Product-type and presmoothed hazard rate estimators with censored data

Ricardo Cao and Ignacio López-de-Ullibarri
Departamento de Matemáticas,
Universidade da Coruña, A Coruña 15071, Spain

February 20, 2004

Abstract

Two new classes of nonparametric hazard estimators for censored data are proposed in this paper. One is based in a formula that expresses the hazard rate of interest as a product of the hazard rate of the observable lifetime and the conditional probability of uncensoring. The second class follows presmoothing ideas already used by Cao, López de Ullibarri, Janssen and Veraverbeke (2003) for the cumulative hazard function and by Cao and Jácome (2004) for the density function. Asymptotic representations for some estimators in these classes are obtained and used to prove their limit distributions. Finally, a simulation study illustrates the comparative behaviour of the estimator studied along the paper.

Abbreviated Title: Product-type and presmoothed hazard rate estimators.
AMS 1991 Subject classifications. Primary: 62G07; secondary: 60F05, 62G20.
Key words and phrases: Bandwidth selection, censored data, kernel smoothing, mean integrated squared error.

1 Introduction and background

Let us consider a right random censorship model. There is a life time T_i (with continuous distribution function F) associated to the i-th individual of a total of n $(i = 1, 2, ..., n)$ and a censoring time C_i (with continuous distribution function G), mutually independent, that are not directly observable. The random vectors $(X_i, C_i), i = 1, 2, ..., n$ are assumed independent as well. The observed data are n pairs $(T_i, \delta_i), i = 1, 2, ..., n$, where $T_i = \min(X_i, C_i)$ and $\delta_i = \mathbf{1}(X_i \leq C_i)$. Thus, if the i-th subject fails then $\delta_i = 1$ and T_i corresponds to the life time, while, if it is censored, $\delta_i = 0$ and T_i is the censoring time. The common distribution of the T_i, $i = 1, 2, ..., n$ is denoted by H and it is immediate to check that

$$1 - H(t) = (1 - F(t))\,(1 - G(t))\,. \tag{1}$$

Along this paper the major interest will be on nonparametric hazard rate estimation. The hazard rate function may be defined as follows

$$\lambda_F(t) = \lim_{\Delta t \to 0+} \frac{P(t \le X < t + \Delta t \,|X \ge t\,)}{\Delta t}.$$

More intuitively, $\lambda_F(t)\Delta t$ is, for small Δt, an approximation of the probability that a subject that did not fail up before time t will fail in a time period of length Δt. Under our absolute continuity assumptions it is easy to check that

$$\lambda_F(t) = \frac{f(t)}{1 - F(t)}. \tag{2}$$

Similar formulas of course hold for the hazard rate of the censoring time, λ_F, and the hazard rate of the observable time, λ_H.

1.1 Product-type estimators

Using expressions (2) and (1), it is straightforward to prove

$$\lambda_F(t) = \lambda_H(t)p(t) \tag{3}$$

which gives some relation between the interest hazard rate, $\lambda_F(t)$, the hazard rate of the observable variable, $\lambda_H(t)$, and the conditional probability of uncensoring, $p(t)$. Equation (3) is extremely convenient for the estimation of $\lambda_F(t)$, since the two terms in the right hand side are populational curves that can be directly estimated without taking into account the censoring.

A new class of nonparametric estimators of $\lambda_F(t)$, can be obtained then by multiplying any nonparametric estimator of $\lambda_H(t)$ by any nonparametric estimator of $p(t)$:

$$\left\{\widehat{\lambda}_F(t) : \widehat{\lambda}_F(t) = \widehat{\lambda}_H(t)\widehat{p}(t), \text{ with } \widehat{\lambda}_H(t) \text{ and } \widehat{p}(t) \text{ estimators of } \lambda_H(t) \text{ and } p(t)\right\}$$

Along this paper this class will be restricted by requiring that the estimators of $\lambda_H(t)$ and $p(t)$ are of the same type. Consequently, we will restrict ourselves to the case of kernel type estimators $\widehat{\lambda}_H(t)$ and $\widehat{p}(t)$.

More specifically, we have considered for $\lambda_H(t)$ the Watson-Leadbetter estimator (see Watson and Leadbetter (1964)):

$$\widehat{\lambda}_{WL}(t) = \frac{1}{n}\sum_{i=1}^{n} \frac{K_{b_1}(t - T_i)}{1 - H_n(T_i) + \frac{1}{n}}$$

and Rice-Rosenblatt estimator:

$$\widehat{\lambda}_{RR}(t)=\frac{\frac{1}{n}\sum_{i=1}^{n}K_{b_1}\left(t-T_i\right)}{1-H_n(t)+\frac{1}{n}},$$

proposed by Rice and Rosenblatt (1976) and very similar to some other estimator previously studied by Murthy (1965).

For the estimation of $p(t)$, we have considered the well-known Nadaraya-Watson estimator (see Nadaraya (1964) and Watson (1964)):

$$\widehat{p}_{NW}(t)=\frac{\frac{1}{n}\sum_{i=1}^{n}K_{b_2}\left(t-T_i\right)\delta_i}{\frac{1}{n}\sum_{i=1}^{n}K_{b_2}\left(t-T_i\right)}$$

and the local linear estimator

$$\widehat{p}_{LL}(t)=\frac{\sum_{i=1}^{n}K_{b_2}(t-T_i)\left(s_{n,2}(t)-(t-T_i)s_{n,1}(t)\right)\delta_i}{\sum_{i=1}^{n}K_{b_2}(t-T_i)\left(s_{n,2}(t)-(t-T_i)s_{n,1}(t)\right)}$$

where, for $j=1,2$, the functions $s_{n,j}$ are defined by

$$s_{n,j}(t)=\sum_{i=1}^{n}K_{b_2}(t-T_i)(t-T_i)^j. \tag{4}$$

Details on the last estimator can be found in Fan and Gijbels (1996), among many other references.

By considering the combination between these two pairs of estimators, Table 1 shows the four product-type estimators that will be studied along this paper.

Table1. Product-type hazard rate estimators to be studied.

estimator of $\lambda_H(t)$	estimator of $p(t)$	estimator of $\lambda_F(t)$
$\widehat{\lambda}_{WL}(t)$	$\widehat{p}_{NW}(t)$	$\widehat{\lambda}_{WLNW}(t)=\widehat{\lambda}_{WL}(t)\widehat{p}_{NW}(t)$
$\widehat{\lambda}_{WL}(t)$	$\widehat{p}_{LL}(t)$	$\widehat{\lambda}_{WLLL}(t)=\widehat{\lambda}_{WL}(t)\widehat{p}_{LL}(t)$
$\widehat{\lambda}_{RR}(t)$	$\widehat{p}_{NW}(t)$	$\widehat{\lambda}_{RRNW}(t)=\widehat{\lambda}_{RR}(t)\widehat{p}_{NW}(t)$
$\widehat{\lambda}_{RR}(t)$	$\widehat{p}_{LL}(t)$	$\widehat{\lambda}_{RRLL}(t)=\widehat{\lambda}_{RR}(t)\widehat{p}_{LL}(t)$

It should be pointed out that any of these estimators needs of two different smoothing parameters, b_1 and b_2, each of these is used to estimate the corresponding factor in the right hand side of (3). On the other hand, the estimators $\widehat{\lambda}_{RRNW}(t)$ and $\widehat{\lambda}_{RRLL}(t)$ are discontinuous functions while $\widehat{\lambda}_{WLNW}(t)$ and $\widehat{\lambda}_{WLLL}(t)$ are continuous, provided that the kernel function is continuous too.

The estimator $\widehat{\lambda}_{RRNW}(t)$ is, in fact, a generalization of the estimator by Blum and Susarla (1980), $\widehat{\lambda}_{BS}(t)$, which can be recovered from $\widehat{\lambda}_{RRNW}(t)$ using the same value for b_1 and b_2.

1.2 Presmoothed Tanner-Wong estimators

This estimator will be motivated following the lines by Cao, López de Ullibarri, Janssen and Veraverbeke (2003), but starting from the hazard rate estimator proposed by Tanner and Wong (1983):

$$\widehat{\lambda}_{TW}(t) = \frac{1}{n}\sum_{i=1}^{n} \frac{K_{b_1}(t-T_i)\,\delta_i}{1-H_n(T_i)+\frac{1}{n}} = \sum_{i=1}^{n} \frac{K_{b_1}(t-T_i)\,\delta_i}{n-R_i+1}$$

which may be rewritten as

$$\widehat{\lambda}_{TW}(t) = \int K_{b_1}(t-x)\,d\Lambda_n^{NA}(x) = (K_{b_1} * \Lambda_n^{NA})(t),$$

i.e., the convolution of a rescaled kernel with the classical Nelson-Aalen estimator (see Nelson (1972) and Aalen (1978)):

$$\Lambda_n^{NA}(t) = \sum_{i:T_i \le t}^{n} \frac{\delta_i}{n-R_i+1}.$$

By using the presmoothed Nelson-Aalen estimator:

$$\Lambda_n^{P}(t) = \sum_{i:T_{(i)} \le t}^{n} \frac{\widehat{p}(T_{(i)})}{n-i+1},$$

proposed by Cao, López de Ullibarri, Janssen and Veraverbeke (2003), instead of the Nelson-Aalen estimator, easy calculations give

$$(K_{b_1} * \Lambda_n^{P})(t) = \frac{1}{n}\sum_{i=1}^{n} \frac{K_{b_1}(t-T_i)\,\widehat{p}(T_i)}{1-H_n(T_i)+\frac{1}{n}}$$

which leads to the class of presmoothed Tanner-Wong estimators depending on the particular choice for the estimator $\widehat{p}$. The only estimator within this class that will be studied in this paper is the one which estimates $p(\cdot)$ by means of the Nadaraya-Watson estimator, $\widehat{p}_{NW}(\cdot)$, with smoothing parameter b_2 (which may be different of b_1). However, for some technical conditions a slightly modified estimator will be considered.

For a given t, we consider two constants, c_1 and c_2, such that $0 < c_1 < t < c_2$, the density h is bounded away from zero in $I = (c_1, c_2)$ and $H(c_2) < 1$. Now, the estimator $\widetilde{p}(\cdot)$ is defined as

$$\widetilde{p}(u) = \begin{cases} 0 & \text{, if } u \notin I \text{ and } u < T_{(1)} \\ \delta_{[i]} & \text{, if } u \notin I \text{ and } u \in \left[T_{(i)}, T_{(i+1)}\right) \\ \delta_{[n]} & \text{, if } u \notin I \text{ and } u \geq T_{(n)} \\ \widehat{p}_{NW}(u) = \frac{\frac{1}{n}\sum\limits_{i=1}^{n} K_{b_2}(u-T_i)\delta_i}{\frac{1}{n}\sum\limits_{i=1}^{n} K_{b_2}(u-T_i)} & \text{, if } u \in I \end{cases} \tag{5}$$

where $\delta_{[i]}$ is the concomitant of the i-th ordered statistic $T_{(i)}$.

From now on, the term presmoothed Tanner-Wong estimator will be used for estimator:

$$\widetilde{\lambda}_{TWP}(t) = \frac{1}{n}\sum_{i=1}^{n} \frac{K_{b_1}(t - T_i)\,\widetilde{p}(T_i)}{1 - H_n(T_i) + \frac{1}{n}}. \tag{6}$$

Using the definition (5) we have

$$\widetilde{p}(T_i) = \begin{cases} \delta_i & \text{, if } T_i \notin I \\ \widehat{p}_{NW}(T_i) & \text{, if } T_i \in I \end{cases},$$

which leads to an alternative expression for the estimator in (6),

$$\widetilde{\lambda}_{TWP}(t) = \frac{1}{n}\sum_{i=1}^{n} \frac{K_{b_1}(t - T_i)\,\widehat{p}_{NW}(T_i)\mathbf{1}(T_i \in I)}{1 - H_n(T_i) + \frac{1}{n}} + \frac{1}{n}\sum_{i=1}^{n} \frac{K_{b_1}(t - T_i)\,\delta_i\mathbf{1}(T_i \notin I)}{1 - H_n(T_i) + \frac{1}{n}}. \tag{7}$$

It should be noted that the presmoothed estimator $\widetilde{\lambda}_{TWP}(t)$ depends on two smoothing parameters, b_1 and b_2. In the limit case, when b_2 tends to zero, the Tanner-Wong estimator, $\widehat{\lambda}_{TW}(t)$, is recovered.

It is clear that similar ideas may be used to define other presmoothed estimators for the hazard rate. For instance, starting from the Blum-Susarla estimator, $\widehat{\lambda}_{BS}(t)$, and replacing the uncensoring indicators by some estimator of the conditional probability of uncensoring results in the presmoothed Blum-Susarla estimator, defined by

$$\widetilde{\lambda}_{BSP}(t) = \frac{\frac{1}{n}\sum\limits_{i=1}^{n} K_{b_1}(t - T_i)\,\widetilde{p}(T_i)}{1 - H_n(t) + \frac{1}{n}}.$$

Although $\widetilde{\lambda}_{BSP}(t)$ could be studied using similar arguments as for $\widetilde{\lambda}_{TWP}(t)$, it will not be considered in this paper.

2 Asymptotic representations

The following assumptions will be needed in the rest of the paper.
(K.1) K is a nonnegative, symmetric, continuous function with support in the interval $[-L, L]$, for some $L > 0$ and $\int_{-L}^{L} K(x)dx = 1$.
(P.1) p is three times differentiable in $[0, \infty)$, with bounded third derivative.
(H.1) There exists some $t_0 > 0$ such that $H(t_0) < 1$ and H is four times differentiable in $[0, t_0]$, with bounded fourth derivative. There also exist some $\delta > 0$ and ε, $0 < \varepsilon < t_0$,such that $H'(t) = h(t) > \delta$, $\forall t \in [\varepsilon, t_0]$.
(V.1) $b_1 \to 0, nb_1^4 \to \infty$ and $nb_1^5 = O(1)$, when $n \to \infty$.
(V.2) The bandwidth sequences b_1 and b_2 are of the same asymptotic order, i.e.,

$$\frac{b_1}{b_2} = a + o(1), \qquad \text{for some } a \in (0, \infty).$$

We also assume, in the following, that $t \in (\varepsilon, t_0)$.

The next six lemmas are needed to obtain asymptotic representations for the estimators $\widehat{\lambda}_{WLNW}(t)$, $\widehat{\lambda}_{RRNW}(t)$, $\widehat{\lambda}_{WLLL}(t)$, $\widehat{\lambda}_{RRLL}(t)$ and $\widetilde{\lambda}_{TWP}(t)$.

Lemma 1 *Under conditions* ***(K.1)***, ***(H.1)*** *and* ***(V.1)***,

$$\widehat{\lambda}_{WL}(t) - \lambda_H(t) = A_n + o_P\left(n^{-1/2} b_1^{-1/2}\right), \tag{8}$$

where

$$A_n = \frac{1}{n}\sum_{i=1}^{n} \frac{K_{b_1}(t - T_i)}{1 - H(T_i) + \frac{1}{n}} - \lambda_H(t). \tag{9}$$

On the other hand

$$E\left[A_n\right] = \frac{1}{2} b_1^2 \mu_K \lambda_H''(t) + o\left(b_1^2\right) \tag{10}$$

and

$$Var\left[A_n\right] = \frac{c_K}{nb_1} \frac{\lambda_H(t)}{1 - H(t)} + o\left(n^{-1} b_1^{-1}\right). \tag{11}$$

Proof. Let us consider the following linearization of $\widehat{\lambda}_{WL}(t)$,

$$\widehat{\lambda}_{WL}(t) = \frac{1}{n}\sum_{i=1}^{n} \frac{K_{b_1}(t - T_i)}{1 - H(T_i) + \frac{1}{n}} + \frac{1}{n}\sum_{i=1}^{n} \frac{K_{b_1}(t - T_i)\left(H_n(T_i) - H(T_i)\right)}{\left(1 - H_n(T_i) + \frac{1}{n}\right)\left(1 - H(T_i) + \frac{1}{n}\right)}. \tag{12}$$

Since, with probability 1,

$$\begin{aligned} 1 - H_n(T_i) + \frac{1}{n} &= 1 - H(T_i) + \frac{1}{n} - \left(H_n(T_i) - H(T_i)\right) \\ &\geq 1 - H(T_i) + \frac{1}{n} - \left|H_n(T_i) - H(T_i)\right| \\ &\geq 1 - H(T_i) + \frac{1}{n} - \left\|H_n - H\right\|_\infty \end{aligned}$$

and

$$\|H_n - H\|_\infty = O\left(\left(\frac{\log\log n}{n}\right)^{1/2}\right) \qquad a.s.,$$

then, for some n on,

$$1 - H_n(T_i) + \frac{1}{n} \geq \frac{1}{2}\left(1 - H(T_i) + \frac{1}{n}\right) \qquad a.s.$$

On the other hand, it is well-known that

$$\|H_n - H\|_\infty = O_P\left(n^{-1/2}\right). \tag{13}$$

Consequently, the second summand in (12) satisfies

$$\begin{aligned}\left|\frac{1}{n}\sum_{i=1}^{n}\frac{K_{b_1}(t-T_i)\left(H_n(T_i)-H(T_i)\right)}{\left(1-H_n(T_i)+\frac{1}{n}\right)\left(1-H(T_i)+\frac{1}{n}\right)}\right| &\leq \left(\frac{1}{2n}\sum_{i=1}^{n}\frac{K_{b_1}(t-T_i)}{\left(1-H(T_i)+\frac{1}{n}\right)^2}\right) \\ &\quad \times O_P\left(n^{-1/2}\right).\end{aligned} \tag{14}$$

We now study the expectation and variance of $\frac{1}{n}\sum_{i=1}^{n}\frac{K_{b_1}(t-T_i)}{\left(1-H(T_i)+\frac{1}{n}\right)^l}$, for $l \in \mathbb{N}$. First

$$\begin{aligned}E\left[\frac{1}{n}\sum_{i=1}^{n}\frac{K_{b_1}(t-T_i)}{\left(1-H(T_i)+\frac{1}{n}\right)^l}\right] &= E\left[\frac{K_{b_1}(t-T_1)}{\left(1-H(T_1)+\frac{1}{n}\right)^l}\right] \\ &= \int_0^\infty \frac{K_{b_1}(t-x)\,h(x)}{\left(1-H(x)+\frac{1}{n}\right)^l}dx \\ &= \int_{-L}^{\frac{t}{b_1}} \frac{K(x_1)\,h(t-b_1x_1)}{\left(1-H(t-b_1x_1)+\frac{1}{n}\right)^l}dx_1\end{aligned}$$

after using the change of variable $\frac{t-x}{b_1} = x_1$. For some n on, $b_1 < \frac{t}{L}$. Thus,

$$E\left[\frac{1}{n}\sum_{i=1}^{n}\frac{K_{b_1}(t-T_i)}{\left(1-H(T_i)+\frac{1}{n}\right)^l}\right] = \int_{-L}^{L} \frac{K(x_1)\,h(t-b_1x_1)}{\left(1-H(t-b_1x_1)+\frac{1}{n}\right)^l}dx_1.$$

Using that for all u such that $H(u) < 1$

$$\begin{aligned}\frac{1}{\left(1-H(u)+\frac{1}{n}\right)^l} &= \frac{1}{\left(1-H(u)\right)^l} - \frac{\left(1-H(u)+\frac{1}{n}\right)^l - \left(1-H(u)\right)^l}{\left(1-H(u)\right)^l\left(1-H(u)+\frac{1}{n}\right)^l} \\ &= \frac{1}{\left(1-H(u)\right)^l} + O(n^{-1}),\end{aligned} \tag{15}$$

which may be proved using a Taylor expansion of $f(x_0+x) = (x_0+x)^l$ around $x_0 = 1 - H(u)$, and defining, for $l \in \mathbb{N}$

$$\varphi_l(x) = \frac{h(x)}{(1-H(x))^l}, \tag{16}$$

we have that, for $b_1 < \frac{t \wedge (t_0 - t)}{L}$,

$$E\left[\frac{1}{n}\sum_{i=1}^{n}\frac{K_{b_1}(t-T_i)}{\left(1-H(T_i)+\frac{1}{n}\right)^l}\right] = \int_{-L}^{L} K(x_1)\,\varphi_l(t-b_1x_1)dx_1 + O(n^{-1}).$$

A Taylor expansion of $\varphi_l(t-b_1x_1)$ around t gives

$$E\left[\frac{1}{n}\sum_{i=1}^{n}\frac{K_{b_1}(t-T_i)}{\left(1-H(T_i)+\frac{1}{n}\right)^l}\right] = \varphi_l(t) + \frac{1}{2}b_1^2\mu_K\varphi_l''(t) + o\left(b_1^2\right) + O(n^{-1}). \tag{17}$$

For the variance,

$$\begin{aligned}
Var\left[\frac{1}{n}\sum_{i=1}^{n}\frac{K_{b_1}(t-T_i)}{\left(1-H(T_i)+\frac{1}{n}\right)^l}\right] &= \frac{1}{n}Var\left[\frac{K_{b_1}(t-T_1)}{\left(1-H(T_1)+\frac{1}{n}\right)^l}\right] \\
&= \frac{1}{n}E\left[\left(\frac{K_{b_1}(t-T_1)}{\left(1-H(T_1)+\frac{1}{n}\right)^l}\right)^2\right] \\
&\quad -\frac{1}{n}\left(E\left[\frac{K_{b_1}(t-T_1)}{\left(1-H(T_1)+\frac{1}{n}\right)^l}\right]\right)^2
\end{aligned}$$

and

$$\begin{aligned}
E\left[\left(\frac{K_{b_1}(t-T_1)}{\left(1-H(T_1)+\frac{1}{n}\right)^l}\right)^2\right] &= \int_0^\infty \frac{K_{b_1}(t-x)^2\,h(x)}{\left(1-H(x)+\frac{1}{n}\right)^{2l}}dx \\
&= \frac{1}{b_1}\int_{-L}^{L}\frac{K(x_1)^2\,h(t-b_1x_1)}{\left(1-H(t-b_1x_1)+\frac{1}{n}\right)^{2l}}dx_1
\end{aligned}$$

after using the change of variable $\frac{t-x}{b_1} = x_1$ and considering $b_1 < \frac{t}{L}$. Equation (15), the definition of φ_l and a Taylor expansion give

$$\begin{aligned}
\frac{1}{b_1}\int_{-L}^{L}\frac{K(x_1)^2\,h(t-b_1x_1)}{\left(1-H(t-b_1x_1)+\frac{1}{n}\right)^{2l}}dx_1 &= \frac{1}{b_1}\int_{-L}^{L}K(x_1)^2\,\varphi_{2l}(t-b_1x_1)dx_1 \\
&\quad + O\left(n^{-1}b_1^{-1}\right) \\
&= \frac{c_K\varphi_{2l}(t)}{b_1} + O\left(b_1 + n^{-1}b_1^{-1}\right).
\end{aligned} \tag{18}$$

From (17) and (18) we conclude

$$Var\left[\frac{1}{n}\sum_{i=1}^{n}\frac{K_{b_1}(t-T_i)}{\left(1-H(T_i)+\frac{1}{n}\right)^l}\right] = \frac{c_K\varphi_{2l}(t)}{nb_1}+O\left(n^{-1}b_1+n^{-2}b_1^{-1}\right) \quad (19)$$
$$= \frac{c_K\varphi_{2l}(t)}{nb_1}+o\left(n^{-1}b_1^{-1}\right).$$

Consequently, applying Tchebychev inequality,

$$\frac{1}{n}\sum_{i=1}^{n}\frac{K_{b_1}(t-T_i)}{\left(1-H(T_i)+\frac{1}{n}\right)^l}=O(1)+O_P\left(n^{-1/2}b_1^{-1/2}\right)=O_P(1) \quad (20)$$

setting $l=2$ and recalling (14) and (12), it is immediate to obtain (8).

