113
$V_{NP} : h_m = 0.50, h_v = 0.10$	0.7573	0.7774	0.8143	5.4589	5.5694	9.4686
$V_{NP} : h_m = 0.50, h_v = 0.25$	0.7686	0.8211	1.1685	5.5099	5.6254	11.353
$V_{NP} : h_m = 0.50, h_v = 0.50$	0.7975	0.9059	3.1313	5.7973	5.9360	15.628
$\hat{V}_{OL}$	5.8269	5.2109	5.7090	11.895	12.193	19.884
$\hat{V}_{NO}$	6.1884	5.5939	5.9858	11.289	11.543	18.841
$\hat{V}_{SRS}$	10.868	9.5747	10.304	11.072	11.342	18.564

Table 22: Comparisons between MSEs with sorting variable $R^2 = 0.25$, $n = 500$ and regression model $R^2 = 0.25$.

Mean function	Linear			Quadr.		
Variance function	Const.	Linear	Quadr.	Const.	Linear	Quadr.
$\hat{V}_L$	0.7474	0.7864	0.7044	4.6722	6.0761	7.7511
$\hat{V}_L^{lin}$	0.7487	0.7680	0.7045	4.5854	5.9466	7.5854
$V_{NP}^{ho} : h_m = 0.25$	0.7864	0.8325	0.7101	1.4846	1.8333	2.1642
$V_{NP}^{ho} : h_m = 0.50$	0.7542	0.8010	0.6856	8.2262	10.634	13.999
$V_{NP} : h_m = 0.10, h_v = 0.10$	1.0381	1.0229	1.0029	1.0105	1.0126	1.0062
$V_{NP} : h_m = 0.10, h_v = 0.25$	1.0240	1.0256	1.0368	1.0074	1.0123	1.0353
$V_{NP} : h_m = 0.10, h_v = 0.50$	1.0347	1.0675	1.2511	1.0083	1.0225	1.1363
$V_{NP} : h_m = 0.25, h_v = 0.10$	0.8202	0.8531	0.6998	1.4994	1.8470	2.1994
$V_{NP} : h_m = 0.25, h_v = 0.25$	0.8074	0.8559	0.7188	1.5177	1.8768	2.4552
$V_{NP} : h_m = 0.25, h_v = 0.50$	0.8188	0.8995	0.9228	1.5396	1.9135	2.9381
$V_{NP} : h_m = 0.50, h_v = 0.10$	0.7887	0.8255	0.6715	7.9801	10.310	13.680
$V_{NP} : h_m = 0.50, h_v = 0.25$	0.7742	0.8256	0.6805	8.2068	10.607	14.733
$V_{NP} : h_m = 0.50, h_v = 0.50$	0.7856	0.8690	0.8758	9.4508	12.219	18.286
$\hat{V}_{OL}$	24.335	22.303	23.892	5.2750	6.4877	8.7567
$\hat{V}_{NO}$	22.897	21.032	22.550	6.0295	7.8604	10.038
$\hat{V}_{SRS}$	64.250	58.331	63.418	4.3018	5.5917	7.1295

Table 23: Comparisons between MSEs with sorting variable $R^2 = 0.75$, $n = 100$ and regression model $R^2 = 0.25$.

Mean function	Linear			Quadr.		
Variance function	Const.	Linear	Quadr.	Const.	Linear	Quadr.
$\hat{V}_L$	0.8841	0.9264	0.9211	0.9277	1.1003	1.1238
$\hat{V}_L^{lin}$	0.8872	0.8890	0.9212	0.9256	1.0688	1.1122
$V_{NP}^{ho} : h_m = 0.25$	0.9030	0.9362	0.9244	0.8428	0.9451	0.9748
$V_{NP}^{ho} : h_m = 0.50$	0.8853	0.9252	0.9192	1.1549	1.3148	1.4586
$V_{NP} : h_m = 0.10, h_v = 0.10$	1.0917	1.0495	1.0047	1.0446	1.0353	1.0041
$V_{NP} : h_m = 0.10, h_v = 0.25$	1.0581	1.0539	1.0697	1.0292	1.0380	1.0576
$V_{NP} : h_m = 0.10, h_v = 0.50$	1.0822	1.1424	1.4820	1.0394	1.0928	1.3578
$V_{NP} : h_m = 0.25, h_v = 0.10$	0.9855	0.9809	0.9044	0.8955	0.9859	0.9854
$V_{NP} : h_m = 0.25, h_v = 0.25$	0.9550	0.9850	0.9402	0.8814	0.9903	1.1071
$V_{NP} : h_m = 0.25, h_v = 0.50$	0.9817	1.0775	1.3365	0.8952	1.0504	1.5575
$V_{NP} : h_m = 0.50, h_v = 0.10$	0.9701	0.9785	0.8937	1.1983	1.3499	1.4963
$V_{NP} : h_m = 0.50, h_v = 0.25$	0.9358	0.9776	0.9093	1.2036	1.3724	1.8274
$V_{NP} : h_m = 0.50, h_v = 0.50$	0.9628	1.0697	1.2892	1.3020	1.5157	2.8031
$\hat{V}_{OL}$	2.2953	2.0817	1.9940	1.2039	1.3724	1.4584
$\hat{V}_{NO}$	2.8031	2.5147	2.4105	1.5601	1.7829	1.8761
$\hat{V}_{SRS}$	2.7954	2.4585	2.2453	0.8549	1.0267	1.0458

Table 24: Comparisons between MSEs with sorting variable $R^2 = 0.25$, $n = 100$ and regression model $R^2 = 0.25$.