Finally (10) and (11) are consequence of (17) and (19) for $l=1$, **(V.1)**, $\varphi_1=\lambda_H$ and $\varphi_2=\frac{\lambda_H}{1-H}$, where the last two facts are obtained from (16). ■

Lemma 2 *Under conditions **(K.1)**, **(H.1)** and **(V.1)***

$$\widehat{\lambda}_{RR}(t)-\lambda_H(t)=B_n+o_P\left(n^{-1/2}b_1^{-1/2}\right) \quad (21)$$

where

$$B_n=\frac{\widehat{h}(t)}{1-H(t)}-\lambda_H(t). \quad (22)$$

The expected value of the dominant term is,

$$E\left[B_n\right]=\frac{1}{2}b_1^2\mu_K\frac{h''(t)}{1-H(t)}+o\left(b_1^2\right) \quad (23)$$

while its variance

$$Var\left[B_n\right]=\frac{1}{nb_1}\frac{c_Kh(t)}{\left(1-H(t)\right)^2}+o\left(n^{-1}b_1^{-1}\right). \quad (24)$$

Proof. Arguing for $\widehat{\lambda}_{RR}(t)$ as done for $\widehat{\lambda}_{WL}(t)$,

$$\widehat{\lambda}_{RR}(t)=\frac{\widehat{h}(t)}{1-H(t)}\left(1+\frac{H_n(t)-H(t)-\frac{1}{n}}{1-H_n(t)+\frac{1}{n}}\right)$$

where $\widehat{h}(t)$ is the Parzen-Rosenblatt estimator of the density of T,

$$\widehat{h}(t)=\frac{1}{n}\sum_{i=1}^{n}K_{b_1}(t-T_i).$$

It is well-known that

$$H_n(t)-H(t)=O_P\left(n^{-1/2}\right)$$

and

$$H_n(t) \stackrel{a.s.}{\to} H(t),$$

which leads to

$$\widehat{\lambda}_{RR}(t) = \frac{\widehat{h}(t)}{1 - H(t)} \left(1 + O_P\left(n^{-1/2}\right)\right)$$

and then to the asymptotic representation (21).

Expressions (23) and (24) are then straight forward consequences of well-known properties of the Parzen-Rosenblatt estimator. ■

Lemma 3 *Conditions **(K.1)**, **(P.1)**, **(H.1)**, **(V.1)** and **(V.2)** imply*

$$\widehat{p}_{NW}(t) - p(t) = C_n \left(1 + o_P(1)\right) \tag{25}$$

where

$$C_n = \frac{1}{nh(t)} \sum_{i=1}^{n} K_{b_2}\left(t - T_i\right)\left(\delta_i - p(t)\right). \tag{26}$$

Its expectation is

$$E\left[C_n\right] = b_2^2 \mu_K \left(\frac{1}{2} p''(t) + \frac{h'(t)p'(t)}{h(t)}\right) + o\left(b_2^2\right) \tag{27}$$

and its variance

$$Var\left[C_n\right] = \frac{1}{nb_2} \frac{c_K p(t)(1-p(t))}{h(t)} + o\left(n^{-1} b_2^{-1}\right). \tag{28}$$

Proof. Let's define $\psi(t) = p(t)h(t)$, $\widehat{\psi}(t) = \frac{1}{n}\sum_{i=1}^{n} K_{b_2}\left(t - T_i\right)\delta_i$ and let's denote by $\widehat{h}(t)$ the Parzen-Rosenblatt estimator of $h(t)$. Then

$$\begin{aligned}
\widehat{p}_{NW}(t) - p(t) &= \frac{\widehat{\psi}(t)}{\widehat{h}(t)} - \frac{\psi(t)}{h(t)} = \frac{1}{h(t)}\left(\widehat{\psi}(t) - \psi(t) - \frac{\widehat{\psi}(t)(\widehat{h}(t) - h(t))}{\widehat{h}(t)}\right) \\
&= \frac{\widehat{\psi}(t) - \widehat{h}(t)p(t)}{h(t)} - \frac{\widehat{\psi}(t) - \widehat{h}(t)p(t)}{h(t)} \frac{\widehat{h}(t) - h(t)}{\widehat{h}(t)} \\
&= \frac{\widehat{\psi}(t) - \widehat{h}(t)p(t)}{h(t)} \left(1 - \frac{\widehat{h}(t) - h(t)}{\widehat{h}(t)}\right) \\
&= C_n \left(1 - \frac{\widehat{h}(t) - h(t)}{\widehat{h}(t)}\right).
\end{aligned}$$

Equation (25) is a straight forward consequence of

$$\widehat{h}(t) - h(t) = O_P\left(b_2^2 + n^{-1/2} b_2^{-1/2}\right),$$

conditions **(V.1)**, **(V.2)** and $h(t) > 0$.

Equation (27) for the expected value of C_n is easy to prove by Taylor expansions, since, for $b_2 < \frac{t}{L}$,

$$\begin{aligned} E[C_n] &= \frac{1}{h(t)} E(K_{b_2}(t - T_1)(\delta_1 - p(t))) \\ &= \frac{1}{h(t)} \int_0^\infty K_{b_2}(t - x)(p(x) - p(t))h(x)dx \\ &= \frac{1}{h(t)} \int_{-L}^{L} K(x_1)(p(t - b_2 x_1) - p(t))h(t - b_2 x_1)dx_1 \\ &= b_2^2 \mu_K \left(\frac{1}{2} p''(t) + \frac{h'(t)p'(t)}{h(t)} \right) + o\left(b_2^2\right). \end{aligned}$$

Similarly,

$$\begin{aligned} Var[C_n] &= \frac{1}{nh^2(t)} Var(K_{b_2}(t - T_1)(\delta_1 - p(t))) \\ &= \frac{1}{nh^2(t)} E\left((K_{b_2}(t - T_1)(\delta_1 - p(t)))^2 \right) \\ &\quad - \frac{1}{nh^2(t)} E^2(K_{b_2}(t - T_1)(\delta_1 - p(t))). \end{aligned}$$

Further Taylor expansions give

$$\begin{aligned} E\left[(K_{b_2}(t - T_1)(\delta_1 - p(t)))^2 \right] &= \int_0^\infty K_{b_2}(t - x)^2 \left(p(x)(1 - 2p(t)) + p(t)^2 \right) h(x)dx \\ &= \frac{1}{b_2} \int_{-L}^{L} K(x_1)^2 \left(p(t - b_2 x_1)(1 - 2p(t)) + p(t)^2 \right) h(t - b_2 x_1)dx_1 \\ &= \frac{1}{b_2} \int_{-L}^{L} K(x_1)^2 dx_1 p(t)(1 - p(t))h(t) + o\left(b_2^{-1}\right). \end{aligned}$$

Now, using (27), the final result (28) follows. ∎

The order of the covariance between A_n and C_n is stated in the following lemma.

Lemma 4 *Under conditions **(K.1)**, **(P.1)**, **(H.1)**, **(V.1)** and **(V.2)**,*

$$Cov[A_n, C_n] = o\left(n^{-1}\right). \tag{29}$$

Proof. Direct calculations show

$$
\begin{aligned}
Cov\left[A_n, C_n\right] = Cov &\left[\frac{1}{n}\sum_{i=1}^{n}\frac{K_{b_1}\left(t-T_i\right)}{1-H(T_i)+\frac{1}{n}} - \lambda_H(t), \frac{1}{nh(t)}\sum_{i=1}^{n}K_{b_2}\left(t-T_i\right)\left(\delta_i - p(t)\right)\right] \\
&= \frac{1}{nh(t)}Cov\left[\frac{K_{b_1}\left(t-T_1\right)}{1-H(T_1)+\frac{1}{n}}, K_{b_2}\left(t-T_1\right)\left(\delta_1-p(t)\right)\right] \\
&= \frac{1}{nh(t)}E\left[\frac{K_{b_1}\left(t-T_1\right)K_{b_2}\left(t-T_1\right)\left(\delta_1-p(t)\right)}{1-H(T_1)+\frac{1}{n}}\right] \\
&- \frac{1}{nh(t)}E\left[\frac{K_{b_1}\left(t-T_1\right)}{1-H(T_1)+\frac{1}{n}}\right]E\left[K_{b_2}\left(t-T_1\right)\left(\delta_1-p(t)\right)\right]. \qquad (30)
\end{aligned}
$$

The expectation of the first summand is

$$
\begin{aligned}
&E\left[\frac{K_{b_1}\left(t-T_1\right)K_{b_2}\left(t-T_1\right)\left(\delta_1-p(t)\right)}{1-H(T_1)+\frac{1}{n}}\right] \\
= &\int_0^\infty \frac{K_{b_1}\left(t-x\right)K_{b_2}\left(t-x\right)\left(p(x)-p(t)\right)h(x)}{1-H(x)+\frac{1}{n}}dx \\
= &\frac{1}{b_2}\int_{-L}^{L}\frac{K\left(x_1\right)K\left(\frac{x_1 b_1}{b_2}\right)\left(p(t-b_1x_1)-p(t)\right)h(t-b_1x_1)}{1-H(t-b_1x_1)+\frac{1}{n}}dx_1
\end{aligned}
$$

after using the change of variable $\frac{t-x}{b_1} = x_1$ and $b_1 < \frac{t}{L}$. Equation (15), the definition (16)and $b_1 < \frac{t\wedge(t_0-t)}{L}$, give

$$
\begin{aligned}
&E\left[\frac{K_{b_1}\left(t-T_1\right)K_{b_2}\left(t-T_1\right)\left(\delta_1-p(t)\right)}{1-H(T_1)+\frac{1}{n}}\right] \\
= &\frac{1}{b_2}\int_{-L}^{L}K\left(x_1\right)K\left(\frac{x_1b_1}{b_2}\right)\left(p(t-b_1x_1)-p(t)\right)\varphi_1(t-b_1x_1)dx_1 + O\left(n^{-1}b_2^{-1}\right).
\end{aligned}
$$

Taylor expansions of $p(t-b_1x_1)$ and $\varphi_1(t-b_1x_1)$ around t can be used to prove

$$
\begin{aligned}
&E\left[\frac{K_{b_1}\left(t-T_1\right)K_{b_2}\left(t-T_1\right)\left(\delta_1-p(t)\right)}{1-H(T_1)+\frac{1}{n}}\right] \\
= &-\frac{b_1}{b_2}p'(t)\varphi_1(t)\int_{-L}^{L}K\left(x_1\right)K\left(\frac{x_1b_1}{b_2}\right)x_1dx_1 + o\left(\frac{b_1}{b_2}\right) + O\left(n^{-1}b_2^{-1}\right) \\
= &\ o\left(\frac{b_1}{b_2}\right),
\end{aligned}
$$

since, as a consequence of **(V.1)**, $nb_1 \to \infty$. Thus, condition **(V.2)** implies

$$
E\left[\frac{K_{b_1}\left(t-T_1\right)K_{b_2}\left(t-T_1\right)\left(\delta_1-p(t)\right)}{1-H(T_1)+\frac{1}{n}}\right] = o\left(1\right).
$$

Using this result in (30), $E\left[\frac{K_{b_1}(t-T_1)}{1-H(T_1)+\frac{1}{n}}\right] = O(1)$ and $E\left[K_{b_2}\left(t-T_1\right)\left(\delta_1-p(t)\right)\right] = o\left(b_2^2\right)$ it is not difficult to prove (29). ■

Lemma 5 *Conditions **(K.1)**, **(P.1)**, **(H.1)**, **(V.1)** and **(V.2)** imply*

$$\widehat{p}_{LL}(t) - p(t) = D_n \left(1 + o_P(1)\right) \tag{31}$$

where

$$D_n = \frac{\frac{1}{n^2 b_2^2} \sum_{i=1}^{n} w_i(t) \left(\delta_i - p(t)\right)}{\mu_K h(t)^2}, \tag{32}$$

$$w_i(t) = K_{b_2}(t - T_i) \left(s_{n,2}(t) - (t - T_i) s_{n,1}(t)\right) \tag{33}$$

and $s_{n,j}, j = 1, 2$, where defined in (4).

On the other hand,

$$E\left[D_n\right] = \frac{1}{2} b_2^2 \mu_K p''(t) + o\left(b_2^2\right) \tag{34}$$

and

$$Var\left[D_n\right] = \left(\frac{1}{nb_2} \frac{c_K p(t)(1 - p(t))}{h(t)}\right) + o\left(n^{-1} b_2^{-1}\right). \tag{35}$$

Proof. To obtain an asymptotic representation of

$$\widehat{p}_{LL}(t) = \frac{\sum_{i=1}^{n} w_i(t) \delta_i}{\sum_{i=1}^{n} w_i(t)}$$

with a linearized dominant term, it is necessary to study the expectation and variance of the denominator. Thus

$$E\left[\sum_{i=1}^{n} w_i(t)\right] = nE\left[w_1(t)\right]$$

and, (33) and (4), imply

$$\begin{aligned} w_i(t) &= K_{b_2}(t - T_i) \left(\sum_{j=1}^{n} K_{b_2}(t - T_j)(t - T_j)^2 - (t - T_i) \sum_{j=1}^{n} K_{b_2}(t - T_j)(t - T_j)\right) \\ &= K_{b_2}(t - T_i) \sum_{j=1}^{n} \left(K_{b_2}(t - T_j)(t - T_j)^2 - K_{b_2}(t - T_j)(t - T_i)(t - T_j)\right) \\ &= \sum_{j=1}^{n} K_{b_2}(t - T_i) K_{b_2}(t - T_j)(t - T_j)(T_i - T_j) \end{aligned} \tag{36}$$

and

$$E\left[w_1(t)\right] = (n-1)E\left[K_{b_2}(t-T_1)K_{b_2}(t-T_2)(t-T_2)(T_1-T_2)\right] = (n-1)M_1$$

where

$$M_1 = E\left[K_{b_2}(t-T_1)K_{b_2}(t-T_2)(t-T_2)(T_1-T_2)\right].$$

Now,

$$\begin{aligned} M_1 &= \int_0^\infty \int_0^\infty K_{b_2}(t-t_1)K_{b_2}(t-t_2)(t-t_2)(t_1-t_2)h(t_1)h(t_2)dt_1dt_2 \\ &= b_2^2 \int_{-\infty}^{\frac{t}{b_2}} \int_{-\infty}^{\frac{t}{b_2}} K(t_{11})K(t_{21})t_{21}\left(t_{21}-t_{11}\right)h(t-b_2t_{11})h(t-b_2t_{21})dt_{11}dt_{21} \end{aligned}$$

after using the change of variable $\frac{t-t_1}{b_2} = t_{11}$, $\frac{t-t_2}{b_2} = t_{21}$. For $b_2 < \frac{t}{L}$,

$$\begin{aligned} M_1 &= b_2^2 \int_{-L}^{L} \int_{-L}^{L} K(t_{11})K(t_{21})t_{21}\left(t_{21}-t_{11}\right)h(t-b_2t_{11})h(t-b_2t_{21})dt_{11}dt_{21} \\ &= b_2^2 \mu_K h(t)^2 + O\left(b_2^4\right) \end{aligned}$$

where the final expression comes from Taylor expansions of $h(t-b_2t_{11})$ and $h(t-b_2t_{21})$ around t. Consequently,

$$E\left[\sum_{i=1}^n w_i(t)\right] = n^2 b_2^2 \mu_K h(t)^2 + O\left(n^2 b_2^4\right).$$

With respect to the variance of $\sum\limits_{i=1}^n w_i(t)$,

$$\begin{aligned} Var\left[\sum_{i=1}^n w_i(t)\right] &= Cov\left[\sum_{i=1}^n w_i(t), \sum_{i=1}^n w_i(t)\right] \\ &= \sum_{i=1}^n\sum_{j=1}^n Cov\left[w_i(t), w_j(t)\right] \\ &= \sum_{i=1}^n\sum_{j=1}^n\sum_{k=1}^n\sum_{l=1}^n Cov\left[K_{b_2}(t-T_i)K_{b_2}(t-T_j)(t-T_j)(T_i-T_j),\right. \\ &\quad \left. K_{b_2}(t-T_k)K_{b_2}(t-T_l)(t-T_l)(T_k-T_l)\right]. \end{aligned}$$

We now study the terms $M_{i,j,k,l}$, $i,j,k,l = 1,2,...,n$, defined by

$$\begin{aligned} M_{i,j,k,l} &= Cov\left[K_{b_2}(t-T_i)K_{b_2}(t-T_j)(t-T_j)(T_i-T_j),\right. \\ &\quad \left. K_{b_2}(t-T_k)K_{b_2}(t-T_l)(t-T_l)(T_k-T_l)\right]. \end{aligned}$$

If $\sharp\{i,j,k,l\}=3$, there are three cases, represented by $M_{1,2,1,3}$, $M_{1,2,3,1}$ and $M_{1,2,3,2}$, for which $M_{i,j,k,l}$ is not trivially zero. For the first of them,

$$\begin{aligned} M_{1,2,1,3} &= E\left[K_{b_2}(t-T_1)^2K_{b_2}(t-T_2)K_{b_2}(t-T_3)(t-T_2)\right. \\ &\quad \left.\times(T_1-T_2)(t-T_3)(T_1-T_3)\right]-M_1^2. \end{aligned}$$

Its first summand is

$$\begin{aligned} &\int_0^\infty\int_0^\infty\int_0^\infty K_{b_2}(t-t_1)^2K_{b_2}(t-t_2)K_{b_2}(t-t_3)(t-t_2)(t_1-t_2) \\ &\times(t-t_3)(t_1-t_3)h(t_1)h(t_2)h(t_3)dt_1dt_2dt_3 \\ =\ & \int_0^\infty K_{b_2}(t-t_1)^2\left(\int_0^\infty K_{b_2}(t-t_2)(t-t_2)(t_1-t_2)h(t_2)dt_2\right)^2h(t_1)dt_1 \\ =\ & b_2^3\int_{-L}^{L}\left(\int_{-L}^{L}K(t_{21})t_{21}(t_{21}-t_{11})h(t-b_2t_{21})dt_{21}\right)^2K(t_{11})^2h(t-b_2t_{11})dt_{11} \end{aligned}$$

after using the change of variable $\frac{t-t_1}{b_2}=t_{11}$, $\frac{t-t_2}{b_2}=t_{21}$ and $b_2<\frac{t}{L}$. Taylor expansions of $h(t-b_2t_{11})$ and $h(t-b_2t_{21})$ around t give

$$\begin{aligned} &\int_{-L}^{L}\left(\int_{-L}^{L}K(t_{21})t_{21}(t_{21}-t_{11})h(t-b_2t_{21})dt_{21}\right)^2K(t_{11})^2h(t-b_2t_{11})dt_{11} \\ =\ & h(t)^3c_K\mu_K^2+o(1), \end{aligned}$$

and, since $M_1=O\left(b_2^2\right)$,

$$M_{1,2,1,3}=b_2^3h(t)^3c_K\mu_K^2+o\left(b_2^3\right). \tag{37}$$

Let's study the term $M_{1,2,3,1}$,

$$\begin{aligned} M_{1,2,3,1} &= E\left[K_{b_2}(t-T_1)^2K_{b_2}(t-T_2)K_{b_2}(t-T_3)(t-T_2)\right. \\ &\quad \left.\times(T_1-T_2)(t-T_1)(T_3-T_1)\right]-M_1^2. \end{aligned}$$

Its first summand is now

$$\begin{aligned} &\int_0^\infty\int_0^\infty\int_0^\infty K_{b_2}(t-t_1)^2K_{b_2}(t-t_2)K_{b_2}(t-t_3)(t-t_2)(t_1-t_2) \\ &\times(t-t_1)(t_3-t_1)h(t_1)h(t_2)h(t_3)dt_1dt_2dt_3 \\ =\ & \int_0^\infty\int_0^\infty\int_0^\infty K_{b_2}(t-t_1)^2K_{b_2}(t-t_2)K_{b_2}(t-t_3)(t-t_2)(t_1-t_2) \\ &\times(t-t_1)(t_3-t_1)h(t_1)h(t_2)h(t_3)dt_1dt_2dt_3 \\ =\ & b_2^3\int_{-L}^{L}\left(\int_{-L}^{L}K(t_{21})(t_{21}-t_{11})t_{21}h(t-b_2t_{21})dt_{21}\right) \\ &\times\left(\int_{-L}^{L}K(t_{31})(t_{11}-t_{31})h(t-b_2t_{31})dt_3\right)K(t_{11})^2t_{11}h(t-b_2t_{11})dt_{11} \end{aligned}$$

using the change $\frac{t-t_1}{b_2} = t_{11}$, $\frac{t-t_2}{b_2} = t_{21}$, $\frac{t-t_3}{b_2} = t_{31}$ and assuming $b_2 < \frac{t}{L}$. Taylor expansions give

$$M_{1,2,3,1} = b_2^3 h(t)^3 \mu_K \int_{-L}^{L} K(u)^2 u^2 du + o\left(b_2^3\right). \tag{38}$$

Using similar arguments for $M_{1,2,3,2}$

$$\begin{aligned} M_{1,2,3,2} &= E\left[K_{b_2}(t-T_1)K_{b_2}(t-T_2)^2 K_{b_2}(t-T_3)(t-T_2)^2\right. \\ &\quad \left.\times(T_1-T_2)(T_3-T_2)\right] - M_1^2 \end{aligned}$$

give, for its first summand,

$$\begin{aligned} &\int_0^\infty \int_0^\infty \int_0^\infty K_{b_2}(t-t_1)K_{b_2}(t-t_2)^2 K_{b_2}(t-t_3)(t-t_2)^2 \\ &\times(t_1-t_2)(t_3-t_2)h(t_1)h(t_2)h(t_3)dt_1dt_2dt_3 \\ = &\int_0^\infty \left(\int_0^\infty K_{b_2}(t-t_1)(t_1-t_2)h(t_1)dt_1\right)^2 K_{b_2}(t-t_2)^2(t-t_2)^2 h(t_2)dt_2 \\ = &\; b_2^3 \int_{-L}^{L} \left(\int_{-L}^{L} K(t_{11})(t_{21}-t_{11})h(t-b_2t_{11})dt_{11}\right)^2 K(t_{21})^2 t_{21}^2 \\ &\times h(t-b_2t_{21})dt_{21}. \end{aligned}$$

Further Taylor expanding leads to

$$\begin{aligned} &\int_{-L}^{L} \left(\int_{-L}^{L} K(t_{11})(t_{21}-t_{11})h(t-b_2t_{11})dt_{11}\right)^2 K(t_{21})^2 t_{21}^2 h(t-b_2t_{21})dt_{21} \\ = &\; h(t)^3 \int_{-L}^{L} K(u)^2 u^4 du + o(1) \end{aligned}$$

and, consequently,

$$M_{1,2,3,2} = b_2^3 h(t)^3 \int_{-L}^{L} K(u)^2 u^4 du + o\left(b_2^3\right). \tag{39}$$

The case $\sharp\{i,j,k,l\} = 2$, only covers two nonzero type of terms, of the form $M_{1,2,1,2}$ and $M_{1,2,2,1}$. The first one is

$$M_{1,2,1,2} = E\left[K_{b_2}(t-T_1)^2 K_{b_2}(t-T_2)^2 (t-T_2)^2 (T_1-T_2)^2\right] - M_1^2,$$

whose first summand equals

$$\begin{aligned} &\int_0^\infty \int_0^\infty K_{b_2}(t-t_1)^2 K_{b_2}(t-t_2)^2 (t-t_2)^2 (t_1-t_2)^2 h(t_1)h(t_2)dt_1dt_2 \\ = &\; b_2^2 \int_{-L}^{L}\int_{-L}^{L} K(t_{11})^2 K(t_{21})^2 t_{21}^2 (t_{21}-t_{11})^2 h(t-b_2t_{11})h(t-b_2t_{21})dt_{11}dt_{21} \\ = &\; O\left(b_2^2\right), \end{aligned}$$

which implies

$$M_{1,2,1,2} = O\left(b_2^2\right). \tag{40}$$

Similarly,

$$M_{1,2,2,1} = -E\left[K_{b_2}(t-T_1)^2 K_{b_2}(t-T_2)^2 (t-T_1)(t-T_2)(T_1-T_2)^2\right] - M_1^2$$

and the first of the two terms is

$$\begin{aligned} & -\int_0^\infty \int_0^\infty K_{b_2}(t-t_1)^2 K_{b_2}(t-t_2)^2 (t-t_1)(t-t_2) \\ & \times (t_1-t_2)^2 h(t_1)h(t_2)dt_1dt_2 \\ = & -b_2^2 \int_{-L}^{L}\int_{-L}^{L} K_{b_2}(t_{11})^2 K_{b_2}(t_{21})^2 t_{11}t_{21}(t_{21}-t_{11})^2 \\ & \times h(t-b_2t_{11})h(t-b_2t_{21})dt_{11}dt_{21} \\ = & \; O\left(b_2^2\right), \end{aligned}$$

As a consequence,

$$M_{1,2,2,1} = O\left(b_2^2\right). \tag{41}$$

Finally, for the case $\sharp\{i,j,k,l\} = 1$, it is immediate to check that $M_{1,1,1,1} = 0$.

Collecting (37)-(41), we have

$$Var\left[\sum_{i=1}^{n} w_i(t)\right] = O\left(n^3b_2^3 + n^2b_2^2\right) = O\left(n^3b_2^3\right)$$

and using Tchebychev inequality,

$$\frac{1}{n^2b_2^2}\sum_{i=1}^{n} w_i(t) = \mu_K h(t)^2 + O\left(b_2^2\right) + O_P\left(n^{-1/2}b_2^{-1/2}\right) = \mu_K h(t)^2 + o_P(1). \tag{42}$$

Now, writing

$$\widehat{p}_{LL}(t) - p(t) = D_n + D_n\left(\frac{\mu_K h(t)^2 - \frac{1}{n^2b_2^2}\sum\limits_{i=1}^{n} w_i(t)}{\frac{1}{n^2b_2^2}\sum\limits_{i=1}^{n} w_i(t)}\right)$$

and using (42) and $h(t) > 0$, equation (31) holds.

The expected value of D_n is

$$E\left[D_n\right] = \frac{1}{nb_2^2\mu_K h(t)^2} E\left(w_1(t)\left(\delta_1 - p(t)\right)\right) \tag{43}$$

and using (36),

$$\begin{aligned}
& E\left[w_1(t)\left(\delta_1 - p(t)\right)\right] \\
= \; & E\left[\sum_{j=1}^{n} K_{b_2}(t-T_1)K_{b_2}(t-T_j)(t-T_j)(T_1-T_j)\left(\delta_1 - p(t)\right)\right] \\
= \; & (n-1)E\left[K_{b_2}(t-T_1)K_{b_2}(t-T_2)(t-T_2)(T_1-T_2)\left(\delta_1 - p(t)\right)\right] \\
= \; & (n-1)E\left[K_{b_2}(t-T_1)K_{b_2}(t-T_2)(t-T_2)(T_1-T_2)\left(p(T_1) - p(t)\right)\right] \quad (44)
\end{aligned}$$

Let's define

$$N_1 = E\left[K_{b_2}(t-T_1)K_{b_2}(t-T_2)(t-T_2)(T_1-T_2)\left(p(T_1) - p(t)\right)\right], \tag{45}$$

then

$$\begin{aligned}
N_1 &= \int_0^\infty \int_0^\infty K_{b_2}(t-t_1)K_{b_2}(t-t_2)(t-t_2)(t_1-t_2) \\
&\quad \times \left(p(t_1) - p(t)\right) h(t_1)h(t_2)dt_1 dt_2 \\
&= b_2^2 \int_{-L}^{L}\int_{-L}^{L} K(t_{11})K(t_{21})t_{21}\left(t_{21}-t_{11}\right)\left(p(t-b_2t_{11}) - p(t)\right) \\
&\quad \times h(t-b_2t_{11})h(t-b_2t_{21})dt_{11}dt_{21}.
\end{aligned}$$

Taylor expansions of $p(t-b_2t_{11})$, $h(t-b_2t_{11})$ and $h(t-b_2t_{21})$ around t, lead to

$$\begin{aligned}
N_1 &= -b_2^3 p'(t)h(t)^2 \int_{-L}^{L}\int_{-L}^{L} K(t_{11})K(t_{21})t_{21}\left(t_{21}-t_{11}\right)t_{11}dt_{11}dt_{21} \\
&\quad + b_2^4 \int_{-L}^{L}\int_{-L}^{L} K(t_{11})K(t_{21})t_{21}\left(t_{21}-t_{11}\right) \\
&\quad \times \left(t_{11}^2\left(\frac{1}{2}p''(t)h(t)^2 + p'(t)h'(t)h(t)\right) + t_{11}t_{21}p'(t)h'(t)h(t)\right)dt_{11}dt_{21} + o\left(b_2^4\right) \\
&= b_2^4\left(\frac{1}{2}p''(t)h(t)^2 + p'(t)h'(t)h(t)\right)\int_{-L}^{L}\int_{-L}^{L} K(t_{11})K(t_{21})t_{11}^2t_{21}^2dt_{11}dt_{21} \\
&\quad - b_2^4 \int_{-L}^{L}\int_{-L}^{L} K(t_{11})K(t_{21})t_{11}^2t_{21}^2 p'(t)h'(t)h(t)dt_{11}dt_{21} + o\left(b_2^4\right) \\
&= b_2^4\frac{1}{2}p''(t)h(t)^2\mu_K^2 + o\left(b_2^4\right) \qquad (46)
\end{aligned}$$

which implies (34).

The variance of D_n is

$$\begin{aligned}
Var\left[D_n\right] &= Cov\left[D_n, D_n\right] \\
&= \frac{1}{n^4 b_2^4}\frac{1}{\mu_K^2 h(t)^4}\sum_{i=1}^{n}\sum_{j=1}^{n} Cov\left[w_i(t)\left(\delta_i - p(t)\right), w_j(t)\left(\delta_j - p(t)\right)\right] \\
&= \frac{1}{n^4 b_2^4}\frac{1}{\mu_K^2 h(t)^4}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} N_{i,j,k,l},
\end{aligned}$$

where we have used the definition

$$
\begin{aligned}
N_{i,j,k,l} &= Cov\left[K_{b_2}(t-T_i)K_{b_2}(t-T_j)(t-T_j)(T_i-T_j)\left(\delta_i-p(t)\right),\right. \\
&\quad \left. K_{b_2}(t-T_k)K_{b_2}(t-T_l)(t-T_l)(T_k-T_l)\left(\delta_k-p(t)\right)\right].
\end{aligned}
$$

The order of $N_{i,j,k,l}$ will be examined depending on the cardinal of $\{i,j,k,l\}$.

If $\sharp\{i,j,k,l\}=3$, there are only three nonzero cases, covered by $N_{1,2,1,3}$, $N_{1,2,3,1}$ and $N_{1,2,3,2}$. The first one is,

$$
\begin{aligned}
N_{1,2,1,3} &= E\left[K_{b_2}(t-T_1)^2K_{b_2}(t-T_2)K_{b_2}(t-T_3)(t-T_2)(T_1-T_2)\right. \\
&\quad \left.\times(t-T_3)(T_1-T_3)\left(p(T_1)(1-2p(t))+p(t)^2\right)\right]-N_1^2,
\end{aligned}
$$

whose first summand is

$$
\begin{aligned}
& \int_0^\infty\int_0^\infty\int_0^\infty K_{b_2}(t-t_1)^2K_{b_2}(t-t_2)K_{b_2}(t-t_3)(t-t_2)(t_1-t_2) \\
& \times(t-t_3)(t_1-t_3)\left(p(t_1)(1-2p(t))+p(t)^2\right)h(t_1)h(t_2)h(t_3)dt_1dt_2dt_3 \\
= & \int_0^\infty\left(\int_0^\infty K_{b_2}(t-t_2)(t-t_2)(t_1-t_2)h(t_2)dt_2\right)^2K_{b_2}(t-t_1)^2 \\
& \times\left(p(t_1)(1-2p(t))+p(t)^2\right)h(t_1)dt_1 \\
= & b_2^3\int_{-L}^{L}\left(\int_{-L}^{L}K(t_{21})t_{21}(t_{21}-t_{11})h(t-b_2t_{21})dt_{21}\right)^2K(t_{11})^2 \\
& \times\left(p(t-b_2t_{11})(1-2p(t))+p(t)^2\right)h(t-b_2t_{11})dt_{11}
\end{aligned}
$$

Taylor expansions give

$$
\begin{aligned}
& \int_{-L}^{L}\left(\int_{-L}^{L}K(t_{21})t_{21}(t_{21}-t_{11})h(t-b_2t_{21})dt_{21}\right)^2K(t_{11})^2 \\
& \times\left(p(t-b_2t_{11})(1-2p(t))+p(t)^2\right)h(t-b_2t_{11})dt_{11} \\
= & p(t)(1-p(t))h(t)^3c_K\mu_K^2+o(1)
\end{aligned}
$$

which implies

$$
N_{1,2,1,3}=b_2^3p(t)(1-p(t))h(t)^3c_K\mu_K^2+o\left(b_2^3\right). \tag{47}
$$

The term $N_{1,2,3,1}$ is

$$
\begin{aligned}
N_{1,2,3,1} &= E\left[K_{b_2}(t-T_1)^2K_{b_2}(t-T_2)K_{b_2}(t-T_3)(t-T_2)(T_1-T_2)\right. \\
&\quad \left.\times(t-T_1)(T_3-T_1)\left(p(t_1)-p(t)\right)\left(p(t_3)-p(t)\right)\right]-N_1^2.
\end{aligned}
$$

Its first summand equals

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \int_0^\infty K_{b_2}(t-t_1)^2 K_{b_2}(t-t_2) K_{b_2}(t-t_3)(t-t_2)(t_1-t_2) \\
&\times (t-t_1)(t_3-t_1)\left(p(t_1)-p(t)\right)\left(p(t_3)-p(t)\right) h(t_1)h(t_2)h(t_3)dt_1 dt_2 dt_3 \\
= \;& b_2^3 \int_{-L}^{L} \int_{-L}^{L} \int_{-L}^{L} K(t_{11})^2 K(t_{21}) K(t_{31}) t_{21}(t_{21}-t_{11}) t_{11}(t_{11}-t_{31}) \\
&\times \left(p(t-b_2 t_{11})-p(t)\right)\left(p(t-b_2 t_{31})-p(t)\right) h(t-b_2 t_{11}) \\
&\times h(t-b_2 t_{21}) h(t-b_2 t_{31}) dt_{11} dt_{21} dt_{31} \\
= \;& o\left(b_2^3\right)
\end{aligned}$$

Now, our standard arguments lead to

$$N_{1,2,3,1} = o\left(b_2^3\right). \tag{48}$$

Similarly,

$$N_{1,2,3,2} = o\left(b_2^3\right), \tag{49}$$

If $\sharp\{i,j,k,l\} = 2$, the only two nonzero terms can be analyzed by means of parallel arguments, giving

$$N_{1,2,1,2} = O\left(b_2^2\right), \tag{50}$$

$$N_{1,2,2,1} = o\left(b_2^2\right). \tag{51}$$

In the case $\sharp\{i,j,k,l\} = 1$, straight forward calculations give $N_{1,1,1,1} = 0$.

Collecting (47)-(51) and using **(V.2)** and **(V.1)** we come to our conclusion (35). ■

Lemma 6 *Under conditions **(K.1)**, **(P.1)**, **(H.1)**, **(V.1)** and **(V.2)**,*

$$Cov\left[A_n, D_n\right] = o\left(n^{-1}\right). \tag{52}$$

Proof. Direct calculations give

$$\begin{aligned}
Cov\left[A_n, D_n\right] &= Cov\left[\frac{1}{n}\sum_{i=1}^n \frac{K_{b_1}(t-T_i)}{1-H(T_i)+\frac{1}{n}} - \lambda_H(t),\right. \\
&\quad \left.\frac{1}{n^2 b_2^2 \mu_K h(t)^2}\sum_{i=1}^n w_i(t)\left(\delta_i - p(t)\right)\right] \\
&= \frac{1}{n^3 b_2^2 \mu_K h(t)^2}\sum_{i=1}^n\sum_{j=1}^n Cov\left[\frac{K_{b_1}(t-T_i)}{1-H(T_i)+\frac{1}{n}}, w_j(t)\left(\delta_j - p(t)\right)\right] \\
&= \frac{1}{n^3 b_2^2 \mu_K h(t)^2}\sum_{i=1}^n\sum_{j=1}^n\sum_{k=1}^n N_{i,j,k}
\end{aligned}$$

where

$$N_{i,j,k} = Cov\left[\frac{K_{b_1}(t-T_i)}{1-H(T_i)+\frac{1}{n}}, K_{b_2}(t-T_j)K_{b_2}(t-T_k)(t-T_k)(T_j-T_k)(\delta_j-p(t))\right].$$

Whenever $\sharp\{i,j,k\}=2$, the only nonzero cases are those of the form $N_{1,1,2}$ and $N_{1,2,1}$. The first one is

$$\begin{aligned} N_{1,1,2} &= E\left[\frac{K_{b_1}(t-T_1)K_{b_2}(t-T_1)K_{b_2}(t-T_2)(t-T_2)(T_1-T_2)(\delta_1-p(t))}{1-H(T_1)+\frac{1}{n}}\right] \\ &\quad -E\left[\frac{K_{b_1}(t-T_1)}{1-H(T_1)+\frac{1}{n}}\right]N_1 \end{aligned}$$

where N_1 was defined in (45). Its first summand is

$$\begin{aligned} &\int_0^\infty\int_0^\infty \frac{1}{1-H(t_1)+\frac{1}{n}}K_{b_1}(t-t_1)K_{b_2}(t-t_1)K_{b_2}(t-t_2) \\ &\times(t-t_2)(t_1-t_2)(p(t_1)-p(t))h(t_1)h(t_2)dt_1dt_2 \\ =\; &\frac{b_1^3}{b_2^2}\int_{-L}^{L}\int_{-L}^{L}\frac{1}{1-H(t-b_1t_{11})+\frac{1}{n}}K(t_{11})K\left(\frac{t_{11}b_1}{b_2}\right)K\left(\frac{t_{21}b_1}{b_2}\right) \\ &\times t_{21}(t_{21}-t_{11})(p(t-b_1t_{11})-p(t))h(t-b_1t_{11})h(t-b_1t_{21})dt_{11}dt_{21}, \end{aligned}$$

after using the change of variable $\frac{t-t_1}{b_1}=t_{11}$, $\frac{t-t_2}{b_1}=t_{21}$ and $b_1<\frac{t}{L}$. Now, if $b_1<\frac{t\wedge(t_0-t)}{L}$

$$\begin{aligned} &\frac{b_1^3}{b_2^2}\int_{-L}^{L}\int_{-L}^{L}\frac{1}{1-H(t-b_1t_{11})+\frac{1}{n}}K(t_{11})K\left(\frac{t_{11}b_1}{b_2}\right)K\left(\frac{t_{21}b_1}{b_2}\right) \\ &\times t_{21}(t_{21}-t_{11})(p(t-b_1t_{11})-p(t))h(t-b_1t_{11})h(t-b_1t_{21})dt_{11}dt_{21} \\ =\; &-\frac{b_1^4}{b_2^2}p'(t)\lambda_H(t)h(t)\int_{-L}^{L}\int_{-L}^{L}K(t_{11})K\left(\frac{t_{11}b_1}{b_2}\right)K\left(\frac{t_{21}b_1}{b_2}\right) \\ &\times t_{21}t_{11}(t_{21}-t_{11})dt_{11}dt_{21}+o\left(\frac{b_1^4}{b_2^2}\right)+O\left(\frac{b_1^3}{b_2^2n}\right), \end{aligned} \tag{53}$$

after using (15), the relation $\lambda_H=\frac{h}{1-H}$ and Taylor expansions of $p(t-b_1t_{11})$, $\lambda_H(t-b_1t_{11})$ and $h(t-b_1t_{21})$ around t. On the other hand, (10) implies

$$E\left[\frac{K_{b_1}(t-T_i)}{1-H(T_i)+\frac{1}{n}}\right]=\lambda_H(t)+\frac{1}{2}b_1^2\mu_K\lambda_H''(t)+o\left(b_1^2\right)+O(n^{-1})$$

and, as proved in (46),

$$N_1=b_2^4\frac{1}{2}p''(t)h(t)^2\mu_K^2+o\left(b_2^4\right)=O\left(b_1^4\right).$$

Since the integral of the first summand in (53) is zero, conditions **(V.2)** and **(V.1)** lead to

$$N_{1,1,2} = o\left(b_1^2\right). \tag{54}$$

Similarly,

$$\begin{aligned} N_{1,2,1} &= E\left[\frac{K_{b_1}(t-T_1)\,K_{b_2}(t-T_2)\,K_{b_2}(t-T_1)(t-T_1)(T_2-T_1)\,(\delta_2 - p(t))}{1-H(T_1)+\frac{1}{n}}\right] \\ &\quad -E\left[\frac{K_{b_1}(t-T_1)}{1-H(T_1)+\frac{1}{n}}\right] N_1, \end{aligned}$$

where the first expectation is

$$\begin{aligned} &\int_0^\infty \int_0^\infty \frac{1}{1-H(t_1)+\frac{1}{n}} K_{b_1}(t-t_1)\,K_{b_2}(t-t_2)\,K_{b_2}(t-t_1) \\ &\times (t-t_1)(t_2-t_1)\,(p(t_2)-p(t))\,h(t_1)h(t_2)dt_1dt_2 \\ = &\ \frac{b_1^3}{b_2^2}\int_{-L}^{L}\int_{-L}^{L} \frac{1}{1-H(t-b_1t_{11})+\frac{1}{n}} K(t_{11})\,K\left(\frac{t_{21}b_1}{b_2}\right) K\left(\frac{t_{11}b_1}{b_2}\right) \\ &\times t_{11}(t_{11}-t_{21})\,(p(t-b_1t_{21})-p(t))\,h(t-b_1t_{11})h(t-b_1t_{21})dt_{11}dt_{21}. \end{aligned}$$

Assuming $b_1 < \frac{t\wedge(t_0-t)}{L}$,

$$\begin{aligned} &\frac{b_1^3}{b_2^2}\int_{-L}^{L}\int_{-L}^{L} \frac{1}{1-H(t-b_1t_{11})+\frac{1}{n}} K(t_{11})\,K\left(\frac{t_{21}b_1}{b_2}\right) K\left(\frac{t_{11}b_1}{b_2}\right) \\ &\times t_{11}(t_{11}-t_{21})\,(p(t-b_1t_{21})-p(t))\,h(t-b_1t_{11})h(t-b_1t_{21})dt_{11}dt_{21} \\ = &\ -\frac{b_1^4}{b_2^2}p'(t)\lambda_H(t)h(t)\int_{-L}^{L}\int_{-L}^{L} K(t_{11})\,K\left(\frac{t_{21}b_1}{b_2}\right) K\left(\frac{t_{11}b_1}{b_2}\right) t_{21}t_{11} \\ &\times (t_{11}-t_{21})dt_{11}dt_{21} + o\left(\frac{b_1^4}{b_2^2}\right) + O\left(\frac{b_1^3}{b_2^2 n}\right) \end{aligned} \tag{55}$$

after using (15), the relation $\lambda_H = \frac{h}{1-H}$ and Taylor expansions of $p(t-b_1t_{11})$, $\lambda_H(t-b_1t_{11})$ and $h(t-b_1t_{21})$ around t. Again, the integral of the first summand in (55) is zero. Now, conditions **(V.2)** and **(V.1)** and equations (10) and (46) give

$$N_{1,2,1} = o\left(b_1^2\right). \tag{56}$$

It is evident that $N_{1,1,1} = 0$. Thus (53) and (55) imply (52). ■

The following theorems give asymptotic representations of $\widehat{\lambda}_{WLNW}(t)$, $\widehat{\lambda}_{RRNW}(t)$, $\widehat{\lambda}_{WLLL}(t)$, $\widehat{\lambda}_{RRLL}(t)$ and $\widetilde{\lambda}_{TWP}(t)$.

Theorem 7 *Under conditions **(K.1)**, **(P.1)**, **(H.1)**, **(V.1)** and **(V.2)**,*

$$\widehat{\lambda}_{WLNW}(t) - \lambda_F(t) = \overline{\lambda}_{WLNW}(t) - \lambda_F(t) + o_P\left(n^{-1/2}b_1^{-1/2}\right) \tag{57}$$

where

$$\overline{\lambda}_{WLNW}(t)=\frac{1}{n}\sum_{i=1}^{n}\left(\frac{K_{b_1}(t-T_i)\,p(t)}{1-H(T_i)+\frac{1}{n}}+\frac{K_{b_2}(t-T_i)\,(\delta_i-p(t))}{1-H(t)}\right). \quad (58)$$

The expectation of the dominant term is

$$\begin{aligned} E\left[\overline{\lambda}_{WLNW}(t)\right] &= \lambda_F(t)+\frac{1}{2}b_1^2\mu_K \quad (59)\\ &\times\left(\lambda_H''(t)p(t)+\frac{1}{a^2}\left(\lambda_H(t)p''(t)+2\left(\lambda_H'(t)-\lambda_H(t)^2\right)p'(t)\right)\right)+o\left(b_1^2\right)\end{aligned}$$

and its variance

$$Var\left[\overline{\lambda}_{WLNW}(t)\right]=\frac{1}{nb_1}\frac{c_K\lambda_H(t)p(t)\left(p(t)+a(1-p(t))\right)}{1-H(t)}+o\left(n^{-1}b_1^{-1}\right). \quad (60)$$

Proof. using the decomposition

$$\begin{aligned}\widehat{\lambda}_{WLNW}(t)-\lambda_F(t) &= \widehat{\lambda}_{WL}(t)\widehat{p}_{NW}(t)-\lambda_H(t)p(t)\\ &= p(t)(\widehat{\lambda}_{WL}(t)-\lambda_H(t))+\lambda_H(t)\left(\widehat{p}_{NW}(t)-p(t)\right)\\ &\quad +(\widehat{\lambda}_{WL}(t)-\lambda_H(t))\left(\widehat{p}_{NW}(t)-p(t)\right),\end{aligned}$$

equation can be obtained (57) by means of the asymptotic representations for $\widehat{\lambda}_{WL}(t)-\lambda_H(t)$ and $\widehat{p}_{NW}(t)-p(t)$ given in Lemmas 1 and 3. In order to do that, let's write $\overline{\lambda}_{WLNW}(t)=p(t)A_n+\lambda_H(t)C_n+\lambda_F(t)$, where A_n and C_n have been defined in (9) and (26), and apply Tchebychev inequality to A_n and C_n, using (10), (11), (27) and (28) and conditions **(V.2)** and **(V.1)**.