Mean function	Linear			Quadr.		
Variance function	Const.	Linear	Quadr.	Const.	Linear	Quadr.
$\hat{V}_L$	0.9996	0.9996	0.9993	14.311	14.495	14.458
$\hat{V}_L^{lin}$	0.9996	0.9996	0.9993	14.311	14.495	14.458
$V_{NP}^{ho} : h_m = 0.25$	0.9996	0.9996	0.9998	1.0042	1.0044	1.0054
$V_{NP}^{ho} : h_m = 0.50$	0.9996	0.9996	0.9997	1.4481	1.4556	1.4607
$V_{NP} : h_m = 0.10, h_v = 0.10$	1.0001	1.0001	1.0003	1.0001	1.0001	1.0003
$V_{NP} : h_m = 0.10, h_v = 0.25$	1.0005	1.0008	1.0052	1.0004	1.0007	1.0049
$V_{NP} : h_m = 0.10, h_v = 0.50$	1.0004	1.0013	1.0261	1.0003	1.0012	1.0255
$V_{NP} : h_m = 0.25, h_v = 0.10$	0.9996	0.9997	0.9998	1.0041	1.0044	1.0036
$V_{NP} : h_m = 0.25, h_v = 0.25$	1.0000	1.0004	1.0034	1.0037	1.0042	0.9994
$V_{NP} : h_m = 0.25, h_v = 0.50$	0.9999	1.0009	1.0225	1.0030	1.0041	1.0067
$V_{NP} : h_m = 0.50, h_v = 0.10$	0.9996	0.9998	0.9997	1.4598	1.4676	1.4554
$V_{NP} : h_m = 0.50, h_v = 0.25$	1.0000	1.0004	1.0029	1.4312	1.4386	1.3603
$V_{NP} : h_m = 0.50, h_v = 0.50$	0.9999	1.0009	1.0214	1.2665	1.2721	1.1303
$\hat{V}_{OL}$	1.0053	1.0071	1.0068	1.0055	1.0073	1.0070
$\hat{V}_{NO}$	1.0087	1.0116	1.0113	1.0090	1.0119	1.0117
$\hat{V}_{SRS}$	17.157	17.389	17.353	14.417	14.600	14.5582

Table 25: Comparisons between MSPEs of with sorting variable $R^2 = 1$, $n = 500$ and regression model $R^2 = 0.75$.

Mean function	Linear			Quadr.		
Variance function	Const.	Linear	Quadr.	Const.	Linear	Quadr.
$\hat{V}_L$	0.9996	0.9996	0.9994	1.1376	1.1407	1.1412
$\hat{V}_L^{lin}$	0.9996	0.9996	0.9994	1.1376	1.1407	1.1412
$V_{NP}^{ho} : h_m = 0.25$	0.9996	0.9996	0.9998	0.9996	0.9997	0.9999
$V_{NP}^{ho} : h_m = 0.50$	0.9996	0.9996	0.9998	1.0041	1.0045	1.0052
$V_{NP} : h_m = 0.10, h_v = 0.10$	1.0001	1.0001	1.0003	1.0001	1.0001	1.0003
$V_{NP} : h_m = 0.10, h_v = 0.25$	1.0004	1.0008	1.0052	1.0004	1.0007	1.0051
$V_{NP} : h_m = 0.10, h_v = 0.50$	1.0003	1.0013	1.0262	1.0003	1.0012	1.0261
$V_{NP} : h_m = 0.25, h_v = 0.10$	0.9996	0.9998	0.9998	0.9996	0.9998	0.9997
$V_{NP} : h_m = 0.25, h_v = 0.25$	1.0000	1.0004	1.0034	1.0000	1.0004	1.0026
$V_{NP} : h_m = 0.25, h_v = 0.50$	0.9999	1.0009	1.0226	0.9998	1.0008	1.0206
$V_{NP} : h_m = 0.50, h_v = 0.10$	0.9997	0.9998	0.9997	1.0043	1.0047	1.0035
$V_{NP} : h_m = 0.50, h_v = 0.25$	1.0000	1.0004	1.0029	1.0043	1.0049	0.9994
$V_{NP} : h_m = 0.50, h_v = 0.50$	0.9999	1.0009	1.0215	1.0024	1.0036	1.0079
$\hat{V}_{OL}$	1.0053	1.0072	1.0069	1.0054	1.0072	1.0070
$\hat{V}_{NO}$	1.0088	1.0117	1.0115	1.0088	1.0118	1.0116
$\hat{V}_{SRS}$	1.1697	1.1716	1.1743	1.1418	1.1439	1.1449

Table 26: Comparisons between MSPEs of with sorting variable $R^2 = 1$, $n = 500$ and regression model $R^2 = 0.25$.