Equation (59) is then an immediate consequence of (10) and (27), by just writing

$$\frac{h'(t)p'(t)}{1-H(t)}=\left(\lambda_H(t)-\lambda_H(t)^2\right)p'(t)$$

and using **(V.2)**.

Finally, since

$$Var\left[\overline{\lambda}_{WLNW}(t)\right]=p(t)^2Var\left[A_n\right]+\lambda_H(t)^2Var\left[C_n\right]+2\lambda_H(t)p(t)Cov\left[A_n,C_n\right],$$

equation (60) can be obtained from (11), (28), (29) and **(V.2)**. ∎

The following theorem gives an asymptotic representation of $\widehat{\lambda}_{RRNW}(t)-\lambda_F(t)$.

Theorem 8 *Let us assume conditions* ***(K.1)****,* ***(P.1)****,* ***(H.1)****,* ***(V.1)*** *and* ***(V.2)****. Then,*

$$\begin{aligned}\widehat{\lambda}_{RRNW}(t)-\lambda_F(t) &= \widehat{\lambda}_{WLNW}(t)-\lambda_F(t)+\frac{1}{2}b_1^2\mu_K\left(\lambda_H(t)^2-3\lambda_H'(t)\right)\lambda_H(t)p(t) \quad (61)\\ &\quad +o_P\left(n^{-1/2}b_1^{-1/2}\right).\end{aligned}$$

Equivalently, defining

$$\overline{\lambda}_{RRNW}(t) = \overline{\lambda}_{WLNW}(t) + \frac{1}{2}b_1^2\mu_K\left(\lambda_H(t)^2 - 3\lambda_H'(t)\right)\lambda_H(t)p(t), \tag{62}$$

where $\overline{\lambda}_{WLNW}(t)$ *can be found in 58, we have*

$$\widehat{\lambda}_{RRNW}(t) - \lambda_F(t) = \overline{\lambda}_{RRNW}(t) - \lambda_F(t) + o_P\left(n^{-1/2}b_1^{-1/2}\right). \tag{63}$$

Proof. Let's consider the following expansion,

$$\begin{aligned}\widehat{\lambda}_{RRNW}(t) - \lambda_F(t) &= \widehat{\lambda}_{WLNW}(t) - \lambda_F(t) + E\left[B_n - A_n\right]\widehat{p}_{NW}(t) \\ &\quad + \left(\widehat{\lambda}_{RR}(t) - \widehat{\lambda}_{WL}(t) - E\left[B_n - A_n\right]\right)\widehat{p}_{NW}(t)\end{aligned} \tag{64}$$

where A_n and B_n were defined in (9) and (22).

An application of Tchebychev inequality to the term C_n, defined in (26), can be used together with the representation (25), $\widehat{p}_{NW}(t) - p(t) = C_n\left(1 + o_P(1)\right)$, (27) and (28) to obtain

$$\widehat{p}_{NW}(t) = p(t) + O_P\left(b_2^2 + n^{-1/2}b_2^{-1/2}\right). \tag{65}$$

On the other hand, (10) and (23) imply

$$E\left[B_n - A_n\right] = \frac{1}{2}b_1^2\mu_K\left(\frac{h''(t)}{1-H(t)} - \lambda_H''(t)\right) + o\left(b_1^2\right). \tag{66}$$

Now (65) and (66), can be used in the second summand of (64)

$$\begin{aligned}E\left[B_n - A_n\right]\widehat{p}_{NW}(t) &= \frac{1}{2}b_1^2\mu_K\left(\frac{h''(t)}{1-H(t)} - \lambda_H''(t)\right)p(t) + o_P\left(b_1^2\right) \\ &\quad + O_P\left(b_1^2\left(b_2^2 + n^{-1/2}b_1^{-1/2}\right)\right)\end{aligned}$$

and, **(V.1)** and **(V.2)** lead to

$$E\left[B_n - A_n\right]\widehat{p}_{NW}(t) = \frac{1}{2}b_1^2\mu_K\left(\frac{h''(t)}{1-H(t)} - \lambda_H''(t)\right)p(t) + o_P\left(n^{-1/2}b_1^{-1/2}\right). \tag{67}$$

As a consequence of (8) and (21), the factor between brackets in the second summand of (64) satisfies

$$\begin{aligned}\widehat{\lambda}_{RR}(t) - \widehat{\lambda}_{WL}(t) - E\left[B_n - A_n\right] &= \widehat{\lambda}_{RR}(t) - \lambda_H(t) - \left(\widehat{\lambda}_{WL}(t) - \lambda_H(t)\right) \\ &\quad - E\left[B_n - A_n\right] \\ &= B_n - A_n - E\left[B_n - A_n\right] + o_P\left(n^{-1/2}b_1^{-1/2}\right).\end{aligned}$$

Since

$$
\begin{aligned}
& E\left[\frac{K_{b_1}\left(t-T_1\right)\left(H(t)-H(T_1)+\frac{1}{n}\right)}{1-H(T_1)+\frac{1}{n}}\right] \\
= & \int_0^{\infty}\frac{K_{b_1}\left(t-x\right)\left(H(t)-H(x)+\frac{1}{n}\right)h(x)}{1-H(x)+\frac{1}{n}}dx \\
= & \int_{-L}^{L}\frac{K\left(x_1\right)\left(H(t)-H(t-b_1x_1)+\frac{1}{n}\right)h(t-b_1x_1)}{1-H(t-b_1x_1)+\frac{1}{n}}dx_1 \\
= & \int_{-L}^{L}K\left(x_1\right)\left(H(t)-H(t-b_1x_1)\right)\varphi_1(t-b_1x_1)dx_1+O(n^{-1}) \\
= & b_1^2\mu_K\left(h(t)\varphi_1'(t)+\frac{1}{2}h'(t)\varphi_1(t)\right)+o\left(b_1^2\right)+O(n^{-1}) \\
= & O\left(b_1^2\right)
\end{aligned}
$$

and

$$
\begin{aligned}
& E\left[\frac{K_{b_1}\left(t-T_1\right)^2\left(H(t)-H(T_1)+\frac{1}{n}\right)^2}{\left(1-H(T_1)+\frac{1}{n}\right)^2}\right] \\
= & \int_0^{\infty}\frac{K_{b_1}\left(t-x\right)^2\left(H(t)-H(x)+\frac{1}{n}\right)^2h(x)}{\left(1-H(x)+\frac{1}{n}\right)^2}dx \\
= & \frac{1}{b_1}\int_{-L}^{L}\frac{K\left(x_1\right)^2\left(H(t)-H(t-b_1x_1)+\frac{1}{n}\right)^2h(t-b_1x_1)}{\left(1-H(t-b_1x_1)+\frac{1}{n}\right)^2}dx_1 \\
= & \frac{1}{b_1}\int_{-L}^{L}K\left(x_1\right)^2\left(H(t)-H(t-b_1x_1)\right)^2\varphi_2(t-b_1x_1)dx_1+O\left(n^{-1}b_1^{-1}\right) \\
= & b_1\int_{-L}^{L}K\left(x_1\right)^2x_1^2dx_1h'(t)^2\varphi_2(t)+O\left(n^{-1}b_1^{-1}\right) \\
= & O(b_1)
\end{aligned}
$$

we have

$$\begin{aligned}
Var\left[(B_n - A_n - E\left[B_n - A_n\right]\right] &= Var\left[B_n - A_n\right] \\
&= Var\left[\frac{1}{n}\sum_{i=1}^{n}\frac{K_{b_1}\left(t - T_i\right)\left(H(t) - H(T_i) + \frac{1}{n}\right)}{\left(1 - H(t)\right)\left(1 - H(T_i) + \frac{1}{n}\right)}\right] \\
&= \frac{1}{n\left(1 - H(t)\right)^2} Var\left[\frac{K_{b_1}\left(t - T_1\right)\left(H(t) - H(T_1) + \frac{1}{n}\right)}{1 - H(T_1) + \frac{1}{n}}\right] \\
&= \frac{1}{n\left(1 - H(t)\right)^2}\left(E\left[\frac{K_{b_1}\left(t - T_1\right)^2\left(H(t) - H(T_1) + \frac{1}{n}\right)^2}{\left(1 - H(T_1) + \frac{1}{n}\right)^2}\right]\right. \\
&\quad \left. - \left(E\left[\frac{K_{b_1}\left(t - T_1\right)\left(H(t) - H(T_1) + \frac{1}{n}\right)}{1 - H(T_1) + \frac{1}{n}}\right]\right)^2\right) \\
&= O(n^{-1}b_1)
\end{aligned}$$

and, consequently,

$$B_n - A_n - E\left[B_n - A_n\right] = O_P\left(n^{-1/2}b_1^{1/2}\right).$$

Thus,

$$\widehat{\lambda}_{RR}(t) - \widehat{\lambda}_{WL}(t) - E\left[B_n - A_n\right] = O_P\left(n^{-1/2}b_1^{1/2}\right) \tag{68}$$

and

$$\begin{aligned}
\left(\widehat{\lambda}_{RR}(t) - \widehat{\lambda}_{WL}(t) - E\left[B_n - A_n\right]\right)\widehat{p}_{NW}(t) &= O_P\left(n^{-1/2}b_1^{1/2}\right)O_P(1) \qquad (69) \\
&= o_P\left(n^{-1/2}b_1^{-1/2}\right).
\end{aligned}$$

Equation (61) is an immediate consequence of (67) and (69), since

$$\frac{h''(t)}{1 - H(t)} - \lambda_H''(t) = \left(\lambda_H(t)^2 - 3\lambda_H'(t)\right)\lambda_H(t).$$

To prove (63) we start from (61) and consider the representation (57) in Theorem 7 and definition (62) of $\overline{\lambda}_{RRNW}(t)$. ■

For $\widehat{\lambda}_{WLLL}(t)$ we have the following theorem.

Theorem 9 *Conditions* ***(K.1)****,* ***(P.1)****,* ***(H.1)****,* ***(V.1)*** *and* ***(V.2)*** *imply*

$$\widehat{\lambda}_{WLLL}(t) - \lambda_F(t) = \overline{\lambda}_{WLLL}(t) - \lambda_F(t) + o_P\left(n^{-1/2}b_1^{-1/2}\right) \tag{70}$$

where

$$\overline{\lambda}_{WLLL}(t) = \frac{1}{n}\sum_{i=1}^{n}\frac{K_{b_1}\left(t - T_i\right)p(t)}{1 - H(T_i) + \frac{1}{n}} + \frac{1}{n^2 b_2^2}\sum_{i=1}^{n}\frac{w_i(t)\left(\delta_i - p(t)\right)}{\mu_K h(t)\left(1 - H(t)\right)} \tag{71}$$

and $w_i(t)$ has been defined in (33).

Further more,

$$E\left[\overline{\lambda}_{WLLL}(t)\right] = \lambda_F(t) + \frac{1}{2}b_1^2\mu_K\left(\lambda_H''(t)p(t) + \frac{\lambda_H(t)p''(t)}{a^2}\right) + o\left(b_1^2\right) \quad (72)$$

and

$$Var\left[\overline{\lambda}_{WLLL}(t)\right] = \frac{1}{nb_1}\frac{c_K\lambda_H(t)p(t)\left(p(t)+a(1-p(t))\right)}{1-H(t)} + o\left(n^{-1}b_1^{-1}\right). \quad (73)$$

Proof. Starting from the following decomposition

$$\begin{aligned}\widehat{\lambda}_{WLLL}(t) - \lambda_F(t) &= p(t)(\widehat{\lambda}_{WL}(t) - \lambda_H(t)) + \lambda_H(t)\left(\widehat{p}_{LL}(t) - p(t)\right) \\ &\quad + (\widehat{\lambda}_{WL}(t) - \lambda_H(t))\left(\widehat{p}_{LL}(t) - p(t)\right)\end{aligned}$$

equation (70) can be obtained. To do it we just use the asymptotic representations for $\widehat{\lambda}_{WL}(t) - \lambda_H(t)$ and $\widehat{p}_{LL}(t) - p(t)$ in lemmas 1 and 5, take into account $\overline{\lambda}_{WLLL}(t) = p(t)A_n + \lambda_H(t)D_n + \lambda_F(t)$, where A_n and D_n have been defined in (9) and (32) and apply Tchebychev inequality to A_n and D_n, using (10), (11), (34), (35), **(V.2)** and **(V.1)**.

Equation (72) is an immediate consequence of (10), (34) and **(V.2)**.

Finally (73) is obtained from (11), (35), (52) and **(V.2)**. ∎

Starting from the previous representation the following theorem gives another representation for $\widehat{\lambda}_{RRLL}(t)$.

Theorem 10 *Under conditions **(K.1)**, **(P.1)**, **(H.1)**, **(V.1)** and **(V.2)**,*

$$\begin{aligned}\widehat{\lambda}_{RRLL}(t) - \lambda_F(t) &= \widehat{\lambda}_{WLLL}(t) - \lambda_F(t) + \frac{1}{2}b_1^2\mu_K\left(\lambda_H(t)^2 - 3\lambda_H'(t)\right)\lambda_H(t)p(t) \quad (74) \\ &\quad + o_P\left(n^{-1/2}b_1^{-1/2}\right).\end{aligned}$$

Equivalently, defining

$$\overline{\lambda}_{RRLL}(t) = \overline{\lambda}_{WLLL}(t) + \frac{1}{2}b_1^2\mu_K\left(\lambda_H(t)^2 - 3\lambda_H'(t)\right)\lambda_H(t)p(t) \quad (75)$$

where$\overline{\lambda}_{WLLL}(t)$ has been defined in (71), we have

$$\widehat{\lambda}_{RRLL}(t) - \lambda_F(t) = \overline{\lambda}_{RRLL}(t) - \lambda_F(t) + o_P\left(n^{-1/2}b_1^{-1/2}\right). \quad (76)$$

Proof. The difference $\widehat{\lambda}_{RRLL}(t) - \lambda_F(t)$ may be expanded as follows:

$$\begin{aligned}\widehat{\lambda}_{RRLL}(t) - \lambda_F(t) &= \widehat{\lambda}_{WLLL}(t) - \lambda_F(t) + E\left[B_n - A_n\right]\widehat{p}_{LL}(t) \\ &\quad + \left(\widehat{\lambda}_{RR}(t) - \widehat{\lambda}_{WL}(t) - E\left[B_n - A_n\right]\right)\widehat{p}_{LL}(t) \quad (77)\end{aligned}$$

where A_n and B_n have been defined in (9) and (22).

An application of Tchebychev inequality to the term D_n, defined in (32), and expressions (34) and (35), for $E[D_n]$ and $Var[D_n]$, give

$$\widehat{p}_{LL}(t) = p(t) + O_P\left(b_2^2 + n^{-1/2} b_2^{-1/2}\right). \tag{78}$$

On the other hand, as proved in (66),

$$E[B_n - A_n] = \frac{1}{2} b_1^2 \mu_K \left(\frac{h''(t)}{1-H(t)} - \lambda''_H(t)\right) + o\left(b_1^2\right). \tag{79}$$

As a consequence, (78) and (79) imply

$$\begin{aligned} E[B_n - A_n]\,\widehat{p}_{LL}(t) &= \frac{1}{2} b_1^2 \mu_K \left(\frac{h''(t)}{1-H(t)} - \lambda''_H(t)\right) p(t) + o_P\left(b_1^2\right) \\ &\quad + O_P\left(b_1^2 \left(b_2^2 + n^{-1/2} b_1^{-1/2}\right)\right) \end{aligned}$$

and **(V.1)** and **(V.2)** give

$$E[B_n - A_n]\,\widehat{p}_{LL}(t) = \frac{1}{2} b_1^2 \mu_K \left(\frac{h''(t)}{1-H(t)} - \lambda''_H(t)\right) p(t) + o_P\left(n^{-1/2} b_1^{-1/2}\right). \tag{80}$$

Similarly, using (68),

$$\widehat{\lambda}_{RR}(t) - \widehat{\lambda}_{WL}(t) - E[B_n - A_n] = O_P\left(n^{-1/2} b_1^{1/2}\right),$$

which leads to

$$\left(\widehat{\lambda}_{RR}(t) - \widehat{\lambda}_{WL}(t) - E[B_n - A_n]\right) \widehat{p}_{LL}(t) = o_P\left(n^{-1/2} b_1^{-1/2}\right). \tag{81}$$

Plugging (80) and (81) into (77) and using

$$\frac{h''(t)}{1-H(t)} - \lambda''_H(t) = \left(\lambda_H(t)^2 - 3\lambda'_H(t)\right) \lambda_H(t)$$

equation (74) is obtained. Finally, (76) is a consequence of (74) by using representation (70), Theorem 9 and (75). ■

The section concludes with a similar result for $\widetilde{\lambda}_{TWP}(t)$.

Theorem 11 *Under conditions **(K.1)**, **(P.1)**, **(H.1)**, **(V.1)** and **(V.2)**,*

$$\widetilde{\lambda}_{TWP}(t) - \lambda_F(t) = \overline{\lambda}_{TWP}(t) - \lambda_F(t) + o_P\left(b_1^2 + n^{-1/2} b_1^{-1/2}\right) \tag{82}$$

where $I = (\varepsilon, t_0)$,

$$\overline{\lambda}_{TWP}(t) = \frac{1}{n} \sum_{i=1}^{n} \frac{K_{b_1}(t - T_i)\left(\psi(T_i) + \widehat{\psi}(T_i) - \widehat{h}(T_i) p(T_i)\right) \mathbf{1}(T_i \in I)}{\left(1 - H(T_i) + \frac{1}{n}\right) h(T_i)}, \tag{83}$$

$$\psi(t) = p(t)h(t)$$

and

$$\widehat{\psi}(t) = \frac{1}{n}\sum_{i=1}^{n} K_{b_2}(t - T_i)\,\delta_i.$$

On the other hand,

$$\begin{aligned} E\left[\overline{\lambda}_{TWP}(t)\right] &= \lambda_F(t) + \frac{1}{2}b_1^2\mu_K\Bigg(\lambda_H''(t)p(t) + 2\lambda_H'(t)p'(t) + \lambda_H(t)p''(t) \\ &\quad + \frac{1}{a^2}\left(\lambda_H(t)p''(t) + 2\left(\lambda_H'(t) - \lambda_H(t)^2\right)p'(t)\right)\Bigg) + o\left(b_1^2\right) \quad (84) \end{aligned}$$

and

$$\begin{aligned} Var\left[\overline{\lambda}_{TWP}(t)\right] &= \frac{1}{nb_1}\frac{\lambda_H(t)p(t)}{1 - H(t)} \\ &\quad \times\left(p(t)c_K + (1 - p(t))a\int_{-L(1+a)}^{L(1+a)}\left(\int_{-L}^{L}K(u)K(v - ua)\,du\right)^2 dv\right) \\ &\quad + o\left(n^{-1}b_1^{-1}\right). \end{aligned} \quad (85)$$

Proof. In (7) some alternative expression for $\widetilde{\lambda}_{TWP}(t)$ was found

$$\widetilde{\lambda}_{TWP}(t) = \frac{1}{n}\sum_{i=1}^{n}\frac{K_{b_1}(t - T_i)\,\widehat{p}_{NW}(T_i)\mathbf{1}(T_i \in I)}{1 - H_n(T_i) + \frac{1}{n}} + \frac{1}{n}\sum_{i=1}^{n}\frac{K_{b_1}(t - T_i)\,\delta_i\mathbf{1}(T_i \notin I)}{1 - H_n(T_i) + \frac{1}{n}}. \quad (86)$$

It is easy to prove that the second term in (86) is negligible with respect to the first one. To do this, let's define $\alpha = (t_0 - t) \wedge (t - \varepsilon) > 0$. For $i = 1, 2, ..., n$, if $\mathbf{1}(T_i \notin I) = 1$ then $|t - T_i| \geq \alpha$. Consequently for n large enough such that $b_1 < \frac{\alpha}{L}$, we have $\frac{|t-T_i|}{b_1} > L$, $K_{b_1}(t - T_i) = 0$ and $\frac{K_{b_1}(t-T_i)\delta_i\mathbf{1}(T_i \notin I)}{1-H_n(T_i)+\frac{1}{n}} = 0$. On the other hand, if $\mathbf{1}(T_i \notin I) = 0$, then, obviously, $\frac{K_{b_1}(t-T_i)\delta_i\mathbf{1}(T_i \notin I)}{1-H_n(T_i)+\frac{1}{n}} = 0$. In summary, for n large enough, the sequence $S_n = \frac{1}{n}\sum_{i=1}^{n}\frac{K_{b_1}(t-T_i)\delta_i\mathbf{1}(T_i \notin I)}{1-H_n(T_i)+\frac{1}{n}}$ is zero with probability 1.

The first term in (86) can be dealt with by linearizing its denominator and using the following decomposition

$$\widehat{p}_{NW}(t) - p(t) = \frac{\widehat{\psi}(t) - \widehat{h}(t)p(t)}{h(t)} - \frac{\left(\widehat{p}_{NW}(t) - p(t)\right)\left(\widehat{h}(t) - h(t)\right)}{h(t)}.$$

Then that term can be written as follows

$$\frac{1}{n}\sum_{i=1}^{n}\frac{K_{b_1}(t - T_i)\,\widehat{p}_{NW}(T_i)\mathbf{1}(T_i \in I)}{1 - H_n(T_i) + \frac{1}{n}} = J_1 + J_2 + J_3 + J_4, \quad (87)$$

where

$$
\begin{aligned}
J_1 &= \frac{1}{n}\sum_{i=1}^{n}\frac{K_{b_1}(t-T_i)\,p(T_i)\mathbf{1}(T_i\in I)}{1-H(T_i)+\frac{1}{n}},\\
J_2 &= \frac{1}{n}\sum_{i=1}^{n}\frac{K_{b_1}(t-T_i)\left(\widehat{\psi}(T_i)-\widehat{h}(T_i)p(T_i)\right)\mathbf{1}(T_i\in I)}{\left(1-H(T_i)+\frac{1}{n}\right)h(T_i)},\\
J_3 &= -\frac{1}{n}\sum_{i=1}^{n}\frac{K_{b_1}(t-T_i)\left(\widehat{p}_{NW}(T_i)-p(T_i)\right)\left(\widehat{h}(T_i)-h(T_i)\right)\mathbf{1}(T_i\in I)}{\left(1-H(T_i)+\frac{1}{n}\right)h(T_i)},\\
J_4 &= \frac{1}{n}\sum_{i=1}^{n}\frac{K_{b_1}(t-T_i)\,\widehat{p}_{NW}(T_i)\left(H_n(T_i)-H(T_i)\right)\mathbf{1}(T_i\in I)}{\left(1-H_n(T_i)+\frac{1}{n}\right)\left(1-H\ (T_i)+\frac{1}{n}\right)}.
\end{aligned}
$$

The term J_3 satisfies

$$
\begin{aligned}
|J_3| &\le \sup_{t\in I}\left|\widehat{p}_{NW}(t)-p(t)\right|\sup_{t\in I}\left|\widehat{h}(t)-h(t)\right|\frac{1}{n}\sum_{i=1}^{n}\frac{K_{b_1}(t-T_i)\,\mathbf{1}(T_i\in I)}{\left(1-H(T_i)+\frac{1}{n}\right)h(T_i)}\\
&\le \sup_{t\in I}\left|\widehat{p}_{NW}(t)-p(t)\right|\sup_{t\in I}\left|\widehat{h}(t)-h(t)\right|\delta^{-1}\frac{1}{n}\sum_{i=1}^{n}\frac{K_{b_1}(t-T_i)}{1-H(T_i)+\frac{1}{n}}
\end{aligned}
$$

where, $\delta>0$ is a lower bound of the density h in the interval I. Now, using some uniform orders in probability for the kernel density and regression estimators (see Silverman (1978) and Mack and Silverman (1982)), expression (20) and condition **(V.1),**

$$
\begin{aligned}
J_3 &= O_P\left(n^{-1/2}b_1^{-1/2}\left(\log\frac{1}{b_1}\right)^{1/2}\right)O_P\left(b_1^2+n^{-1/2}b_1^{-1/2}\left(\log\frac{1}{b_1}\right)^{1/2}\right)O_P(1)\\
&= o_P\left(b_1^2+n^{-1/2}b_1^{-1/2}\right)
\end{aligned}
$$

Concerning J_4,

$$
\begin{aligned}
|J_4| &\le \frac{\sup_{t\in I}\left|H_n(t)-H(t)\right|}{1-H(t_0)}\frac{1}{n}\sum_{i=1}^{n}\frac{K_{b_1}(t-T_i)\,\widehat{p}_{NW}(T_i)\mathbf{1}(T_i\in I)}{1-H_n(T_i)+\frac{1}{n}}\\
&\le \frac{\sup_{t\in I}\left|H_n(t)-H(t)\right|}{1-H(t_0)}\frac{1}{n}\sum_{i=1}^{n}\frac{K_{b_1}(t-T_i)}{1-H_n(T_i)+\frac{1}{n}}.
\end{aligned}
$$

Now recalling, as mentioned in (13),

$$
\sup_{t\in I}\left|H_n(t)-H(t)\right| = O_P\left(n^{-1/2}\right)
$$

the definition of $\widehat{\lambda}_{WL}$, and equations (8), (10) and (11) which imply $\widehat{\lambda}_{WL} = O_P(1)$, we obtain the final expression for this term

$$
J_4 = O_P\left(n^{-1/2}\right) = o_P\left(b_1^2+n^{-1/2}b_1^{-1/2}\right).
$$

Using the previous results and $\overline{\lambda}_{TWP}(t) = J_1 + J_2$, we have (82).
We now consider the terms J_1 and J_2. The expectation of J_1 is

$$\begin{aligned} E[J_1] &= E\left[\frac{K_{b_1}(t-T_1)\, p(T_1)\mathbf{1}(T_1 \in I)}{1-H(T_1)+\frac{1}{n}}\right] \\ &= \int_{\varepsilon}^{t_0} \frac{K_{b_1}(t-t_1)\, p(t_1)h(t_1)}{1-H(t_1)+\frac{1}{n}} dt_1 \\ &= \int_{\frac{t-t_0}{b_1}}^{\frac{t-\varepsilon}{b_1}} \frac{K(t_{11})\, p(t-b_1t_{11})h(t-b_1t_{11})}{1-H(t-b_1t_{11})+\frac{1}{n}} dt_{11} \\ &= \int_{-L}^{L} K(t_{11})\, \lambda_F(t-b_1t_{11})dt_{11} + O(n^{-1}) \\ &= \lambda_F(t) + \frac{1}{2}b_1^2\mu_K\left(\lambda_H''(t)p(t) + 2\lambda_H'(t)p'(t) + \lambda_H(t)p''(t)\right) \\ &\quad +o\left(b_1^2\right) \end{aligned} \tag{88}$$

where we have used the change of variable $\frac{t-t_1}{b_1} = t$, the fact $b_1 < \frac{(t_0-t)\wedge(t-\varepsilon)}{L}$, equation (15), some Taylor expansion of the function λ_F around t, λ_F'' has been written in terms of $\lambda_H(t)$, $p(t)$ and its derivatives and condition **(V.1)** has also been used.

Its variance is

$$\begin{aligned} Var[J_1] &= \frac{1}{n}E\left[\frac{K_{b_1}(t-T_1)^2\, p(T_1)^2\mathbf{1}(T_1 \in I)}{\left(1-H(T_1)+\frac{1}{n}\right)^2}\right] - \frac{1}{n}\left(E[U_1]\right)^2 \\ &= \frac{1}{n}\int_{\varepsilon}^{t_0} \frac{K_{b_1}(t-t_1)^2\, p(t_1)^2h(t_1)}{\left(1-H(t_1)+\frac{1}{n}\right)^2} dt_1 - \frac{1}{n}\left(E[U_1]\right)^2 \\ &= \frac{1}{nb_1}\int_{\frac{t-t_0}{b_1}}^{\frac{t-\varepsilon}{b_1}} \frac{K(t_{11})^2\, p(t-b_1t_{11})^2h(t-b_1t_{11})}{\left(1-H(t-b_1t_{11})+\frac{1}{n}\right)^2} dt_{11} - \frac{1}{n}\left(E[U_1]\right)^2 \\ &= \frac{1}{nb_1}\frac{c_K p(t)^2h(t)}{\left(1-H(t)\right)^2} + o\left(nb_1^{-1}\right). \end{aligned} \tag{89}$$

The term J_2 may be written as

$$J_2 = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{K_{b_1}(t-T_i)\, K_{b_2}(T_i-T_j)\,(\delta_j - p(T_i))\,\mathbf{1}(T_i \in I)}{\left(1-H(T_i)+\frac{1}{n}\right)h(T_i)}$$

whose expectation is

$$\begin{aligned} E[J_2] &= \frac{n-1}{n}E\left[\frac{K_{b_1}(t-T_1)\, K_{b_2}(T_1-T_2)\,(p(T_2)-p(T_1))\,\mathbf{1}(T_1 \in I)}{\left(1-H(T_1)+\frac{1}{n}\right)h(T_1)}\right] \\ &= \frac{n-1}{n}\int_0^{\infty}\int_{\varepsilon}^{t_0} \frac{K_{b_1}(t-t_1)\, K_{b_2}(t_1-t_2)\,(p(t_2)-p(t_1))\, h(t_2)}{1-H(t_1)+\frac{1}{n}} dt_1dt_2. \end{aligned}$$

Using the change of variable $\frac{t-t_1}{b_1} = t_{11}$, $\frac{t-t_2}{b_2} = t_{21}$,

$$\begin{aligned}
E\left[J_2\right] &= \frac{n-1}{n} \int_{-\infty}^{\frac{t}{b_2}} \int_{\frac{t-t_0}{b_1}}^{\frac{t-\varepsilon}{b_1}} K\left(t_{11}\right) K\left(t_{21} - \frac{t_{11} b_1}{b_2}\right) \\
&\quad \times \left(p(t-b_2 t_{21}) - p(t-b_1 t_{11})\right) h(t-b_2 t_{21}) \\
&\quad \times \left(1 - H(t-b_1 t_{11}) + \frac{1}{n}\right)^{-1} dt_{11} dt_{21} \\
&= \frac{n-1}{n} \int_{-L\left(1+\frac{b_1}{b_2}\right)}^{L\left(1+\frac{b_1}{b_2}\right)} \int_{-L}^{L} K\left(t_{11}\right) K\left(t_{21} - \frac{t_{11} b_1}{b_2}\right) \\
&\quad \times \left(p(t-b_2 t_{21}) - p(t-b_1 t_{11})\right) \\
&\quad \times h(t-b_2 t_{21}) \left(1 - H(t-b_1 t_{11}) + \frac{1}{n}\right)^{-1} dt_{11} dt_{21} \qquad (90)
\end{aligned}$$

where we have used $b_1 < \frac{(t_0-t)\wedge(t-\varepsilon)\wedge t/2}{L}$ and $b_2 < \frac{t}{2L}$.Let us define $\zeta = \frac{1}{1-H}$. Now, taking into account (15) and performing Taylor expansions of $p(t-b_2 t_{21})$, $p(t-b_1 t_{11})$, $h(t-b_2 t_{21})$ and $\zeta(t-b_1 t_{11})$ around t,the integral of (90) is

$$\begin{aligned}
& p'(t)h(t)\zeta(t)\left(b_1 J_{21} - b_2 J_{22}\right) + \frac{1}{2} p''(t)h(t)\zeta(t)\left(b_2^2 J_{24} - b_1^2 J_{23}\right) \\
& + p'(t)h'(t)\zeta(t)\left(b_2^2 J_{24} - b_1 b_2 J_{25}\right) \\
& + p'(t)h(t)\zeta'(t)\left(b_1 b_2 J_{25} - b_1^2 J_{23}\right) + o\left((b_1+b_2)^2\right) + O(n^{-1})
\end{aligned}$$

where

$$\begin{aligned}
J_{21} &= \int_{-L\left(1+\frac{b_1}{b_2}\right)}^{L\left(1+\frac{b_1}{b_2}\right)} \int_{-L}^{L} K\left(t_{11}\right) K\left(t_{21} - \frac{t_{11} b_1}{b_2}\right) t_{11} dt_{11} dt_{21}, \\
J_{22} &= \int_{-L\left(1+\frac{b_1}{b_2}\right)}^{L\left(1+\frac{b_1}{b_2}\right)} \int_{-L}^{L} K\left(t_{11}\right) K\left(t_{21} - \frac{t_{11} b_1}{b_2}\right) t_{21} dt_{11} dt_{21}, \\
J_{23} &= \int_{-L\left(1+\frac{b_1}{b_2}\right)}^{L\left(1+\frac{b_1}{b_2}\right)} \int_{-L}^{L} K\left(t_{11}\right) K\left(t_{21} - \frac{t_{11} b_1}{b_2}\right) t_{11}^2 dt_{11} dt_{21}, \\
J_{24} &= \int_{-L\left(1+\frac{b_1}{b_2}\right)}^{L\left(1+\frac{b_1}{b_2}\right)} \int_{-L}^{L} K\left(t_{11}\right) K\left(t_{21} - \frac{t_{11} b_1}{b_2}\right) t_{21}^2 dt_{11} dt_{21}, \\
J_{25} &= \int_{-L\left(1+\frac{b_1}{b_2}\right)}^{L\left(1+\frac{b_1}{b_2}\right)} \int_{-L}^{L} K\left(t_{11}\right) t_{11} K\left(t_{21} - \frac{t_{11} b_1}{b_2}\right) t_{21} dt_{11} dt_{21}
\end{aligned}$$

The change of variable $t_{11} = u$, $t_{21} - \frac{t_{11} b_1}{b_2} = v$ gives, for $u \in [-L, L]$,

$-L\left(1+\frac{b_1}{b_2}\right)-\frac{ub_1}{b_2}\leq -L$ and $L\left(1+\frac{b_1}{b_2}\right)-\frac{ub_1}{b_2}\geq L$, which implies

$$\begin{aligned} J_{21} &= \int_{-L}^{L}\int_{-L\left(1+\frac{b_1}{b_2}\right)-\frac{ub_1}{b_2}}^{L\left(1+\frac{b_1}{b_2}\right)-\frac{ub_1}{b_2}} K\left(u\right)K\left(v\right)udvdu \qquad (91)\\ &= \int_{-L}^{L}K\left(u\right)udu\int_{-L}^{L}K\left(v\right)dv=0, \end{aligned}$$

$$\begin{aligned} J_{22} &= \int_{-L}^{L}\int_{-L\left(1+\frac{b_1}{b_2}\right)-\frac{ub_1}{b_2}}^{L\left(1+\frac{b_1}{b_2}\right)-\frac{ub_1}{b_2}} K\left(u\right)K\left(v\right)\left(\frac{ub_1}{b_2}+v\right)dvdu\\ &= \left(1+\frac{b_1}{b_2}\right)\int_{-L}^{L}K\left(u\right)du\int_{-L}^{L}K\left(v\right)vdv=0, \qquad (92) \end{aligned}$$

$$\begin{aligned} J_{23} &= \int_{-L}^{L}\int_{-L\left(1+\frac{b_1}{b_2}\right)-\frac{ub_1}{b_2}}^{L\left(1+\frac{b_1}{b_2}\right)-\frac{ub_1}{b_2}} K\left(u\right)K\left(v\right)u^2dvdu \qquad (93)\\ &= \int_{-L}^{L}K\left(u\right)u^2du\int_{-L}^{L}K\left(v\right)dv=\mu_K, \end{aligned}$$

$$\begin{aligned} J_{24} &= \int_{-L}^{L}\int_{-L\left(1+\frac{b_1}{b_2}\right)-\frac{ub_1}{b_2}}^{L\left(1+\frac{b_1}{b_2}\right)-\frac{ub_1}{b_2}} K\left(u\right)K\left(v\right)\left(\frac{ub_1}{b_2}+v\right)^2dvdu\\ &= \frac{b_1^2}{b_2^2}\int_{-L}^{L}K\left(u\right)u^2du\int_{-L}^{L}K\left(v\right)dv+\frac{2b_1}{b_2}\int_{-L}^{L}K\left(u\right)udu\int_{-L}^{L}K\left(v\right)vdv\\ &\quad+\int_{-L}^{L}K\left(u\right)du\int_{-L}^{L}K\left(v\right)v^2dv\\ &= \mu_K\left(\frac{b_1^2}{b_2^2}+1\right) \qquad (94) \end{aligned}$$

and

$$\begin{aligned} J_{25} &= \int_{-L}^{L}\int_{-L\left(1+\frac{b_1}{b_2}\right)-\frac{ub_1}{b_2}}^{L\left(1+\frac{b_1}{b_2}\right)-\frac{ub_1}{b_2}} K\left(u\right)K\left(v\right)\left(\frac{ub_1}{b_2}+v\right)udvdu\\ &= \frac{b_1}{b_2}\int_{-L}^{L}\int_{-L}^{L}K\left(u\right)K\left(v\right)u^2dvdu+\int_{-L}^{L}K\left(u\right)udu\int_{-L}^{L}K\left(v\right)vdv\\ &= \frac{\mu_K b_1}{b_2}. \qquad (95) \end{aligned}$$

Consequently, using (91)-(95),

$$\begin{aligned} E\left[J_2\right] &= b_2^2\mu_K\left(\frac{1}{2}h(t)p''(t)\zeta(t)+h'(t)p'(t)\zeta(t)\right)+o\left((b_1+b_2)^2\right)+O(n^{-1})\\ &= \frac{b_1^2\mu_K}{a^2}\left(\frac{1}{2}\lambda_H(t)p''(t)+\left(\lambda_H(t)-\lambda_H(t)^2\right)p'(t)\right)+o\left(b_1^2\right) \qquad (96) \end{aligned}$$

where the last equation is obtained from

$$\frac{h'(t)p'(t)}{1-H(t)} = \left(\lambda_H(t) - \lambda_H(t)^2\right) p'(t)$$

and conditions **(V.2)** and **(V.1)**.

The variance of J_2 is

$$\begin{aligned} Var\left[J_2\right] &= Cov\left[J_2, J_2\right] = \frac{1}{n^4}\left(nM_{1,1,1,1} + n(n-1)\left(2M_{1,2,2,2} + 2M_{1,2,1,1}\right.\right. \\ &\quad \left. + M_{1,2,1,2} + M_{1,2,2,1}\right) + n(n-1)(n-2)\left(M_{1,2,1,3} + M_{1,2,3,1}\right. \\ &\quad \left.\left. + M_{1,2,2,3} + M_{1,2,3,2}\right)\right.\end{aligned}$$

where

$$\begin{aligned} M_{i,j,k,l} &= Cov\left[\frac{K_{b_1}\left(t-T_i\right)K_{b_2}\left(T_i-T_j\right)\left(\delta_j - p(T_i)\right)\mathbf{1}(T_i \in I)}{\left(1-H(T_i)+\frac{1}{n}\right)h(T_i)},\right. \\ &\quad \left.\frac{K_{b_1}\left(t-T_k\right)K_{b_2}\left(T_k-T_l\right)\left(\delta_l - p(T_k)\right)\mathbf{1}(T_k \in I)}{\left(1-H(T_k)+\frac{1}{n}\right)h(T_k)}\right].\end{aligned}$$

To study these covariances we start with the case $\#\{i,j,k,l\} = 3$. For $M_{1,2,3,2}$ we have, after some change of variables,

$$\begin{aligned} & E\left[\frac{K_{b_1}(t-T_1)K_{b_2}\left(T_1-T_2\right)\left(\delta_2 - p(T_1)\right)\mathbf{1}(T_1 \in I)}{\left(1-H(T_1)+\frac{1}{n}\right)h(T_1)}\right. \\ & \left.\times\frac{K_{b_1}\left(t-T_3\right)K_{b_2}\left(T_3-T_2\right)\left(\delta_2 - p(T_3)\right)\mathbf{1}(T_3 \in I)}{\left(1-H(T_3)+\frac{1}{n}\right)h(T_3)}\right] \\ = & \int_{\varepsilon}^{t_0}\int_0^{\infty}\int_{\varepsilon}^{t_0}\frac{1}{1-H(t_1)+\frac{1}{n}}\frac{1}{1-H(t_3)+\frac{1}{n}} \\ & \times K_{b_1}(t-t_1)K_{b_2}\left(t_1-t_2\right)K_{b_1}\left(t-t_3\right)K_{b_2}\left(t_3-t_2\right) \\ & \times\left(p(t_2)(1-p(t_1)-p(t_3)) + p(t_1)p(t_3)\right)h(t_2)dt_1dt_2dt_3 \\ = & \frac{1}{b_2}\int_{\frac{t-t_0}{b_1}}^{\frac{t-\varepsilon}{b_1}}\int_{-\infty}^{\frac{t}{b_2}}\int_{\frac{t-t_0}{b_1}}^{\frac{t-\varepsilon}{b_1}}\frac{1}{1-H(t-b_1t_{11})+\frac{1}{n}}\frac{1}{1-H(t-b_1t_{31})+\frac{1}{n}} \\ & \times K(t_{11})K\left(t_{21}-\frac{t_{11}b_1}{b_2}\right)K\left(t_{31}\right)K\left(t_{21}-\frac{t_{31}b_1}{b_2}\right) \\ & \times\left(p(t-b_2t_{21})(1-p(t-b_1t_{11})-p(t-b_1t_{31})) + p(t-b_1t_{11})p(t-b_1t_{31})\right) \\ & \times h(t-b_2t_{21})dt_{11}dt_{21}dt_{31}\end{aligned}$$

Taking $b_1 < \frac{(t_0-t)\wedge(t-\varepsilon)\wedge t/2}{L}$, $b_2 < \frac{t}{2L}$,the last expression leads to

$$\begin{aligned}
& \frac{1}{b_2}\int_{-L}^{L}\int_{-L\left(1+\frac{b_1}{b_2}\right)}^{L\left(1+\frac{b_1}{b_2}\right)}\int_{-L}^{L}\frac{1}{1-H(t-b_1t_{11})+\frac{1}{n}}\frac{1}{1-H(t-b_1t_{31})+\frac{1}{n}} \\
& \times K(t_{11})K\left(t_{21}-\frac{t_{11}b_1}{b_2}\right)K\left(t_{31}\right)K\left(t_{21}-\frac{t_{31}b_1}{b_2}\right) \\
& \times\left(p(t-b_2t_{21})(1-p(t-b_1t_{11})-p(t-b_1t_{31}))+p(t-b_1t_{11})p(t-b_1t_{31})\right) \\
& \times h(t-b_2t_{21})dt_{11}dt_{21}dt_{31} \\
= & \frac{1}{b_2}\frac{p(t)(1-p(t))h(t)}{(1-H(t))^2}\int_{-L\left(1+\frac{b_1}{b_2}\right)}^{L\left(1+\frac{b_1}{b_2}\right)}\left(\int_{-L}^{L}K(t_{11})K\left(t_{21}-\frac{t_{11}b_1}{b_2}\right)dt_{11}\right)^2 dt_{21} \\
& +o\left(b_2^{-1}\right)+O\left(b_2^{-1}n^{-1}\right) \\
= & \frac{1}{b_1}\frac{ap(t)(1-p(t))h(t)}{(1-H(t))^2}\int_{-L(1+a)}^{L(1+a)}\left(\int_{-L}^{L}K(t_{11})K\left(t_{21}-t_{11}a\right)dt_{11}\right)^2 dt_{21} \\
& +o\left(b_1^{-1}\right).
\end{aligned}$$

This expression has been obtained by using some Taylor expansions, condition **(V.2)** and, applying the dominated convergence theorem,

$$\begin{aligned}
& \int_{-L\left(1+\frac{b_1}{b_2}\right)}^{L\left(1+\frac{b_1}{b_2}\right)}\left(\int_{-L}^{L}K(t_{11})K\left(t_{21}-\frac{t_{11}b_1}{b_2}\right)dt_{11}\right)^2 dt_{21} \\
= & \int_{-L(1+a)}^{L(1+a)}\left(\int_{-L}^{L}K(t_{11})K\left(t_{21}-t_{11}a\right)dt_{11}\right)^2 dt_{21}+o(1).
\end{aligned}$$

Now (96), leads to

$$\begin{aligned}
M_{1,2,3,2} = & \frac{1}{b_1}\frac{ap(t)(1-p(t))h(t)}{(1-H(t))^2}\int_{-L(1+a)}^{L(1+a)}\left(\int_{-L}^{L}K(t_{11})K\left(t_{21}-t_{11}a\right)dt_{11}\right)^2 dt_{21} \\
& +o\left(b_1^{-1}\right). \qquad (97)
\end{aligned}$$

For $M_{1,2,1,3}$, we have

$$E\left[\frac{K_{b_1}(t-T_1)^2K_{b_2}(T_1-T_2)K_{b_2}(T_1-T_3)(\delta_2-p(T_1))(\delta_3-p(T_1))\mathbf{1}(T_1\in I)}{\left(1-H(T_1)+\frac{1}{n}\right)^2h(T_1)^2}\right]$$

$$= \int_0^\infty\int_0^\infty\int_\varepsilon^{t_0}K_{b_1}(t-t_1)^2K_{b_2}(t_1-t_2)K_{b_2}(t_1-t_3)(p(t_2)-p(t_1))$$
$$\times(p(t_3)-p(t_1))\left(1-H(t_1)+\frac{1}{n}\right)^{-2}h(t_1)^{-1}h(t_2)h(t_3)dt_1dt_2dt_3$$
$$= \frac{1}{b_1}\int_{-\infty}^{\frac{t}{b_2}}\int_{-\infty}^{\frac{t}{b_2}}\int_{\frac{t-t_0}{b_1}}^{\frac{t-\varepsilon}{b_1}}K(t_{11})^2K\left(t_{21}-\frac{t_{11}b_1}{b_2}\right)K\left(t_{31}-\frac{t_{11}b_1}{b_2}\right)$$
$$\times(p(t-b_2t_{21})-p(t-b_1t_{11}))(p(t-b_2t_{31})-p(t-b_1t_{11}))$$
$$\times\left(1-H(t-b_1t_{11})+\frac{1}{n}\right)^{-2}h(t-b_1t_{11})^{-1}h(t-b_2t_{21})h(t-b_2t_{31})dt_{11}dt_{21}dt_{31}$$
$$= \frac{1}{b_1}\int_{\frac{t-t_0}{b_1}}^{\frac{t-\varepsilon}{b_1}}K(t_{11})^2\left(1-H(t-b_1t_{11})+\frac{1}{n}\right)^{-2}h(t-b_1t_{11})^{-1}$$
$$\times\left(\int_{-\infty}^{\frac{t}{b_2}}K\left(t_{21}-\frac{t_{11}b_1}{b_2}\right)(p(t-b_2t_{21})-p(t-b_1t_{11}))h(t-b_2t_{21})dt_{21}\right)^2dt_{11}$$
$$= \frac{1}{b_1}\int_{-L}^{L}K(t_{11})^2\left(1-H(t-b_1t_{11})+\frac{1}{n}\right)^{-2}h(t-b_1t_{11})^{-1}$$
$$\times\left(\int_{-L\left(1+\frac{b_1}{b_2}\right)}^{L\left(1+\frac{b_1}{b_2}\right)}K\left(t_{21}-\frac{t_{11}b_1}{b_2}\right)(p(t-b_2t_{21})-p(t-b_1t_{11}))h(t-b_2t_{21})dt_{21}\right)^2dt_{11}$$

Some further application of the dominated convergence theorem and condition **(V.2)** imply

$$E\left[\frac{K_{b_1}(t-T_1)^2K_{b_2}(T_1-T_2)K_{b_2}(T_1-T_3)(\delta_2-p(T_1))(\delta_3-p(T_1))\mathbf{1}(T_1\in I)}{\left(1-H(T_1)+\frac{1}{n}\right)^2h(T_1)^2}\right]$$
$$= o\left(b_1^{-1}\right).$$

Now, using (96)

$$E\left[\frac{K_{b_1}(t-T_1)K_{b_2}(T_1-T_2)(\delta_2-p(T_1))\mathbf{1}(T_1\in I)}{\left(1-H(T_1)+\frac{1}{n}\right)h(T_1)}\right]=O\left(b_1^2\right),$$

and we can conclude

$$M_{1,2,1,3}=o\left(b_1^{-1}\right). \tag{98}$$

Similar arguments can be used to obtain,

$$M_{1,2,3,1}=o\left(b_1^{-1}\right) \tag{99}$$

and

$$M_{1,2,2,3} = o\left(b_1^{-1}\right). \tag{100}$$

If $\# \{i,j,k,l\} < 3$, it suffices the following bound

$$\begin{aligned} |M_{i,j,k,l}| &\leq \Bigg| E\Bigg[K_{b_1}(t-T_i) K_{b_2}(T_i - T_j) K_{b_1}(t-T_k) K_{b_2}(T_k - T_l) \\ &\quad \times (\delta_j - p(T_i))(\delta_l - p(T_k)) \left(1 - H(T_i) + \frac{1}{n}\right)^{-1} \\ &\quad \times \left(1 - H(T_k) + \frac{1}{n}\right)^{-1} h(T_i)^{-1} h(T_k)^{-1} \mathbf{1}(T_i \in I) \mathbf{1}(T_k \in I) \Bigg] \Bigg| \\ &\quad + \Big| E\left[K_{b_1}(t-T_i) K_{b_2}(T_i - T_j)(\delta_j - p(T_i)) \right. \\ &\quad + \left. \left(1 - H(T_i) + \frac{1}{n}\right) h(T_i) \mathbf{1}(T_i \in I) \right] \Big|^2 \\ &\leq \frac{1}{b_1^2 b_2^2} \frac{2\|K\|_\infty^4}{(1-H(t_0))^2 \delta^2} \end{aligned}$$

where $\delta > 0$ is a lower bound of the density h in the interval I.