Mean function	Linear			Quadr.		
Variance function	Const.	Linear	Quadr.	Const.	Linear	Quadr.
$\hat{V}_L$	0.9407	0.9445	0.9554	68.138	65.756	66.590
$\hat{V}_L^{lin}$	0.9406	0.9445	0.9553	68.141	65.759	66.597
$V_{NP}^{ho} : h_m = 0.25$	0.9510	0.9527	0.9631	0.9345	0.9361	0.9542
$V_{NP}^{ho} : h_m = 0.50$	0.9429	0.9460	0.9585	2.9939	2.9169	3.0102
$V_{NP} : h_m = 0.10, h_v = 0.10$	1.0075	1.0084	1.0113	1.0080	1.0086	1.0119
$V_{NP} : h_m = 0.10, h_v = 0.25$	1.0115	1.0218	1.0631	1.0127	1.0237	1.0668
$V_{NP} : h_m = 0.10, h_v = 0.50$	1.0219	1.0466	1.2073	1.0225	1.0492	1.2170
$V_{NP} : h_m = 0.25, h_v = 0.10$	0.9529	0.9562	0.9671	0.9341	0.9369	0.9500
$V_{NP} : h_m = 0.25, h_v = 0.25$	0.9572	0.9705	0.9926	0.9370	0.9512	0.9430
$V_{NP} : h_m = 0.25, h_v = 0.50$	0.9680	0.9966	1.0963	0.9469	0.9779	0.9991
$V_{NP} : h_m = 0.50, h_v = 0.10$	0.9440	0.9492	0.9604	3.0453	2.9664	2.9893
$V_{NP} : h_m = 0.50, h_v = 0.25$	0.9476	0.9628	0.9765	2.8964	2.8369	2.5381
$V_{NP} : h_m = 0.50, h_v = 0.50$	0.9586	0.9896	1.0672	2.1133	2.1000	1.4528
$\hat{V}_{OL}$	1.0232	1.0446	1.0523	1.0210	1.0443	1.0556
$\hat{V}_{NO}$	1.0947	1.1410	1.1376	1.0970	1.1452	1.1410
$\hat{V}_{SRS}$	79.946	77.001	77.932	70.576	68.077	68.796

Table 27: Comparisons between MSPEs of with sorting variable $R^2 = 1$, $n = 100$ and regression model $R^2 = 0.75$.

Mean function	Linear			Quadr.		
Variance function	Const.	Linear	Quadr.	Const.	Linear	Quadr.
$\hat{V}_L$	0.9382	0.9429	0.9527	1.6121	1.5850	1.6005
$\hat{V}_L^{lin}$	0.9381	0.9429	0.9526	1.6120	1.5851	1.6010
$V_{NP}^{ho} : h_m = 0.25$	0.9486	0.9508	0.9608	0.9442	0.9466	0.9573
$V_{NP}^{ho} : h_m = 0.50$	0.9404	0.9442	0.9559	0.9398	0.9429	0.9604
$V_{NP} : h_m = 0.10, h_v = 0.10$	1.0079	1.0087	1.0119	1.0080	1.0086	1.0120
$V_{NP} : h_m = 0.10, h_v = 0.25$	1.0124	1.0234	1.0670	1.0127	1.0237	1.0677
$V_{NP} : h_m = 0.10, h_v = 0.50$	1.0225	1.0488	1.2180	1.0224	1.0491	1.2193
$V_{NP} : h_m = 0.25, h_v = 0.10$	0.9506	0.9543	0.9649	0.9460	0.9498	0.9607
$V_{NP} : h_m = 0.25, h_v = 0.25$	0.9555	0.9700	0.9924	0.9513	0.9660	0.9854
$V_{NP} : h_m = 0.25, h_v = 0.50$	0.9660	0.9968	1.1009	0.9616	0.9929	1.0896
$V_{NP} : h_m = 0.50, h_v = 0.10$	0.9416	0.9474	0.9580	0.9403	0.9451	0.9547
$V_{NP} : h_m = 0.50, h_v = 0.25$	0.9456	0.9624	0.9756	0.9423	0.9586	0.9385
$V_{NP} : h_m = 0.50, h_v = 0.50$	0.9563	0.9899	1.0703	0.9481	0.9818	0.9892
$\hat{V}_{OL}$	1.0173	1.0403	1.0508	1.0175	1.0411	1.0517
$\hat{V}_{NO}$	1.0906	1.1385	1.1349	1.0905	1.1386	1.1356
$\hat{V}_{SRS}$	1.8658	1.8293	1.8138	1.7202	1.6837	1.6891

Table 28: Comparisons between MSPEs of with sorting variable $R^2 = 1$, $n = 100$ and regression model $R^2 = 0.25$.