Therefore, using **(V.2)**, for any i,j,k,l,

$$M_{i,j,k,l} = O\left(b_1^{-4}\right). \tag{101}$$

Collecting (97)-(101), we have

$$\begin{aligned} Var[J_2] &= \frac{1}{nb_1} \frac{a \int_{-L(1+a)}^{L(1+a)} \left(\int_{-L}^{L} K(t_{11}) K(t_{21} - t_{11} a) \, dt_{11} \right)^2 dt_{21} p(t)(1-p(t)) h(t)}{(1-H(t))^2} \qquad (102) \\ &\quad + o\left(n^{-1} b_1^{-1}\right). \qquad (103) \end{aligned}$$

Finally, the covariance between J_1 and J_2 is

$$\begin{aligned} Cov[J_1, J_2] &= \frac{1}{n^3} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} Cov\left[\frac{K_{b_1}(t-T_i) p(T_i) \mathbf{1}(T_i \in I)}{1 - H(T_i) + \frac{1}{n}}, \right. \\ &\quad \left. \frac{K_{b_1}(t-T_j) K_{b_2}(T_j - T_k)(\delta_k - p(T_j)) \mathbf{1}(T_j \in I)}{\left(1 - H(T_j) + \frac{1}{n}\right) h(T_j)} \right] \\ &= \frac{1}{n^3} \left(n M_{1,1,1} + n(n-1) M_{1,2,1} + n(n-1) M_{1,1,2} \right) \end{aligned}$$

where

$$\begin{aligned} M_{i,j,k} &= Cov\left[\frac{K_{b_1}(t-T_i) p(T_i) \mathbf{1}(T_i \in I)}{1 - H(T_i) + \frac{1}{n}}, \right. \\ &\quad \left. \frac{K_{b_1}(t-T_j) K_{b_2}(T_j - T_k)(\delta_k - p(T_j)) \mathbf{1}(T_j \in I)}{\left(1 - H(T_j) + \frac{1}{n}\right) h(T_j)} \right]. \end{aligned}$$

Since

$$E\left[\frac{K_{b_1}(t-T_1)K_{b_2}(0)(\delta_1-p(T_1))\mathbf{1}(T_1\in I)}{\left(1-H(T_1)+\frac{1}{n}\right)h(T_1)}\right]=0$$

and

$$E\left[\frac{K_{b_1}(t-T_1)^2K_{b_2}(0)p(T_1)(\delta_1-p(T_1))\mathbf{1}(T_1\in I)}{\left(1-H(T_i)+\frac{1}{n}\right)^2h(T_1)}\right]=0,$$

we have

$$M^{1,1,1}=0. \tag{104}$$

For $M_{1,1,2}$,

$$\begin{aligned}
&E\left[\frac{K_{b_1}(t-T_1)^2K_{b_2}(T_1-T_2)p(T_1)(\delta_2-p(T_1))\mathbf{1}(T_1\in I)}{\left(1-H(T_1)+\frac{1}{n}\right)^2h(T_1)}\right]\\
&=\int_0^\infty\int_\varepsilon^{t_0}\frac{K_{b_1}(t-t_1)^2K_{b_2}(t_1-t_2)p(t_1)(p(t_2)-p(t_1))h(t_2)}{\left(1-H(t_1)+\frac{1}{n}\right)^2}dt_1dt_2\\
&=\frac{1}{b_1}\int_{-\infty}^{\frac{t}{b_2}}\int_{\frac{t-t_0}{b_1}}^{\frac{t-\varepsilon}{b_1}}K(t_{11})^2K\left(t_{21}-\frac{b_1t_{11}}{b_2}\right)p(t-b_1t_{11})\\
&\times(p(t-b_2t_{21})-p(t-b_1t_{11}))\frac{h(t-b_2t_{21})}{\left(1-H(t-b_1t_{11})+\frac{1}{n}\right)^2}dt_{11}dt_{21}\\
&=\frac{1}{b_1}\int_{-L\left(1+\frac{b_1}{b_2}\right)}^{L\left(1+\frac{b_1}{b_2}\right)}\int_{-L}^{L}K(t_{11})^2K\left(t_{21}-\frac{b_1t_{11}}{b_2}\right)p(t-b_1t_{11})\\
&\times(p(t-b_2t_{21})-p(t-b_1t_{11}))\frac{h(t-b_2t_{21})}{\left(1-H(t-b_1t_{11})+\frac{1}{n}\right)^2}dt_{11}dt_{21}\\
&=o\left(b_1^{-1}\right)
\end{aligned}$$

after using the change of variable $\frac{t-t_1}{b_1}=t_{11}$, $\frac{t-t_2}{b_2}=t_{21}$, conditions $b_1<\frac{(t_0-t)\wedge(t-\varepsilon)\wedge t/2}{L}$, $b_2<\frac{t}{2L}$, the dominated convergence theorem and condition **(V.2)**. As proved in (88) and (96),

$$E\left[\frac{K_{b_1}(t-T_1)p(T_1)\mathbf{1}(T_1\in I)}{1-H(T_1)+\frac{1}{n}}\right]=O(1)$$

and

$$E\left[\frac{K_{b_1}(t-T_1)K_{b_2}(T_1-T_2)(p(T_2)-p(T_1))\mathbf{1}(T_1\in I)}{\left(1-H(T_1)+\frac{1}{n}\right)h(T_1)}\right]=O(b_1),$$

and consequently

$$M_{1,1,2} = o\left(b_1^{-1}\right). \tag{105}$$

Similarly, it is easy to conclude

$$M_{1,2,1} = o\left(b_1^{-1}\right). \tag{106}$$

From (104)-(106), we have

$$Cov\left[J_1, J_2\right] = o\left(n^{-1} b_1^{-1}\right). \tag{107}$$

Finally, (88) and (96) imply (84) and, since

$$Var\left[\overline{\lambda}_{TWP}(t)\right] = Var\left[J_1\right] + 2Cov\left[J_1, J_2\right] + Var\left[J_2\right],$$

(89), (102) and (107) we obtain (85). ■

Remark 12 *If, in addition to condition **(K.1)**, the kernel function, K, is concave,*

$$\begin{aligned} \int_{-L}^{L} K(u) K(v - ua)\, du &= \int_{0}^{L} K(u) \left(K(v + ua) + K(v - ua)\right) du \\ &\leq 2 \int_{0}^{L} K(u) K(v)\, du = K(v). \end{aligned}$$

Thus,

$$\int_{-L(1+a)}^{L(1+a)} \left(\int_{-L}^{L} K(u) K(v - ua)\, du \right)^2 dv \leq \int_{-L(1+a)}^{L(1+a)} K(v)^2\, dv = c_K,$$

and the dominant term in the variance of $\overline{\lambda}_{TWP}(t)$ is than the leading terms in the variances of $\overline{\lambda}_{WLNW}(t)$, $\overline{\lambda}_{WLLL}(t)$, $\overline{\lambda}_{WLNW}(t)$ and $\overline{\lambda}_{RRLL}(t)$.

Remark 13 *As pointed out in Section 1.2, if the smoothing parameter b_2 tends to zero 0 the presmoothed Tanner-Wong estimator tends to the Tanner-Wong estimator. As a consequence, the leading terms in the bias and variance of $\widehat{\lambda}_{TW}(t)$ coincide with the limit, when a tends to ∞, of the dominant terms of the bias and variance of $\overline{\lambda}_{TWP}(t)$.*

3 Asymptotic normality

The following result by van der Vaart (1998) will be used along this section.

Theorem 14 *Let $\mathcal{S}_n, n = 1, 2, ...,$ be vector spaces of random variables with finite second moment, that contain the constants. Let T_n be random variables with projections $\widehat{S}_n$ on $\mathcal{S}_n$. If*

$$\lim_{n\to\infty} \frac{Var\,[T_n]}{Var\left[\widehat{S}_n\right]} = 1,$$

then

$$\frac{T_n - E\,[T_n]}{\sqrt{Var\,[T_n]}} - \frac{\widehat{S}_n - E\left[\widehat{S}_n\right]}{\sqrt{Var\left[\widehat{S}_n\right]}} \xrightarrow{P} 0.$$

Theorems 15-21 give the asymptotic normality of the estimators $\widehat{\lambda}_{WLNW}(t)$, $\widehat{\lambda}_{RRNW}(t)$, $\widehat{\lambda}_{WLLL}(t)$, $\widehat{\lambda}_{RRLL}(t)$ and $\widetilde{\lambda}_{TWP}(t)$, suitably normalized.

Theorem 15 *Assume conditions **(K.1)**, **(P.1)**, **(H.1)**, **(V.1)** and **(V.2)** and the existence of the limit $\lim_{n\to\infty} nb_1^5$, then*

$$\sqrt{nb_1}\left(\widehat{\lambda}_{WLNW}(t) - \lambda_F(t)\right) \xrightarrow{d} N(l_0\xi_1(t), \xi_2(t)) \tag{108}$$

where

$$\xi_1(t) = \frac{1}{2}\mu_K\left(\lambda''_H(t)p(t) + \frac{1}{a^2}\left(\lambda_H(t)p''(t) + 2\left(\lambda'_H(t) - \lambda_H(t)^2\right)p'(t)\right)\right), \tag{109}$$

$$\xi_2(t) = \sqrt{\frac{c_K\lambda_F(t)\,(p(t) + a(1 - p(t)))}{1 - H(t)}} \tag{110}$$

and

$$\lim_{n\to\infty} nb_1^5 = l_0^2, \qquad l_0 \in [0, \infty)\,. \tag{111}$$

Proof. In Theorem 7 the following asymptotic expression was found

$$\widehat{\lambda}_{WLNW}(t) - \lambda_F(t) = \overline{\lambda}_{WLNW}(t) - \lambda_F(t) + o_P\left(n^{-1/2}b_1^{-1/2}\right), \tag{112}$$

where $\overline{\lambda}_{WLNW}(t)$ was introduced in (58).

Defining

$$U_{n,i} = \frac{1}{n}\left(\frac{K_{b_1}(t - T_i)\,p(t)}{1 - H(T_i) + \frac{1}{n}} + \frac{K_{b_2}(t - T_i)\,(\delta_i - p(t))}{1 - H(t)}\right),$$

it is immediate to check

$$\overline{\lambda}_{WLNW}(t)=\sum_{i=1}^{n}U_{n,i},$$

where $U_{n,i}, i=1,2,...,n;\ n=1,2,...,$ constitutes a triangular array of iid random variables.

We now prove the asymptotic normality of $\frac{\overline{\lambda}_{WLNW}(t)-E\left[\overline{\lambda}_{WLNW}(t)\right]}{\sqrt{Var\left[\overline{\lambda}_{WLNW}(t)\right]}}$. Using Liapunov Theorem, it suffices to prove that

$$\lim_{n\to\infty}\frac{nE\left[\left|U_{n,1}-E(U_{n,1})\right|^{3}\right]}{\left(Var\left[\overline{\lambda}_{WLNW}(t)\right]\right)^{3/2}}=0.$$

The c_p inequality, for $p\geq 1$, gives

$$E\left[\left|U_{n,1}-E(U_{n,1})\right|^{p}\right]\leq 2^{p-1}\left(E\left[\left|U_{n,1}\right|^{p}\right]+\left|E\left[U_{n,1}\right]\right|^{p}\right)$$

and, for $1\leq q\leq r$,

$$\left(E\left[\left|U_{n,1}\right|^{q}\right]\right)^{\frac{1}{q}}\leq\left(E\left[\left|U_{n,1}\right|^{r}\right]\right)^{\frac{1}{r}}.$$

Thus

$$\frac{nE\left[\left|U_{n,1}-E(U_{n,1})\right|^{3}\right]}{\left(Var\left[\overline{\lambda}_{WLNW}(t)\right]\right)^{3/2}}\leq\frac{8nE\left[\left|U_{n,1}\right|^{3}\right]}{\left(Var\left[\overline{\lambda}_{WLNW}(t)\right]\right)^{3/2}}. \tag{113}$$

The support of the kernel K is $[-L,L]$, which implies

$$\begin{aligned}U_{n,1} &= \frac{1}{n}\left(\frac{K_{b_1}(t-T_1)\,p(t)}{1-H(T_1)+\frac{1}{n}}+\frac{K_{b_2}(t-T_1)\,(\delta_1-p(t))}{1-H(t)}\right)\mathbf{1}\left\{T_1\leq t+b_1L\right\}\\ &\quad+\frac{1}{n}\left(\frac{K_{b_1}(t-T_1)\,p(t)}{1-H(T_1)+\frac{1}{n}}+\frac{K_{b_2}(t-T_1)\,(\delta_1-p(t))}{1-H(t)}\right)\mathbf{1}\left\{T_1> t+b_1L\right\}\\ &= \frac{1}{n}\left(\frac{K_{b_1}(t-T_1)\,p(t)}{1-H(T_1)+\frac{1}{n}}+\frac{K_{b_2}(t-T_1)\,(\delta_1-p(t))}{1-H(t)}\right)\mathbf{1}\left\{T_1\leq t+b_1L\right\}\\ &\quad+\frac{1}{n}\frac{K_{b_2}(t-T_1)\,(\delta_1-p(t))}{1-H(t)}\mathbf{1}\left\{T_1> t+b_1L\right\}.\end{aligned}$$

Whenever $b_1<\frac{t_0-t}{L}$, we have $t+b_1L<t_0$, which leads to

$$|U_{n,1}|\leq\frac{\|K\|_{\infty}}{n}\left(\frac{1}{b_1(1-H(t_0))}+\frac{2}{b_2(1-H(t))}\right)\leq\frac{1}{n}\frac{\|K\|_{\infty}}{1-H(t_0)}\left(\frac{1}{b_1}+\frac{2}{b_2}\right)\ a.s.$$

and

$$E\left[\left|U_{n,1}\right|^{3}\right]\leq\frac{1}{n^3}\frac{\|K\|_{\infty}^{3}}{\left(1-H(t_0)\right)^{3}}\left(\frac{1}{b_1}+\frac{2}{b_2}\right)^{3}.$$

Using this equation, **(V.2)** and (60),

$$\frac{nE\left[\left|U_{n,1}-E(U_{n,1})\right|^3\right]}{\left(Var\left[\overline{\lambda}_{WLNW}(t)\right]\right)^{3/2}} = O(n)O\left(n^{3/2}b_1^{3/2}\right)O\left(n^{-3}b_1^{-3}\right) = O\left(n^{-1/2}b_1^{-3/2}\right).$$

Now condition **(V.1)** gives

$$\lim_{n\to\infty}\frac{nE\left[\left|U_{n,1}-E(U_{n,1})\right|^3\right]}{\left(Var\left[\overline{\lambda}_{WLNW}(t)\right]\right)^{3/2}} = 0.$$

Therefore, it has been proved that

$$\begin{aligned} & \frac{\overline{\lambda}_{WLNW}(t)-E\left[\overline{\lambda}_{WLNW}(t)\right]}{\sqrt{Var\left[\overline{\lambda}_{WLNW}(t)\right]}} \\ = \; & \frac{\overline{\lambda}_{WLNW}(t)-\lambda_F(t)-E\left[\overline{\lambda}_{WLNW}(t)-\lambda_F(t)\right]}{\sqrt{Var\left[\overline{\lambda}_{WLNW}(t)\right]}} \xrightarrow{d} N(0,1). \end{aligned}$$

We now use (59), (60) and definitions (109) and (110) to obtain

$$\begin{aligned} & \frac{\overline{\lambda}_{WLNW}(t)-\lambda_F(t)-E\left[\overline{\lambda}_{WLNW}(t)-\lambda_F(t)\right]}{\sqrt{Var\left[\overline{\lambda}_{WLNW}(t)\right]}} \\ = \; & \sqrt{nb_1}\frac{\overline{\lambda}_{WLNW}(t)-\lambda_F(t)-b_1^2\xi_1(t)+o\left(b_1^2\right)}{\xi_2(t)+o(1)} \\ = \; & \sqrt{nb_1}\frac{\overline{\lambda}_{WLNW}(t)-\lambda_F(t)-b_1^2\xi_1(t)}{\xi_2(t)+o(1)} \\ & +o\left(n^{1/2}b_1^{5/2}\right) \\ = \; & \sqrt{nb_1}\frac{\overline{\lambda}_{WLNW}(t)-\lambda_F(t)-b_1^2\xi_1(t)}{\xi_2(t)}\frac{\xi_2(t)}{\xi_2(t)+o(1)} \\ & +o\left(n^{1/2}b_1^{5/2}\right), \end{aligned}$$

which gives,

$$\begin{aligned} & \sqrt{nb_1}\frac{\overline{\lambda}_{WLNW}(t)-\lambda_F(t)-b_1^2\xi_1(t)}{\xi_2(t)} \\ = \; & \frac{\overline{\lambda}_{WLNW}(t)-\lambda_F(t)-E\left[\overline{\lambda}_{WLNW}(t)-\lambda_F(t)\right]}{\sqrt{Var\left[\overline{\lambda}_{WLNW}(t)\right]}}\frac{\xi_2(t)+o(1)}{\xi_2(t)} \\ & +o\left(n^{1/2}b_1^{5/2}\right)O(1). \end{aligned}$$

Equation (111) implies $nb_1^5 = O(1)$. This fact together with $\frac{\xi_2(t)+o(1)}{\xi_2(t)} \to 1$ and Slutsky's Theorem lead to

$$\sqrt{nb_1}\frac{\overline{\lambda}_{WLNW}(t) - \lambda_F(t) - b_1^2\xi_1(t)}{\xi_2(t)} \xrightarrow{d} N(0,1).$$

Slutsky's Theorem can be further applied to (112) to obtain

$$\sqrt{nb_1}\frac{\widehat{\lambda}_{WLNW}(t) - \lambda_F(t) - b_1^2\xi_1(t)}{\xi_2(t)} \xrightarrow{d} N(0,1)$$

and, finally, (108). ■

Theorem 16 *Let us assume conditions* ***(K.1)****,* ***(P.1)****,* ***(H.1)****,* ***(V.1)*** *and* ***(V.2)*** *and the existence of the limit* $\lim_{n\to\infty} nb_1^5$*. Then*

$$\sqrt{nb_1}\left(\widehat{\lambda}_{RRNW}(t) - \lambda_F(t)\right) \xrightarrow{d} N(l_0\xi_3(t), \xi_2(t)) \tag{114}$$

where $\xi_2(t)$ *and* l_0 *have been defined in (110) and (111), respectively, and*

$$\begin{aligned}\xi_3(t) &= \frac{1}{2}\mu_K\left(\lambda_H''(t)p(t) + \left(\lambda_H(t)^2 - 3\lambda_H'(t)\right)\lambda_F(t)\right. \\ &\left.+\frac{1}{a^2}\left(\lambda_H(t)p''(t) + 2\left(\lambda_H'(t) - \lambda_H(t)^2\right)p'(t)\right)\right).\end{aligned} \tag{115}$$

Proof. The asymptotic representation (61) in Theorem 8, gives

$$\begin{aligned}\sqrt{nb_1}\left(\widehat{\lambda}_{RRNW}(t) - \lambda_F(t)\right) &= \sqrt{nb_1}\left(\widehat{\lambda}_{WLNW}(t) - \lambda_F(t)\right) \\ &+\frac{1}{2}\sqrt{nb_1^5}\mu_K\left(\lambda_H(t)^2 - 3\lambda_H'(t)\right)\lambda_F(t) + o_P(1)\end{aligned} \tag{116}$$

Now, using Theorem 15,

$$\sqrt{nb_1}\left(\widehat{\lambda}_{WLNW}(t) - \lambda_F(t)\right) \xrightarrow{d} N(l_0\xi_1(t), \xi_2(t))$$

where $\xi_1(t)$ and $\xi_2(t)$ have been defined in (109) and (110).

Applying Slutsky's Theorem to (116), we have

$$\begin{aligned}&\sqrt{nb_1}\left(\widehat{\lambda}_{RRNW}(t) - \lambda_F(t)\right) \xrightarrow{d} \\ &N\left(l_0\left(\xi_1(t) + \frac{1}{2}\mu_K\left(\lambda_H(t)^2 - 3\lambda_H'(t)\right)\lambda_F(t)\right), \xi_2(t)\right),\end{aligned}$$

or, equivalently, using the definition (115)

$$\xi_3(t) = \xi_1(t) + \frac{1}{2}\mu_K\left(\lambda_H(t)^2 - 3\lambda_H'(t)\right)\lambda_F(t),$$

the final result (114). ■

The following lemma will be used in the proof of Theorem 18.

Lemma 17 *Under conditions* ***(K.1)***, ***(P.1)***, ***(H.1)***, ***(V.1)*** *and* ***(V.2)***, *and using the notation $\overline{\lambda}^*_{WLLL}(t)$ for the Hájek projection of $\overline{\lambda}_{WLLL}(t)$, we have*

$$\frac{Var\left[\overline{\lambda}^*_{WLLL}(t)\right]}{Var\left[\overline{\lambda}_{WLLL}(t)\right]} \rightarrow 1. \tag{117}$$

Proof. Let us define $Z_i = (T_i, \delta_i)$, for $i = 1, 2, ..., n$. The Hájek projection of $\overline{\lambda}_{WLLL}(t)$ on the vector space $\mathcal{S}_n$ of all random variables of the form $\sum_{i=1}^{n} \tau_{ni}(Z_i)$, for arbitrary measurable functions τ_{ni} with $E\left[\tau_{ni}(Z_i)^2\right] < \infty$, is given by

$$\overline{\lambda}^*_{WLLL}(t) = \sum_{i=1}^{n} E\left[\overline{\lambda}_{WLLL}(t)\,|Z_i\right] - (n-1)E\left[\overline{\lambda}_{WLLL}(t)\right].$$

Since $\overline{\lambda}_{WLLL}(t) = p(t)A_n + \lambda_H(t)D_n + \lambda_F(t)$ and $p(t)A_n + \lambda_F(t)$ is an element of $\mathcal{S}_n$, it suffices to find the Hájek projection of D_n,

$$D^*_n = \sum_{i=1}^{n} E\left[D_n\,|Z_i\right] - (n-1)E\left[D_n\right],$$

which leads tot he Hájek projection of $\overline{\lambda}_{WLLL}(t)$,

$$\overline{\lambda}^*_{WLLL}(t) = p(t)A_n + \lambda_F(t) + \lambda_H(t)D^*_n.$$

We introduce the notation

$$D_{n,k} = \frac{1}{n^2 b_2^2 \mu_K h(t)^2} \sum_{j=1}^{n} K_{b_2}(t-T_k)K_{b_2}(t-T_j)(t-T_j)(T_k-T_j)\left(\delta_k - p(t)\right)$$

in order to write

$$D_n = \sum_{k=1}^{n} D_{n,k}.$$

For $k \neq i$,

$$\begin{aligned} E\left[D_{n,k}\,|Z_i\right] &= \frac{n-2}{n^2 b_2^2 \mu_K h(t)^2} \\ &\quad \times E\left[K_{b_2}(t-T_k)\left(\delta_k - p(t)\right)K_{b_2}(t-T_j)(t-T_j)(T_k-T_j)\right] \\ &\quad + \frac{1}{n^2 b_2^2 \mu_K h(t)^2} K_{b_2}(t-T_i)(t-T_i) \\ &\quad \times E\left[K_{b_2}(t-T_k)\left(\delta_k - p(t)\right)(T_k - T_i)\,|Z_i\right] \\ &= \frac{n-2}{n^2 b_2^2 \mu_K h(t)^2} \\ &\quad \times E\left[K_{b_2}(t-T_k)\left(\delta_k - p(t)\right)K_{b_2}(t-T_j)(t-T_j)(T_k-T_j)\right] \\ &\quad + \frac{1}{n^2 b_2^2 \mu_K h(t)^2} K_{b_2}(t-T_i)(t-T_i) \\ &\quad \times \left(E\left[K_{b_2}(t-T_k)\left(\delta_k - p(t)\right)T_k\right] - T_i E\left[K_{b_2}(t-T_k)\left(\delta_k - p(t)\right)\right]\right) \end{aligned} \tag{118}$$

whereas, for $k = i$,

$$\begin{aligned} E\left[D_{n,i}\left|Z_i\right.\right] &= \frac{n-1}{n^2 b_2^2 \mu_K h(t)^2} \\ &\quad \times K_{b_2}(t-T_i)\left(\delta_i - p(t)\right) E\left[K_{b_2}(t-T_j)(t-T_j)(T_i-T_j)\left|Z_i\right.\right] \\ &= \frac{n-1}{n^2 b_2^2 \mu_K h(t)^2} K_{b_2}(t-T_i)\left(\delta_i - p(t)\right) \\ &\quad \times \left(T_i E\left[K_{b_2}(t-T_j)(t-T_j)\right] - E\left[K_{b_2}(t-T_j)(t-T_j)T_j\right]\right) \quad (119) \end{aligned}$$

where, both in (118) and (119) we have assumed $i \neq j$.

Thus,

$$\begin{aligned} E\left[D_n\left|Z_i\right.\right] &= \frac{n-1}{n^2 b_2^2 \mu_K h(t)^2} \\ &\quad \times \left((n-2) E\left[K_{b_2}(t-T_1)\left(\delta_1 - p(t)\right) K_{b_2}(t-T_2)(t-T_2)(T_1-T_2)\right]\right. \\ &\quad + K_{b_2}(t-T_i)(t-T_i)\left(M_{101}(b_2) - T_i M_{100}(b_2)\right) \\ &\quad \left. - K_{b_2}(t-T_i)\left(\delta_i - p(t)\right)\left(M_{011}(b_2) - T_i M_{010}(b_2)\right)\right) \end{aligned}$$

where, for $r, s, t = 0, 1$, we use the notation

$$M_{rst}(b_2) = E\left[K_{b_2}(t-T_1)\left(\delta_1 - p(t)\right)^r (t-T_1)^s T_1^t\right].$$

Since, using (43) and (44),

$$E\left[D_n\right] = \frac{n-1}{n b_2^2 \mu_K h(t)^2} E\left[K_{b_2}(t-T_1) K_{b_2}(t-T_2)(t-T_2)(T_1-T_2)\left(\delta_1 - p(t)\right)\right],$$

then the Hájek projection of D_n is

$$\begin{aligned} D_n^* &= \frac{n-1}{n^2 b_2^2 \mu_K h(t)^2} \sum_{i=1}^n K_{b_2}(t-T_i)\left((t-T_i)\left(M_{101}(b_2) - T_i M_{100}(b_2)\right)\right. \\ &\quad \left. - \left(\delta_i - p(t)\right)\left(M_{011}(b_2) - T_i M_{010}(b_2)\right)\right) - E\left[D_n\right] \end{aligned}$$

and the Hájek projection of $\overline{\lambda}_{WLLL}(t)$,

$$\begin{aligned} \overline{\lambda}_{WLLL}^*(t) &= \sum_{i=1}^n \left(\frac{p(t)}{n} \frac{K_{b_1}(t-T_i)}{1 - H(T_i) + \frac{1}{n}} + \frac{(n-1)\lambda_H(t)}{n^2 b_2^2 \mu_K h(t)^2} K_{b_2}(t-T_i)\right. \\ &\quad \left. \times \left((t-T_i)\left(M_{101}(b_2) - T_i M_{100}(b_2)\right) - \left(\delta_i - p(t)\right)\left(M_{011}(b_2) - T_i M_{010}(b_2)\right)\right)\right) \\ &\quad - \lambda_H(t) E\left[D_n\right]. \end{aligned} \quad (120)$$

To study the variance of $\overline{\lambda}_{WLLL}^*(t)$,

$$Var\left[\overline{\lambda}_{WLLL}^*(t)\right] = p(t)^2 Var\left[A_n\right] + 2p(t)\lambda_H(t) Cov\left[A_n, D_n^*\right] + \lambda_H(t)^2 Var\left[D_n^*\right],$$

we start from the variance of D_n^*. We first denote

$$D_{1,i}^* = \frac{n-1}{n^2 b_2^2 \mu_K h(t)^2} K_{b_2}(t-T_i)(t-T_i)\left(M_{101}(b_2) - T_i M_{100}(b_2)\right),$$
$$D_{2,i}^* = \frac{n-1}{n^2 b_2^2 \mu_K h(t)^2} K_{b_2}(t-T_i)\left(\delta_i - p(t)\right)\left(M_{011}(b_2) - T_i M_{010}(b_2)\right),$$

and write

$$D_n^* = \sum_{i=1}^{n} \left(D_{1,i}^* - D_{2,i}^*\right) - E\left[D_n\right]$$

which leads to

$$Var\left[D_n^*\right] = n\left(Var\left[D_{1,1}^*\right] - 2Cov\left[D_{1,1}^*, D_{2,1}^*\right] + Var\left[D_{2,1}^*\right]\right). \tag{121}$$

Before analyzing these variances and covariances, we study the order of the terms $M_{rst}(b_2)$. Using the change of variable $\frac{t-t_1}{b_2} = t_{11}$, assuming $b_2 < \frac{t}{L}$ and by means of some Taylor expansions, it is not difficult to reach at

$$\begin{aligned} M_{010}(b_2) &= E\left[K_{b_2}(t-T_1)(t-T_1)\right] = \int_0^\infty K_{b_2}(t-t_1)(t-t_1)h(t_1)dt_1 \\ &= b_2 \int_{-L}^{L} K(t_{11})t_{11}h(t-b_2t_{11})dt_{11} = -b_2^2\mu_K h'(t) + o\left(b_2^2\right), \end{aligned} \tag{122}$$

$$\begin{aligned} M_{011}(b_2) &= E\left[K_{b_2}(t-T_1)(t-T_1)T_1\right] = \int_0^\infty K_{b_2}(t-t_1)(t-t_1)t_1h(t_1)dt_1 \\ &= b_2 \int_{-L}^{L} K(t_{11})t_{11}(t-b_2t_{11})h(t-b_2t_{11})dt_{11} \\ &= -b_2^2\mu_K\left(th'(t) + h(t)\right) + o\left(b_2^2\right), \end{aligned} \tag{123}$$

$$\begin{aligned} M_{100}(b_2) &= E\left[K_{b_2}(t-T_1)\left(\delta_1 - p(t)\right)\right] = \int_0^\infty K_{b_2}(t-t_1)(p(t_1)-p(t))h(t_1)dt_1 \\ &= \int_{-L}^{L} K(t_{11})(p(t-b_2t_{11})-p(t))h(t-b_2t_{11})dt_{11} \\ &= b_2^2\mu_K\left(\frac{1}{2}p''(t)h(t) + h'(t)p'(t)\right) + o\left(b_2^2\right) \end{aligned} \tag{124}$$

and

$$\begin{aligned} M_{101}(b_2) &= E\left[K_{b_2}(t-T_1)\left(\delta_1 - p(t)\right)T_1\right] \\ &= \int_0^\infty K_{b_2}(t-t_1)(p(t_1)-p(t))t_1h(t_1)dt_1 \\ &= \int_{-L}^{L} K(t_{11})(p(t-b_2t_{11})-p(t))(t-b_2t_{11})h(t-b_2t_{11})dt_{11} \\ &= b_2^2\mu_K\left(\frac{1}{2}p''(t)h(t)t + p'(t)h'(t)t + p'(t)h(t)\right) + o\left(b_2^2\right). \end{aligned} \tag{125}$$

Coming back to the first variance in the right hand side of (121),

$$
\begin{aligned}
Var\left[D^*_{1,1}\right] &= \frac{(n-1)^2}{n^4 b_2^4 \mu_K^2 h(t)^4} \\
&\quad \times E\left[K_{b_2}(t-T_1)^2 (t-T_1)^2 \left(M_{101}(b_2) - T_1 M_{100}(b_2)\right)^2\right] \\
&\quad - \frac{(n-1)^2}{n^4 b_2^4 \mu_K^2 h(t)^4} \\
&\quad \times \left(E\left[K_{b_2}(t-T_1)(t-T_1)\left(M_{101}(b_2) - T_1 M_{100}(b_2)\right)\right]\right)^2 \\
&= \frac{(n-1)^2}{n^4 b_2^4 \mu_K^2 h(t)^4} \\
&\quad \times E\left[K_{b_2}(t-T_1)^2 (t-T_1)^2 \left(M_{101}(b_2) - T_1 M_{100}(b_2)\right)^2\right] \\
&\quad - \frac{(n-1)^2}{n^4 b_2^4 \mu_K^2 h(t)^4} \\
&\quad \times \left(M_{010}(b_2) M_{101}(b_2) - M_{011}(b_2) M_{100}(b_2)\right)^2 \\
&= \frac{(n-1)^2}{n^4 b_2^4 \mu_K^2 h(t)^4} E\left[K_{b_2}(t-T_1)^2 (t-T_1)^2 \left(M_{101}(b_2) - T_1 M_{100}(b_2)\right)^2\right] \\
&\quad - O\left(n^{-2} b_2^4\right),
\end{aligned}
$$

where the last bound is a consequence of (122)-(125). Some further change of variable, Taylor expansions and (122)-(125), lead to

$$
\begin{aligned}
& E\left[K_{b_2}(t-T_1)^2 (t-T_1)^2 \left(M_{101}(b_2) - T_1 M_{100}(b_2)\right)^2\right] \\
= & \int_0^\infty K_{b_2}(t-t_1)^2 (t-t_1)^2 \left(M_{101}(b_2) - t_1 M_{100}(b_2)\right)^2 h(t_1) dt_1 \\
= & \; b_2 \int_{-L}^{L} K(t_{11})^2 t_{11}^2 \left(M_{101}(b_2) - t M_{100}(b_2) + b_2 t_{11} M_{100}(b_2)\right)^2 h(t - b_2 t_{11}) dt_{11} \\
= & \; b_2^5 \mu_K^2 \int_{-L}^{L} K(t_{11})^2 t_{11}^2 dt_{11} p'(t)^2 h(t)^3 + o\left(b_2^5\right)
\end{aligned}
$$

which implies

$$
Var\left[D^*_{1,1}\right] = \frac{b_2}{n^2} \int_{-L}^{L} K(t_{11})^2 t_{11}^2 dt_{11} p'(t)^2 h(t)^{-1} + o(n^{-2} b_2). \qquad (126)
$$

For the second variance in the right hand side of (121),

$$
\begin{aligned}
Var\left[D^*_{2,1}\right] &= \frac{(n-1)^2}{n^4 b_2^4 \mu_K^2 h(t)^4} \\
&\quad \times E\left[K_{b_2}(t-T_1)^2(\delta_1-T_1)^2\left(M_{011}(b_2)-T_1 M_{010}(b_2)\right)^2\right] \\
&\quad - \frac{(n-1)^2}{n^4 b_2^4 \mu_K^2 h(t)^4} \\
&\quad \times \left(E\left[K_{b_2}(t-T_1)(\delta_1-T_1)\left(M_{011}(b_2)-T_1 M_{010}(b_2)\right)\right]\right)^2 \\
&= \frac{(n-1)^2}{n^4 b_2^4 \mu_K^2 h(t)^4} \\
&\quad \times E\left[K_{b_2}(t-T_1)^2(\delta_1-T_1)^2\left(M_{011}(b_2)-T_1 M_{010}(b_2)\right)^2\right] \\
&\quad - \frac{(n-1)^2}{n^4 b_2^4 \mu_K^2 h(t)^4} \\
&\quad \times \left(M_{100}(b_2)M_{011}(b_2)-M_{101}(b_2)M_{010}(b_2)\right)^2 \\
&= \frac{(n-1)^2}{n^4 b_2^4 \mu_K^2 h(t)^4} E\left[K_{b_2}(t-T_1)^2(\delta_1-T_1)^2\left(M_{011}(b_2)-T_1 M_{010}(b_2)\right)^2\right] \\
&\quad - O\left(n^{-2} b_2^4\right)
\end{aligned}
$$

where the results (122)-(125) have been used. Similar arguments to the previous case give

$$
\begin{aligned}
&E\left[K_{b_2}(t-T_1)^2(\delta_1-T_1)^2\left(M_{011}(b_2)-T_1 M_{010}(b_2)\right)^2\right] \\
&= \int_0^\infty K_{b_2}(t-t_1)^2\left(p(t_1)\left(1-2p(t)\right)+p(t)^2\right)\left(M_{011}(b_2)-t_1 M_{010}(b_2)\right)^2 h(t_1)dt_1 \\
&= \frac{1}{b_2}\int_{-L}^{L} K(t_{11})^2\left(p(t-b_2 t_{11})\left(1-2p(t)\right)+p(t)^2\right) \\
&\times \left(M_{011}(b_2)-tM_{010}(b_2)+b_2 t_{11} M_{010}(b_2)\right)^2 h(t-b_2 t_{11})dt_{11} \\
&= b_2^3 \mu_K^2 c_K p(t)\left(1-p(t)\right)h(t)^3 + o\left(b_2^3\right)
\end{aligned}
$$

and, therefore,

$$
Var\left[D^*_{2,1}\right] = \frac{1}{n^2 b_2} c_K p(t)\left(1-p(t)\right)h(t)^{-1} + o\left(n^{-2} b_2^{-1}\right). \qquad (127)
$$

Finally, the covariance in (121) is

$$\begin{aligned}
Cov\left[D_{1,1}^*, D_{2,1}^*\right] &= \frac{(n-1)^2}{n^4 b_2^4 \mu_K^2 h(t)^4} E\left[K_{b_2}(t-T_1)^2 (t-T_1)\left(\delta_1 - p(t)\right)\right. \\
&\quad \times \left(M_{101}(b_2) - T_1 M_{100}(b_2)\right)\left.\left(M_{011}(b_2) - T_1 M_{010}(b_2)\right)\right] \\
&\quad - \frac{(n-1)^2}{n^4 b_2^4 \mu_K^2 h(t)^4}\left(M_{010}(b_2) M_{101}(b_2) - M_{011}(b_2) M_{100}(b_2)\right) \\
&\quad \times \left(M_{100}(b_2) M_{011}(b_2) - M_{101}(b_2) M_{010}(b_2)\right) \\
&= \frac{(n-1)^2}{n^4 b_2^4 \mu_K^2 h(t)^4} E\left[K_{b_2}(t-T_1)^2 (t-T_1)\left(\delta_1 - p(t)\right)\right. \\
&\quad \times \left(M_{101}(b_2) - T_1 M_{100}(b_2)\right)\left.\left(M_{011}(b_2) - T_1 M_{010}(b_2)\right)\right] - O\left(n^{-2} b_2^4\right),
\end{aligned}$$

obtained using, once more, (122)-(125). Standard arguments give

$$\begin{aligned}
& E\left[K_{b_2}(t-T_1)^2 (t-T_1)\left(\delta_1 - p(t)\right)\right. \\
& \left(M_{101}(b_2) - T_1 M_{100}(b_2)\right)\left.\left(M_{011}(b_2) - T_1 M_{010}(b_2)\right)\right] \\
& = \int_0^\infty K_{b_2}(t-t_1)^2 (t-t_1)\left(p(t_1) - p(t)\right)\left(M_{101}(b_2) - t_1 M_{100}(b_2)\right) \\
& \times \left(M_{011}(b_2) - t_1 M_{010}(b_2)\right) h(t_1) dt_1 \\
& = \int_{-L}^{L} K(t_{11})^2 t_{11}\left(p(t - b_2 t_{11}) - p(t)\right)\left(M_{101}(b_2) - t M_{100}(b_2) + b_2 t_{11} M_{100}(b_2)\right) \\
& \times \left(M_{011}(b_2) - t M_{010}(b_2) + b_2 t_{11} M_{010}(b_2)\right) h(t - b_2 t_{11}) dt_{11} \\
& = b_2^5 \mu_K^2 p'(t)^2 h(t)^3 \int_{-L}^{L} K(t_{11})^2 t_{11}^2 dt_{11} + o\left(b_2^5\right),
\end{aligned}$$

which implies

$$Cov\left[D_{1,1}^*, D_{2,1}^*\right] = \frac{b_2}{n^2} \int_{-L}^{L} K(t_{11})^2 t_{11}^2 dt_{11} p'(t)^2 h(t)^3 + o(n^{-2} b_2). \tag{128}$$

Collecting (126), (127) and (128), we conclude

$$Var\left[D_n^*\right] = \frac{1}{n b_2} \frac{c_K p(t)\left(1 - p(t)\right)}{h(t)} + o\left(n^{-1} b_2^{-1}\right). \tag{129}$$

The covariance between A_n and D_n^* is

$$Cov\left[A_n, D_n^*\right] = Cov\left[A_n, \sum_{i=1}^n D_{1,i}^*\right] - Cov\left[A_n, \sum_{i=1}^n D_{2,i}^*\right].$$

Using Cauchy-Schwarz inequality,

$$\left|Cov\left[A_n, \sum_{i=1}^n D_{1,i}^*\right]\right| \leq \left(Var\left[A_n\right] Var\left[\sum_{i=1}^n D_{1,i}^*\right]\right)^{1/2} = \left(Var\left[A_n\right] n Var\left[D_{1,1}^*\right]\right)^{1/2}$$

and condition **(V.2)**, we reach to

$$Cov\left[A_n, \sum_{i=1}^{n} D_{1,i}^{*}\right] = O\left(n^{-1/2} b_1^{-1/2}\right) O\left(n^{1/2}\right) O\left(n^{-1} b_1^{1/2}\right) = O\left(n^{-1}\right). \tag{130}$$

On the other hand,

$$\begin{aligned} Cov\left[A_n, \sum_{i=1}^{n} D_{2,i}^{*}\right] &= nCov\left[\frac{1}{n}\frac{K_{b_1}(t-T_1)}{1-H(T_1)+\frac{1}{n}},\right. \\ &\left. \frac{n-1}{n^2 b_2^2 \mu_K h(t)^2} K_{b_2}(t-T_1)(\delta_1 - p(t))(M_{011}(b_2) - T_1 M_{010}(b_2))\right] \\ &= \frac{n-1}{n^2 b_2^2 \mu_K h(t)^2} E\left[\frac{K_{b_1}(t-T_1)}{1-H(T_1)+\frac{1}{n}}\right. \\ &\left. \times K_{b_2}(t-T_1)(\delta_1 - p(t))(M_{011}(b_2) - T_1 M_{010}(b_2))\right] \\ &- \frac{n-1}{n^2 b_2^2 \mu_K h(t)^2} E\left[\frac{K_{b_1}(t-T_1)}{1-H(T_1)+\frac{1}{n}}\right] \\ &\times E\left[K_{b_2}(t-T_1)(\delta_1 - p(t))(M_{011}(b_2) - T_1 M_{010}(b_2))\right]. \end{aligned}$$

By means of the usual arguments,

$$\begin{aligned} & E\left[\frac{K_{b_1}(t-T_1)}{1-H(T_1)+\frac{1}{n}} K_{b_2}(t-T_1)(\delta_1 - p(t))(M_{011}(b_2) - T_1 M_{010}(b_2))\right] \\ =& \int_0^{\infty} \frac{1}{1-H(t_1)+\frac{1}{n}} K_{b_1}(t-t_1) K_{b_2}(t-t_1)(p(t_1) - p(t)) \\ & \times (M_{011}(b_2) - t_1 M_{010}(b_2)) h(t_1) dt_1 \\ =& \frac{1}{b_2}\int_{-L}^{L} \frac{1}{1-H(t-b_1 t_{11})+\frac{1}{n}} K(t_{11}) K\left(\frac{t_{11} b_1}{b_2}\right)(p(t-b_1 t_{11}) - p(t)) \\ & \times (M_{011}(b_2) - (t - b_1 t_{11}) M_{010}(b_2)) h(t - b_1 t_{11}) dt_{11} \\ =& \frac{1}{b_2}\int_{-L}^{L} K(t_{11}) K\left(\frac{t_{11} b_1}{b_2}\right)(p(t-b_1 t_{11}) - p(t)) \\ & \times (M_{011}(b_2) - (t - b_1 t_{11}) M_{010}(b_2)) \varphi_1(t - b_1 t_{11}) dt_{11} + O(n^{-1} b_2) \\ =& b_1 b_2 \mu_K \int_{-L}^{L} K(t_{11}) K(c t_{11}) dt_{11} p'(t) \varphi_1(t) h(t) + o(b_1 b_2) + O(n^{-1} b_2) \\ =& O\left(b_2^2\right), \end{aligned}$$

which is a consequence of **(V.2)**, **(V.1)**, (15), (16), (122) and (123).