Mean function	Linear			Quadr.		
Variance function	Const.	Linear	Quadr.	Const.	Linear	Quadr.
$\hat{V}_L$	0.9960	0.9944	0.9987	4.8099	4.8297	4.7704
$\hat{V}_L^{lin}$	0.9960	0.9943	0.9987	4.8107	4.8299	4.7718
$V_{NP}^{ho} : h_m = 0.25$	0.9964	0.9954	0.9987	0.9942	0.9949	0.9927
$V_{NP}^{ho} : h_m = 0.50$	0.9960	0.9944	0.9985	1.0207	1.0197	1.0204
$V_{NP} : h_m = 0.10, h_v = 0.10$	1.0000	1.0005	1.0002	1.0002	1.0004	1.0003
$V_{NP} : h_m = 0.10, h_v = 0.25$	1.0002	1.0004	1.0025	1.0002	1.0000	1.0031
$V_{NP} : h_m = 0.10, h_v = 0.50$	1.0005	1.0010	1.0128	1.0004	1.0005	1.0144
$V_{NP} : h_m = 0.25, h_v = 0.10$	0.9964	0.9958	0.9988	0.9943	0.9952	0.9929
$V_{NP} : h_m = 0.25, h_v = 0.25$	0.9966	0.9957	1.0010	0.9946	0.9952	0.9976
$V_{NP} : h_m = 0.25, h_v = 0.50$	0.9968	0.9963	1.0110	0.9952	0.9962	1.0128
$V_{NP} : h_m = 0.50, h_v = 0.10$	0.9960	0.9949	0.9986	1.0233	1.0224	1.0200
$V_{NP} : h_m = 0.50, h_v = 0.25$	0.9962	0.9948	1.0006	1.0181	1.0171	1.0051
$V_{NP} : h_m = 0.50, h_v = 0.50$	0.9964	0.9954	1.0106	0.9966	0.9966	0.9914
$\hat{V}_{OL}$	1.0588	1.0560	1.0611	2.0024	2.0004	1.9920
$\hat{V}_{NO}$	1.0686	1.0657	1.0693	2.0322	2.0307	2.0190
$\hat{V}_{SRS}$	6.6464	6.6933	6.4948	4.8623	4.8820	4.8201

Table 29: Comparisons between MSPEs of with sorting variable $R^2 = 0.75$, $n = 500$ and regression model $R^2 = 0.75$.

Mean function	Linear			Quadr.		
Variance function	Const.	Linear	Quadr.	Const.	Linear	Quadr.
$\hat{V}_L$	0.9984	0.9980	0.9989	1.0703	1.0704	1.0694
$\hat{V}_L^{lin}$	0.9983	0.9977	0.9989	1.0703	1.0700	1.0696
$V_{NP}^{ho} : h_m = 0.25$	0.9986	0.9985	0.9994	0.9980	0.9984	0.9979
$V_{NP}^{ho} : h_m = 0.50$	0.9984	0.9980	0.9992	0.9977	0.9979	0.9978
$V_{NP} : h_m = 0.10, h_v = 0.10$	1.0002	1.0008	1.0005	1.0003	1.0008	1.0006
$V_{NP} : h_m = 0.10, h_v = 0.25$	1.0003	1.0006	1.0051	1.0003	1.0004	1.0053
$V_{NP} : h_m = 0.10, h_v = 0.50$	1.0006	1.0014	1.0248	1.0006	1.0012	1.0254
$V_{NP} : h_m = 0.25, h_v = 0.10$	0.9988	0.9992	0.9998	0.9983	0.9991	0.9983
$V_{NP} : h_m = 0.25, h_v = 0.25$	0.9990	0.9990	1.0041	0.9983	0.9988	1.0031
$V_{NP} : h_m = 0.25, h_v = 0.50$	0.9992	0.9998	1.0234	0.9986	0.9996	1.0238
$V_{NP} : h_m = 0.50, h_v = 0.10$	0.9986	0.9988	0.9996	0.9979	0.9986	0.9976
$V_{NP} : h_m = 0.50, h_v = 0.25$	0.9988	0.9986	1.0036	0.9979	0.9982	0.9993
$V_{NP} : h_m = 0.50, h_v = 0.50$	0.9990	0.9994	1.0227	0.9981	0.9990	1.0183
$\hat{V}_{OL}$	1.0052	1.0043	1.0061	1.0228	1.0220	1.0229
$\hat{V}_{NO}$	1.0089	1.0089	1.0092	1.0288	1.0285	1.0278
$\hat{V}_{SRS}$	1.1088	1.1083	1.1043	1.0734	1.0727	1.0721

Table 30: Comparisons between MSPEs of with sorting variable $R^2 = 0.75$, $n = 500$ and regression model $R^2 = 0.25$.

Mean function	Linear			Quadr.		
Variance function	Const.	Linear	Quadr.	Const.	Linear	Quadr.
$\hat{\mathrm{V}}_L$	0.9682	0.9623	0.9641	17.670	17.627	17.920
$\hat{\mathrm{V}}_L^{lin}$	0.9684	0.9609	0.9644	17.687	17.637	17.942
$\mathrm{V}_{NP}^{ho} : h_m = 0.25$	0.9740	0.9689	0.9721	0.9812	0.9843	1.0055
$\mathrm{V}_{NP}^{ho} : h_m = 0.50$	0.9701	0.9652	0.9656	1.0139	1.0198	1.0507
$\mathrm{V}_{NP} : h_m = 0.10, h_v = 0.10$	1.0118	1.0118	1.0108	1.0094	1.0096	1.0087
$\mathrm{V}_{NP} : h_m = 0.10, h_v = 0.25$	1.0078	1.0125	1.0253	1.0073	1.0111	1.0243
$\mathrm{V}_{NP} : h_m = 0.10, h_v = 0.50$	1.0086	1.0228	1.0814	1.0104	1.0206	1.0822
$\mathrm{V}_{NP} : h_m = 0.25, h_v = 0.10$	0.9829	0.9787	0.9798	0.9881	0.9904	1.0128
$\mathrm{V}_{NP} : h_m = 0.25, h_v = 0.25$	0.9804	0.9806	0.9905	0.9903	0.9966	1.0421
$\mathrm{V}_{NP} : h_m = 0.25, h_v = 0.50$	0.9820	0.9915	1.0432	0.9997	1.0126	1.1286
$\mathrm{V}_{NP} : h_m = 0.50, h_v = 0.10$	0.9788	0.9755	0.9725	1.0472	1.0515	1.0732
$\mathrm{V}_{NP} : h_m = 0.50, h_v = 0.25$	0.9759	0.9768	0.9800	1.0160	1.0252	1.0059
$\mathrm{V}_{NP} : h_m = 0.50, h_v = 0.50$	0.9776	0.9879	1.0294	0.9615	0.9778	0.9997
$\hat{\mathrm{V}}_{OL}$	1.6106	1.5705	1.6202	5.8889	5.8274	6.0001
$\hat{\mathrm{V}}_{NO}$	1.6869	1.6432	1.6914	5.8425	5.7817	5.8906
$\hat{\mathrm{V}}_{SRS}$	33.084	32.119	31.961	18.694	18.636	18.908

Table 31: Comparisons between MSPEs of with sorting variable $R^2 = 0.75$, $n = 100$ and regression model $R^2 = 0.75$.

Mean function	Linear			Quadr.		
Variance function	Const.	Linear	Quadr.	Const.	Linear	Quadr.
$\hat{V}_L$	0.9761	0.9770	0.9779	1.3124	1.3097	1.3014
$\hat{V}_L^{lin}$	0.9765	0.9745	0.9784	1.3132	1.3061	1.3033
$V_{NP}^{ho} : h_m = 0.25$	0.9832	0.9819	0.9856	0.9829	0.9839	0.9906
$V_{NP}^{ho} : h_m = 0.50$	0.9790	0.9786	0.9816	0.9710	0.9748	0.9822
$V_{NP} : h_m = 0.10, h_v = 0.10$	1.0193	1.0192	1.0175	1.0178	1.0180	1.0159
$V_{NP} : h_m = 0.10, h_v = 0.25$	1.0129	1.0216	1.0421	1.0124	1.0208	1.0400
$V_{NP} : h_m = 0.10, h_v = 0.50$	1.0161	1.0401	1.1427	1.0172	1.0393	1.1403
$V_{NP} : h_m = 0.25, h_v = 0.10$	0.9979	0.9975	0.9980	0.9967	0.9980	1.0021
$V_{NP} : h_m = 0.25, h_v = 0.25$	0.9940	1.0018	1.0163	0.9936	1.0029	1.0232
$V_{NP} : h_m = 0.25, h_v = 0.50$	0.9986	1.0216	1.1114	1.0007	1.0239	1.1246
$V_{NP} : h_m = 0.50, h_v = 0.10$	0.9932	0.9948	0.9927	0.9851	0.9894	0.9911
$V_{NP} : h_m = 0.50, h_v = 0.25$	0.9886	0.9981	1.0052	0.9792	0.9916	0.9926
$V_{NP} : h_m = 0.50, h_v = 0.50$	0.9934	1.0182	1.0944	0.9894	1.0155	1.0861
$\hat{V}_{OL}$	1.0946	1.1005	1.1087	1.1993	1.2007	1.2109
$\hat{V}_{NO}$	1.1709	1.1856	1.1871	1.2797	1.2939	1.2821
$\hat{V}_{SRS}$	1.6102	1.5852	1.5493	1.3854	1.3763	1.3625

Table 32: Comparisons between MSPEs of with sorting variable $R^2 = 0.75$, $n = 100$ and regression model $R^2 = 0.25$.

Mean function	Linear			Quadr.		
Variance function	Const.	Linear	Quadr.	Const.	Linear	Quadr.
$\hat{V}_L$	0.9836	0.9841	0.9929	3.8817	3.9367	3.9779
$\hat{V}_L^{lin}$	0.9836	0.9839	0.9928	3.8805	3.9356	3.9765
$V_{NP}^{ho} : h_m = 0.25$	0.9954	0.9933	0.9984	1.1077	1.1072	1.1163
$V_{NP}^{ho} : h_m = 0.50$	0.9893	0.9878	0.9967	2.3199	2.3436	2.3692
$V_{NP} : h_m = 0.10, h_v = 0.10$	0.9998	0.9997	0.9989	1.0000	0.9999	0.9998
$V_{NP} : h_m = 0.10, h_v = 0.25$	0.9998	0.9999	0.9998	1.0000	0.9999	1.0005
$V_{NP} : h_m = 0.10, h_v = 0.50$	1.0001	1.0003	1.0073	1.0000	1.0000	1.0026
$V_{NP} : h_m = 0.25, h_v = 0.10$	0.9953	0.9930	0.9973	1.1077	1.1071	1.1181
$V_{NP} : h_m = 0.25, h_v = 0.25$	0.9951	0.9932	0.9981	1.1086	1.1081	1.1274
$V_{NP} : h_m = 0.25, h_v = 0.50$	0.9955	0.9937	1.0056	1.1096	1.1091	1.1442
$V_{NP} : h_m = 0.50, h_v = 0.10$	0.9892	0.9876	0.9954	2.3115	2.3350	2.3670
$V_{NP} : h_m = 0.50, h_v = 0.25$	0.9891	0.9878	0.9963	2.3235	2.3472	2.4073
$V_{NP} : h_m = 0.50, h_v = 0.50$	0.9895	0.9883	1.0038	2.3984	2.4236	2.5374
$\hat{V}_{OL}$	2.7953	2.8376	2.8273	4.0687	4.1268	4.1707
$\hat{V}_{NO}$	2.7637	2.8091	2.7912	3.8865	3.9411	3.9838
$\hat{V}_{SRS}$	4.9478	5.0429	5.0024	3.8681	3.9228	3.9644

Table 33: Comparisons between MSPEs of with sorting variable $R^2 = 0.25$, $n = 500$ and regression model $R^2 = 0.75$.

Mean function	Linear			Quadr.		
Variance function	Const.	Linear	Quadr.	Const.	Linear	Quadr.
$\hat{V}_L$	0.9968	0.9967	0.9998	1.1828	1.1869	1.1946
$\hat{V}_L^{lin}$	0.9968	0.9963	0.9998	1.1827	1.1868	1.1945
$V_{NP}^{ho} : h_m = 0.25$	0.9989	0.9984	0.9998	1.0010	0.9994	1.0049
$V_{NP}^{ho} : h_m = 0.50$	0.9976	0.9973	1.0001	1.0773	1.0785	1.0864
$V_{NP} : h_m = 0.10, h_v = 0.10$	0.9998	0.9997	0.9983	0.9999	0.9997	0.9990
$V_{NP} : h_m = 0.10, h_v = 0.25$	0.9995	0.9999	1.0017	0.9998	0.9999	1.0004
$V_{NP} : h_m = 0.10, h_v = 0.50$	1.0005	1.0010	1.0205	1.0002	1.0003	1.0079
$V_{NP} : h_m = 0.25, h_v = 0.10$	0.9987	0.9982	0.9980	1.0009	0.9992	1.0053
$V_{NP} : h_m = 0.25, h_v = 0.25$	0.9984	0.9984	1.0012	1.0009	0.9995	1.0119
$V_{NP} : h_m = 0.25, h_v = 0.50$	0.9994	0.9995	1.0201	1.0014	1.0000	1.0290
$V_{NP} : h_m = 0.50, h_v = 0.10$	0.9975	0.9970	0.9982	1.0767	1.0778	1.0895
$V_{NP} : h_m = 0.50, h_v = 0.25$	0.9972	0.9972	1.0014	1.0775	1.0789	1.1104
$V_{NP} : h_m = 0.50, h_v = 0.50$	0.9981	0.9984	1.0204	1.0831	1.0847	1.1584
$\hat{V}_{OL}$	1.0409	1.0407	1.0424	1.1956	1.1991	1.2077
$\hat{V}_{NO}$	1.0429	1.0442	1.0428	1.1841	1.1875	1.1965
$\hat{V}_{SRS}$	1.0860	1.0868	1.0853	1.1798	1.1837	1.1920

Table 34: Comparisons between MSPEs of with sorting variable $R^2 = 0.25$, $n = 500$ and regression model $R^2 = 0.25$.

Mean function	Linear			Quadr.		
Variance function	Const.	Linear	Quadr.	Const.	Linear	Quadr.
$\hat{V}_L$	0.9691	0.9712	0.9638	1.9308	2.0803	2.1188
$\hat{V}_L^{lin}$	0.9688	0.9691	0.9636	1.9085	2.0521	2.0914
$V_{NP}^{ho} : h_m = 0.25$	0.9730	0.9764	0.9614	1.1149	1.1697	1.1874
$V_{NP}^{ho} : h_m = 0.50$	0.9703	0.9738	0.9602	2.8365	3.0546	3.1473
$V_{NP} : h_m = 0.10, h_v = 0.10$	1.0053	1.0038	1.0031	1.0027	1.0017	1.0021
$V_{NP} : h_m = 0.10, h_v = 0.25$	1.0024	1.0036	1.0072	1.0026	1.0028	1.0069
$V_{NP} : h_m = 0.10, h_v = 0.50$	1.0058	1.0114	1.0384	1.0019	1.0041	1.0210
$V_{NP} : h_m = 0.25, h_v = 0.10$	0.9776	0.9798	0.9627	1.1189	1.1716	1.1946
$V_{NP} : h_m = 0.25, h_v = 0.25$	0.9752	0.9800	0.9654	1.1241	1.1791	1.2371
$V_{NP} : h_m = 0.25, h_v = 0.50$	0.9785	0.9878	0.9951	1.1286	1.1859	1.3146
$V_{NP} : h_m = 0.50, h_v = 0.10$	0.9751	0.9779	0.9612	2.7748	2.9852	3.0973
$V_{NP} : h_m = 0.50, h_v = 0.25$	0.9723	0.9777	0.9624	2.8331	3.0496	3.2724
$V_{NP} : h_m = 0.50, h_v = 0.50$	0.9755	0.9853	0.9909	3.1509	3.3948	3.8597
$\hat{V}_{OL}$	3.9904	4.0662	3.9840	2.0904	2.2482	2.2903
$\hat{V}_{NO}$	3.8220	3.8876	3.8274	2.2820	2.4616	2.5006
$\hat{V}_{SRS}$	9.1338	9.2800	9.1638	1.8362	1.9774	2.0160

Table 35: Comparisons between MSPEs of with sorting variable $R^2 = 0.25$, $n = 100$ and regression model $R^2 = 0.75$.

Mean function	Linear			Quadr.		
Variance function	Const.	Linear	Quadr.	Const.	Linear	Quadr.
$\hat{V}_L$	0.9820	0.9866	0.9866	0.9776	1.0186	1.0241
$\hat{V}_L^{lin}$	0.9817	0.9790	0.9861	0.9769	1.0105	1.0219
$V_{NP}^{ho} : h_m = 0.25$	0.9843	0.9875	0.9853	0.9613	0.9855	0.9935
$V_{NP}^{ho} : h_m = 0.50$	0.9820	0.9860	0.9859	1.0231	1.0615	1.0820
$V_{NP} : h_m = 0.10, h_v = 0.10$	1.0145	1.0091	1.0058	1.0093	1.0053	1.0042
$V_{NP} : h_m = 0.10, h_v = 0.25$	1.0086	1.0102	1.0176	1.0074	1.0082	1.0140
$V_{NP} : h_m = 0.10, h_v = 0.50$	1.0157	1.0298	1.0984	1.0090	1.0190	1.0662
$V_{NP} : h_m = 0.25, h_v = 0.10$	0.9967	0.9953	0.9862	0.9722	0.9915	0.9992
$V_{NP} : h_m = 0.25, h_v = 0.25$	0.9921	0.9970	0.9934	0.9707	0.9951	1.0217
$V_{NP} : h_m = 0.25, h_v = 0.50$	0.9994	1.0171	1.0711	0.9728	1.0068	1.1014
$V_{NP} : h_m = 0.50, h_v = 0.10$	0.9950	0.9956	0.9860	1.0332	1.0672	1.0940
$V_{NP} : h_m = 0.50, h_v = 0.25$	0.9895	0.9962	0.9892	1.0352	1.0740	1.1548
$V_{NP} : h_m = 0.50, h_v = 0.50$	0.9966	1.0160	1.0637	1.0549	1.1031	1.3317
$\hat{V}_{OL}$	1.2041	1.2187	1.1865	1.0370	1.0772	1.0874
$\hat{V}_{NO}$	1.2954	1.3066	1.2741	1.1137	1.1630	1.1658
$\hat{V}_{SRS}$	1.2862	1.2918	1.2351	0.9624	1.0022	1.0099

Table 36: Comparisons between MSPEs of with sorting variable $R^2 = 0.25$, $n = 100$ and regression model $R^2 = 0.25$.

	$\bar{Y}_S$	$\hat{V}_{SRS}$	$\hat{V}_{ST}$	$\hat{V}_{NP0.5}$	$\hat{V}_{NP0.2}$	$\hat{V}_{NP0.1}$
BIOMASS	14.5	0.46	0.36	0.40	0.38	0.37
CRCOV	22.5	0.71	0.62	0.64	0.62	0.59
BA	48.5	3.87	3.19	3.40	3.30	3.12
NVOLTOT	906.9	1886	1538	1645	1584	1511
FOREST (%)	54.8	2.46	1.89	2.16	2.05	1.91

Table 37: Mean and variance estimates for the five response variables for FIA data, using estimators $\hat{V}_{SRS}$, $\hat{V}_{ST}$ and $\hat{V}_{NP}$ under model (14) with span = 0.5, 0.2 and 0.1.

	$\hat{V}_{NP0.5}$	$\hat{V}_{NP0.2}$	$\hat{V}_{NP0.1}$	$\hat{V}_{NP(0.1,0.3)}$
BIOMASS	0.36	0.34	0.33	0.34
CRCOV	0.59	0.55	0.53	0.55
BA	3.11	2.96	2.78	2.87
NVOLTOT	1487	1417	1342	1396
FOREST (%)	1.92	1.77	1.65	1.71

Table 38: Variance estimates for five response variables for FIA data, using nonparametric estimator for additive model (15) with same span used for both variables (span = 0.5, 0.2 and 0.1), and span 0.1 for location and 0.3 for elevation.

	$\hat{V}_{NP0.5}$	$\hat{V}_{NP0.2}$	$\hat{V}_{NP0.1}$
BIOMASS	0.40	0.38	0.36
CRCOV	0.64	0.62	0.58
BA	3.40	3.29	3.10
NVOLTOT	1643	1580	1507
FOREST (%)	2.15	2.04	1.89

Table 39: Variance component of variance estimates for five response variables. $\hat{V}_{NP}$ under model (14) is considered with span = 0.5, 0.2 and 0.1, respectively.

	$\hat{V}_{NP0.5}$	$\hat{V}_{NP0.2}$	$\hat{V}_{NP0.1}$	$\hat{V}_{NP(0.1,0.3)}$
BIOMASS	0.34	0.32	0.31	0.32
CRCOV	0.58	0.54	0.51	0.53
BA	3.04	2.85	2.67	2.75
NVOLTOT	1446	1350	1280	1324
FOREST (%)	1.57	1.70	1.57	1.62

Table 40: Variance component of variance estimates for five response variables. $\hat{V}_{NP}$ under model (15) is considered with same span for both **LOC** and ELEV. (span = 0.5, 0.2 and 0.1), and span 0.1 for location and 0.3 for elevation.

	$\bar{Y}_S$	$\hat{\mathrm{V}}_{SRS}$	$\hat{\mathrm{V}}_{ST}$	$\hat{\mathrm{V}}_{NP0.2}$
BIOMASS	14.5	0.40	0.36	0.38
CRCOV	22.5	0.64	0.61	0.62
BA	48.5	3.41	3.16	3.29
NVOLTOT	906.9	1635	1528	1580
FOREST (%)	54.8	2.11	1.87	2.04

Table 41: Model assisted mean and variance estimates for five response variables. Three variance estimators are considered: $\hat{\mathrm{V}}_{SRS}$, $\hat{\mathrm{V}}_{ST}$ and $\hat{\mathrm{V}}_{NP}$ under model (14) with span = 0.2.

	$\bar{Y}_S$	$\hat{V}_{SRS}$	$\hat{V}_{ST}$	$\hat{V}_{NP0.2}$
BIOMASS	14.39	0.33	0.33	0.32
CRCOV	22.4	0.56	0.56	0.54
BA	48.1	2.95	2.88	2.85
NVOLTOT	901.3	1397	1371	1350
FOREST (%)	54.6	1.76	1.67	1.70

Table 42: Model assisted mean and variance estimates for five response variables. Three variance estimators are considered: $\hat{V}_{SRS}$, $\hat{V}_{ST}$ and $\hat{V}_{NP}$ under model (15) with span = 0.2.

	$\hat{\text{V}}_{NP}$	$h_m = 0.5$	$h_m = 0.2$	$h_m = 0.1$
BIOMASS	$h_v = 0.2$	0.40	0.39	0.37
	$h_v = 0.4$	0.40	0.39	0.37
	$h_v = 0.6$	0.40	0.39	0.37
CRCOV	$h_v = 0.2$	0.64	0.62	0.59
	$h_v = 0.4$	0.64	0.62	0.59
	$h_v = 0.6$	0.65	0.62	0.59
BA	$h_v = 0.2$	3.41	3.31	3.12
	$h_v = 0.4$	3.41	3.31	3.13
	$h_v = 0.6$	3.41	3.31	3.13
NVOLTOT	$h_v = 0.2$	1653	1592	1519
	$h_v = 0.4$	1655	1594	1521
	$h_v = 0.6$	1653	1592	1519
FOREST (%)	$h_v = 0.2$	2.16	2.05	1.91
	$h_v = 0.4$	2.16	2.05	1.91
	$h_v = 0.6$	2.16	2.05	1.91

Table 43: Variance estimates for five response variables using $\hat{\text{V}}_{NP}$ under model (16) with span = 0.5, 0.2 and 0.1 for the regression function and span= 0.2, 0.4 and 0.6 for the variance function.

	$\hat{V}_{NP}$	$h_m = 0.5$	$h_m = 0.2$	$h_m = 0.1$	$h_{X,Y} = 0.1, h_{elev} = 0.3$
	$h_v = 0.2$	0.36	0.35	0.33	0.34
BIOMASS	$h_v = 0.4$	0.36	0.35	0.33	0.34
	$h_v = 0.6$	0.36	0.35	0.33	0.34
	$h_v = 0.2$	0.59	0.56	0.53	0.55
CRCOV	$h_v = 0.4$	0.59	0.55	0.53	0.55
	$h_v = 0.6$	0.59	0.56	0.53	0.55
	$h_v = 0.2$	3.11	2.96	2.79	2.89
BA	$h_v = 0.4$	3.11	2.96	2.78	2.87
	$h_v = 0.6$	3.11	2.96	2.78	2.87
	$h_v = 0.2$	1516	1448	1372	1425
NVOLTOT	$h_v = 0.4$	1504	1435	1358	1411
	$h_v = 0.6$	1499	1429	1353	1406
	$h_v = 0.2$	1.91	1.76	1.64	1.70
FOREST (%)	$h_v = 0.4$	1.91	1.76	1.64	1.70
	$h_v = 0.6$	1.92	1.77	1.64	1.70

Table 44: Variance estimates for five response variables using $\hat{V}_{NP}$ under model (17) with variance (18), with same span for location and elevation (0.5, 0.2 and 0.1) and also 0.1 for the location and 0.3 for the elevation, for the regression function, and span= 0.2, 0.4 and 0.6 for the variance function.

	$\hat{\text{V}}_{NP}$	$h_m = 0.5$	$h_m = 0.2$	$h_m = 0.1$	$h_{X,Y} = 0.1, h_{elev} = 0.3$
	$h_v = 0.2$	0.36	0.34	0.33	0.33
BIOMASS	$h_v = 0.4$	0.36	0.34	0.32	0.34
	$h_v = 0.6$	0.36	0.34	0.32	0.33
	$h_v = 0.2$	0.59	0.56	0.53	0.55
CRCOV	$h_v = 0.4$	0.59	0.56	0.53	0.55
	$h_v = 0.6$	0.59	0.56	0.53	0.55
	$h_v = 0.2$	3.11	2.96	2.78	2.87
BA	$h_v = 0.4$	3.10	2.95	2.77	2.87
	$h_v = 0.6$	3.10	2.95	2.78	2.87
	$h_v = 0.2$	1492	1427	1351	1404
NVOLTOT	$h_v = 0.4$	1487	1418	1342	1395
	$h_v = 0.6$	1486	1417	1341	1395
	$h_v = 0.2$	1.91	1.76	1.64	1.70
FOREST (%)	$h_v = 0.4$	1.92	1.77	1.65	1.70
	$h_v = 0.6$	1.92	1.77	1.65	1.71

Table 45: Variance estimates for five response variables using $\hat{\text{V}}_{NP}(\bar{Y}_S)$ under model (17) with variance (19), with same span for location and elevation (0.5, 0.2 and 0.1) and also 0.1 for the location and 0.3 for the elevation, for the regression function, and span= 0.2, 0.4 and 0.6 for the variance function.