Now, using (10),

$$E\left[\frac{K_{b_1}(t-T_1)}{1-H(T_1)+\frac{1}{n}}\right] = E\left[A_n + \lambda_H(t)\right] = \lambda_H(t) + o(1)$$

and, according to (122)-(125),

$$\begin{aligned}
& E\left[K_{b_2}(t-T_1)\left(\delta_1-p(t)\right)\left(M_{011}(b_2)-T_1M_{010}(b_2)\right)\right] \\
= \; & M_{100}(b_2)M_{011}(b_2)-M_{101}(b_2)M_{010}(b_2) \\
= \; & O\left(b_2^4\right)=O\left(b_1^4\right),
\end{aligned}$$

which leads to

$$\begin{aligned}
Cov\left[A_n,\sum_{i=1}^{n}D_{2,i}^*\right] \;=\; & \frac{n-1}{n^2b_2^2\mu_K h(t)^2}E\left[\frac{K_{b_1}(t-T_1)}{1-H(T_1)+\frac{1}{n}}\right. \\
& \left.\times K_{b_2}(t-T_1)\left(\delta_1-p(t)\right)\left(M_{011}(b_2)-T_1M_{010}(b_2)\right)\right] \\
& -\frac{n-1}{n^2b_2^2\mu_K h(t)^2}E\left[\frac{K_{b_1}(t-T_1)}{1-H(T_1)+\frac{1}{n}}\right] \\
& \times E\left[K_{b_2}(t-T_1)\left(\delta_1-p(t)\right)\left(M_{011}(b_2)-T_1M_{010}(b_2)\right)\right] \\
= \; & O(n^{-1})+O\left(n^{-1}b_1^2\right)=O(n^{-1}). \qquad (131)
\end{aligned}$$

From (130) and (131) we conclude

$$Cov\left[A_n,D_n^*\right]=O(n^{-1}). \qquad (132)$$

Collecting (11), (129) and (132) we end up with the following asymptotic expression for the variance of $\overline{\lambda}_{WLLL}^*(t)$

$$Var\left[\overline{\lambda}_{WLLL}^*(t)\right]=\frac{1}{nb_1}\frac{c_Kp(t)^2h(t)}{\left(1-H(t)\right)^2}+\frac{1}{nb_2}\frac{c_K\lambda_H(t)^2p(t)\left(1-p(t)\right)}{h(t)}+O(n^{-1})$$

and, using **(V.2)**,

$$Var\left[\overline{\lambda}_{WLLL}^*(t)\right]=\frac{1}{nb_1}\frac{c_K\lambda_F(t)\left(p(t)+a(1-p(t))\right)}{1-H(t)}+O(n^{-1}). \qquad (133)$$

The asymptotic expressions for the variances of $\overline{\lambda}_{WLLL}(t)$ and its Hájek projection, $\overline{\lambda}_{WLLL}^*(t)$, given in Theorem 9 and equation (133), immediately lead to (117). ∎

Theorem 18 *Le us assume conditions **(K.1)**, **(P.1)**, **(H.1)**, **(V.1)** and **(V.2)** and the existence of the limit $\lim_{n\to\infty}nb_1^5$. Then*

$$\sqrt{nb_1}\left(\widehat{\lambda}_{WLLL}(t)-\lambda_F(t)\right)\xrightarrow{d}N(l_0\xi_4(t),\xi_2(t)) \qquad (134)$$

where $\xi_2(t)$ and l_0 have been defined in (110) and (111), and

$$\xi_4(t)=\frac{1}{2}\mu_K\left(\lambda_H''(t)p(t)+\frac{\lambda_H(t)p''(t)}{a^2}\right). \qquad (135)$$

Proof. Theorem 9 includes the asymptotic expression

$$\widehat{\lambda}_{WLLL}(t) - \lambda_F(t) = \overline{\lambda}_{WLLL}(t) - \lambda_F(t) + o_P\left(n^{-1/2} b_1^{-1/2}\right) \tag{136}$$

where $\overline{\lambda}_{WLLL}(t)$ has been defined in (71).

Using Lemma 17 the Hájek projection of $\overline{\lambda}_{WLLL}(t)$, $\overline{\lambda}^*_{WLLL}(t)$, satisfies

$$\frac{Var\left[\overline{\lambda}^*_{WLLL}(t)\right]}{Var\left[\overline{\lambda}_{WLLL}(t)\right]} \to 1.$$

Thus, Theorem 14 can be used to conclude

$$\frac{\overline{\lambda}_{WLLL}(t) - E\left[\overline{\lambda}_{WLLL}(t)\right]}{\sqrt{Var\left[\overline{\lambda}_{WLLL}(t)\right]}} - \frac{\overline{\lambda}^*_{WLLL}(t) - E\left[\overline{\lambda}^*_{WLLL}(t)\right]}{\sqrt{Var\left[\overline{\lambda}^*_{WLLL}(t)\right]}} = o_P(1).$$

Thus, the asymptotic normality of $\frac{\overline{\lambda}_{WLLL}(t) - E\left[\overline{\lambda}_{WLLL}(t)\right]}{\sqrt{Var\left[\overline{\lambda}_{WLLL}(t)\right]}}$ is a consequence of Slutsky's Theorem and the asymptotic normality of $\frac{\overline{\lambda}^*_{WLLL}(t) - E\left[\overline{\lambda}^*_{WLLL}(t)\right]}{\sqrt{Var\left[\overline{\lambda}^*_{WLLL}(t)\right]}}$, which will be proven next.

Recall (120) and define

$$\begin{aligned} V^*_{n,i} &= \frac{1}{n}\frac{p(t) K_{b_1}(t - T_i)}{1 - H(T_i) + \frac{1}{n}} \\ &+ \frac{(n-1)\lambda_H(t)}{n^2 b_2^2 \mu_K h(t)^2} K_{b_2}(t - T_i)\left((t - T_i)\left(M_{101}(b_2) - T_i M_{100}(b_2)\right)\right. \\ &\left. - (\delta_i - p(t))\left(M_{011}(b_2) - T_i M_{010}(b_2)\right)\right) - \frac{1}{n} E\left[D_n\right], \end{aligned}$$

to write

$$\overline{\lambda}^*_{WLLL}(t) = \sum_{i=1}^{n} V^*_{n,i}$$

where $V^*_{n,i}, i = 1, 2, ..., n; n = 1, 2, ...,$ is a triangular array of iid random variables. In order to prove the asymptotic normality of $\frac{\overline{\lambda}^*_{WLLL}(t) - E\left[\overline{\lambda}^*_{WLLL}(t)\right]}{\sqrt{Var\left[\overline{\lambda}^*_{WLLL}(t)\right]}}$,we will use Liapunov Theorem and check

$$\lim_{n \to \infty} \frac{nE\left[\left|V^*_{n,1} - E(V^*_{n,1})\right|^3\right]}{\left(Var\left[\overline{\lambda}^*_{WLLL}(t)\right]\right)^{3/2}} = 0. \tag{137}$$

The arguments used to obtain the inequality (113) can also be used to derive

$$\frac{nE\left[\left|V_{n,1}^{*}-E(V_{n,1}^{*})\right|^{3}\right]}{\left(Var\left[\overline{\lambda}_{WLLL}^{*}(t)\right]\right)^{3/2}}\leq\frac{8nE\left[\left|V_{n,1}^{*}\right|^{3}\right]}{\left(Var\left[\overline{\lambda}_{WLLL}^{*}(t)\right]\right)^{3/2}}. \tag{138}$$

Since the support of the kernel is $[-L,L]$, we have

$$\begin{aligned}
V_{n,1}^{*} &= V_{n,1}^{*}\mathbf{1}\left\{T_{1}\leq t+b_{1}L\right\}+V_{n,1}^{*}\mathbf{1}\left\{T_{1}>t+b_{1}L\right\} \\
&= \frac{1}{n}\frac{p(t)K_{b_{1}}\left(t-T_{1}\right)}{1-H(T_{1})+\frac{1}{n}}\mathbf{1}\left\{T_{1}\leq t+b_{1}L\right\} \\
&\quad+\frac{(n-1)\,\lambda_{H}(t)}{n^{2}b_{2}^{2}\mu_{K}h(t)^{2}}K_{b_{2}}\left(t-T_{1}\right)\Big(\left(t-T_{1}\right)\left(M_{101}(b_{2})-T_{1}M_{100}(b_{2})\right) \\
&\quad-\left(\delta_{1}-p(t)\right)\left(M_{011}(b_{2})-T_{1}M_{010}(b_{2})\right)\Big)-\frac{1}{n}E(D_{n})
\end{aligned}$$

which, provided that $b_1<\frac{t_0-t}{L}$, implies

$$\begin{aligned}
\left|V_{n,1}^{*}\right| &\leq \left\|K\right\|_{\infty}\Big(\frac{1}{nb_{1}}\frac{1}{1-H(t_{0})}+\frac{(n-1)\,\lambda_{H}(t)}{n^{2}b_{2}^{3}\mu_{K}h(t)^{2}} \\
&\quad\times\left(2t_{0}\left(\left|M_{101}(b_{2})\right|+t_{0}\left|M_{100}(b_{2})\right|\right)+\left|M_{011}(b_{2})\right|+t_{0}\left|M_{010}(b_{2})\right|\right)\Big) \\
&\quad+\frac{1}{n}\left|E(D_{n})\right| \qquad a.s.
\end{aligned}$$

Thus, using **(V.2)**, (34) and (122)-(125),

$$E\left[\left|V_{n,1}^{*}\right|^{3}\right]=O\left(n^{-3}b_{1}^{-3}\right)+O\left(n^{-3}b_{1}^{6}\right)=O\left(n^{-3}b_{1}^{-3}\right)$$

and, equations (138) and (133) lead to

$$\frac{nE\left[\left|V_{n,1}^{*}-E(V_{n,1}^{*})\right|^{3}\right]}{\left(Var\left[\overline{\lambda}_{WLLL}^{*}(t)\right]\right)^{3/2}}=O(n)O\left(n^{3/2}b_{1}^{3/2}\right)O\left(n^{-3}b_{1}^{-3}\right)=O\left(n^{-1/2}b_{1}^{-3/2}\right).$$

Now condition **(V.1)** implies Liapunov condition in (137) and, therefore,

$$\frac{\overline{\lambda}_{WLLL}^{*}(t)-E\left[\overline{\lambda}_{WLLL}^{*}(t)\right]}{\sqrt{Var\left[\overline{\lambda}_{WLLL}^{*}(t)\right]}}\xrightarrow{d}N(0,1).$$

Similarly,as argued above,

$$\begin{aligned}
& \frac{\overline{\lambda}_{WLLL}(t) - \lambda_F(t) - E\left[\overline{\lambda}_{WLLL}(t) - \lambda_F(t)\right]}{\sqrt{Var\left[\overline{\lambda}_{WLLL}(t)\right]}} \\
= & \frac{\overline{\lambda}_{WLLL}(t) - E\left[\overline{\lambda}_{WLLL}(t)\right]}{\sqrt{Var\left[\overline{\lambda}_{WLLL}(t)\right]}} \xrightarrow{d} N(0,1).
\end{aligned}$$

Now, definitions (110) and (135) of ξ_2 and ξ_4, give

$$\begin{aligned}
& \frac{\overline{\lambda}_{WLLL}(t) - \lambda_F(t) - E\left[\overline{\lambda}_{WLLL}(t) - \lambda_F(t)\right]}{\sqrt{Var\left[\overline{\lambda}_{WLLL}(t)\right]}} \\
= & \sqrt{nb_1}\frac{\overline{\lambda}_{WLLL}(t) - \lambda_F(t) - b_1^2\xi_4(t) + o\left(b_1^2\right)}{\xi_2(t) + o(1)} \\
= & \sqrt{nb_1}\frac{\overline{\lambda}_{WLLL}(t) - \lambda_F(t) - b_1^2\xi_4(t)}{\xi_2(t) + o(1)} + o\left(n^{1/2}b_1^{5/2}\right) \\
= & \sqrt{nb_1}\frac{\overline{\lambda}_{WLLL}(t) - \lambda_F(t) - b_1^2\xi_4(t)}{\xi_2(t)}\frac{\xi_2(t)}{\xi_2(t) + o(1)} \\
& + o\left(n^{1/2}b_1^{5/2}\right),
\end{aligned}$$

i.e.,

$$\begin{aligned}
& \sqrt{nb_1}\frac{\overline{\lambda}_{WLLL}(t) - \lambda_F(t) - b_1^2\xi_4(t)}{\xi_2(t)} \\
= & \frac{\overline{\lambda}_{WLLL}(t) - \lambda_F(t) - E\left[\overline{\lambda}_{WLLL}(t) - \lambda_F(t)\right]}{\sqrt{Var\left[\overline{\lambda}_{WLLL}(t)\right]}}\frac{\xi_2(t) + o(1)}{\xi_2(t)} \\
& + o\left(n^{1/2}b_1^{5/2}\right)O(1).
\end{aligned}$$

Condition **(V.1),** $\frac{\xi_2(t)+o(1)}{\xi_2(t)} \rightarrow 1$ and Slutsky's Theorem imply

$$\sqrt{nb_1}\frac{\overline{\lambda}_{WLLL}(t) - \lambda_F(t) - b_1^2\xi_4(t)}{\xi_2(t)} \xrightarrow{d} N(0,1).$$

A further application of Slutsky's Theorem and representation (136) lead to

$$\sqrt{nb_1}\frac{\widehat{\lambda}_{WLLL}(t) - \lambda_F(t) - b_1^2\xi_4(t)}{\xi_2(t)} \xrightarrow{d} N(0,1),$$

or, equivalently, (134), by just using the definition of l_0. ■

Theorem 19 *Conditions* ***(K.1)****,* ***(P.1)****,* ***(H.1)****,* ***(V.1)*** *and* ***(V.2)*** *and the existence of* $\lim_{n\to\infty} nb_1^5$*, imply*

$$\sqrt{nb_1}\left(\widehat{\lambda}_{RRLL}(t)-\lambda_F(t)\right) \xrightarrow{d} N(l_0\xi_5(t),\xi_2(t)) \tag{139}$$

where $\xi_2(t)$ *and* l_0 *have been defined in (110) and (111), respectively, and*

$$\xi_5(t)=\frac{1}{2}\mu_K\left(\lambda_H''(t)p(t)+\left(\lambda_H(t)^2-3\lambda_H'(t)\right)\lambda_F(t)+\frac{\lambda_H(t)p''(t)}{a^2}\right). \tag{140}$$

Proof. Multiplying by $\sqrt{nb_1}$ (74) in Theorem 10,

$$\begin{aligned}\sqrt{nb_1}\left(\widehat{\lambda}_{RRLL}(t)-\lambda_F(t)\right) &= \sqrt{nb_1}\left(\widehat{\lambda}_{WLLL}(t)-\lambda_F(t)\right)\\ &+\frac{1}{2}\sqrt{nb_1^5}\mu_K\left(\lambda_H(t)^2-3\lambda_H'(t)\right)\lambda_F(t)+o_P(1)\end{aligned} \tag{141}$$

It has been proven in Theorem 18 the following result

$$\sqrt{nb_1}\left(\widehat{\lambda}_{WLLL}(t)-\lambda_F(t)\right) \xrightarrow{d} N(l_0\xi_4(t),\xi_2(t))$$

where $\xi_4(t)$ and $\xi_2(t)$ have been defined in (135) and (110).

Slutsky's Theorem and (141) give

$$\sqrt{nb_1}\left(\widehat{\lambda}_{RRLL}(t)-\lambda_F(t)\right) \xrightarrow{d} N\left(l_0\left(\xi_4(t)+\frac{1}{2}\mu_K\left(\lambda_H(t)^2-3\lambda_H'(t)\right)\lambda_F(t)\right),\xi_2(t)\right)$$

and, using definition (140), it is obtained (139). ■

In the proof of Theorem 21 the following lemma will be needed.

Lemma 20 *Under conditions* ***(K.1)****,* ***(P.1)****,* ***(H.1)****,* ***(V.1)*** *and* ***(V.2)****, and denoting by* $\overline{\lambda}_{TWP}^*(t)$ *Hájek projection of* $\overline{\lambda}_{TWP}(t)$*, defined in (83), we have*

$$\frac{Var\left[\overline{\lambda}_{TWP}^*(t)\right]}{Var\left[\overline{\lambda}_{TWP}(t)\right]}\to 1. \tag{142}$$

Proof. In the proof of Theorem 11 the term $\overline{\lambda}_{TWP}(t)$ was split as the sum of two terms

$$J_1=\frac{1}{n}\sum_{i=1}^{n}\frac{K_{b_1}(t-T_i)\,p(T_i)\mathbf{1}(T_i\in I)}{1-H(T_i)+\frac{1}{n}}$$

and

$$J_2=\frac{1}{n}\sum_{i=1}^{n}\frac{K_{b_1}(t-T_i)\left(\widehat{\psi}(T_i)-\widehat{h}(T_i)p(T_i)\right)\mathbf{1}(T_i\in I)}{\left(1-H(T_i)+\frac{1}{n}\right)h(T_i)}.$$

Since J_1 is a sum of iid random variables, the Hájek projection of $\overline{\lambda}_{TWP}(t)$ is

$$\overline{\lambda}^*_{TWP}(t) = J_1 + J_2^*$$

where J_2^* denotes the Hájek projection of J_2, given by

$$J_2^* = \sum_{i=1}^{n} E\left[J_2 \left| Z_i\right.\right] - (n-1)E\left[J_2\right]$$

where Z_i denotes the random vector (T_i, δ_i).

The definition

$$J_{2,k} = \frac{1}{n^2}\sum_{j=1}^{n}\frac{K_{b_1}(t-T_k)\,K_{b_2}(T_k-T_j)\,(\delta_j - p(T_k))\,\mathbf{1}(T_k\in I)}{\left(1-H(T_k)+\frac{1}{n}\right)h(T_k)},$$

can be used to write

$$J_2 = \sum_{k=1}^{n} J_{2,k}.$$

Now, for $k \neq i$,

$$\begin{aligned} E\left[J_{2,k}\left|Z_i\right.\right] &= \frac{n-2}{n^2}E\left[\frac{K_{b_1}(t-T_k)\,K_{b_2}(T_k-T_j)\,(\delta_j - p(T_k))\,\mathbf{1}(T_k\in I)}{\left(1-H(T_k)+\frac{1}{n}\right)h(T_k)}\right] \\ &+\frac{1}{n^2}E\left[\frac{K_{b_1}(t-T_k)\,K_{b_2}(T_k-T_i)\,(\delta_i - p(T_k))\,\mathbf{1}(T_k\in I)}{\left(1-H(T_k)+\frac{1}{n}\right)h(T_k)}\left|Z_i\right.\right] \end{aligned} \tag{143}$$

and for $k = i$,

$$\begin{aligned} E\left[J_{2,i}\left|Z_i\right.\right] &= \frac{n-1}{n^2}E\left[K_{b_1}(t-T_i)\,K_{b_2}(T_i-T_j)\,(\delta_j - p(T_i))\,\mathbf{1}(T_i\in I)\right. \\ &\times\left.\left(1-H(T_i)+\frac{1}{n}\right)^{-1} h(T_i)^{-1}\left|Z_i\right.\right] \\ &+\frac{1}{n^2}K_{b_1}(t-T_i)\,K_{b_2}(0)\,(\delta_i - p(T_i))\,\mathbf{1}(T_i\in I) \\ &\times\left(1-H(T_i)+\frac{1}{n}\right)^{-1} h(T_i)^{-1} \end{aligned} \tag{144}$$

where, in (143) and (144), we have assumed $i \neq j$.

Thus,

$$
\begin{aligned}
E\left[J_2 | Z_i\right] &= \frac{(n-1)(n-2)}{n^2} E\left[\frac{K_{b_1}(t-T_1) K_{b_2}(T_1-T_2)(\delta_2 - p(T_1)) \mathbf{1}(T_1 \in I)}{\left(1-H(T_1)+\frac{1}{n}\right) h(T_1)}\right] \\
&+ \frac{n-1}{n^2} E\left[\frac{K_{b_1}(t-T_j) K_{b_2}(T_j-T_i)(\delta_i - p(T_j)) \mathbf{1}(T_j \in I)}{\left(1-H(T_j)+\frac{1}{n}\right) h(T_j)} | Z_i\right] \\
&+ \frac{n-1}{n^2} E\left[\frac{K_{b_1}(t-T_i) K_{b_2}(T_i-T_j)(\delta_j - p(T_i)) \mathbf{1}(T_i \in I)}{\left(1-H(T_i)+\frac{1}{n}\right) h(T_i)} | Z_i\right] \\
&+ \frac{1}{n^2} \frac{K_{b_1}(t-T_i) K_{b_2}(0)(\delta_i - p(T_i)) \mathbf{1}(T_i \in I)}{\left(1-H(T_i)+\frac{1}{n}\right) h(T_i)}
\end{aligned}
$$

where $i \neq j$.

Since

$$
E\left[J_2\right] = \frac{n-1}{n} E\left[\frac{K_{b_1}(t-T_1) K_{b_2}(T_1-T_2)(p(T_2) - p(T_1)) \mathbf{1}(T_1 \in I)}{\left(1-H(T_1)+\frac{1}{n}\right) h(T_1)}\right],
$$

the Hájek projection of J_2 is

$$
J_2^* = J_{2,1}^* + J_{2,2}^* + J_{2,3}^* - E\left[J_2\right]
$$

after having defined

$$
\begin{aligned}
J_{2,1}^* &= \frac{n-1}{n^2} \sum_{i=1}^{n} E\left[\frac{K_{b_1}(t-T_j) K_{b_2}(T_j-T_i)(\delta_i - p(T_j)) \mathbf{1}(T_j \in I)}{\left(1-H(T_j)+\frac{1}{n}\right) h(T_j)} | Z_i\right], \\
J_{2,2}^* &= \frac{n-1}{n^2} \sum_{i=1}^{n} E\left[\frac{K_{b_1}(t-T_i) K_{b_2}(T_i-T_j)(\delta_j - p(T_i)) \mathbf{1}(T_i \in I)}{\left(1-H(T_i)+\frac{1}{n}\right) h(T_i)} | Z_i\right], \\
J_{2,3}^* &= \frac{1}{n^2} \sum_{i=1}^{n} \frac{K_{b_1}(t-T_i) K_{b_2}(0)(\delta_i - p(T_i)) \mathbf{1}(T_i \in I)}{\left(1-H(T_i)+\frac{1}{n}\right) h(T_i)}
\end{aligned}
$$

and assumed $i \neq j$.

The variance of $\overline{\lambda}_{TWP}^*(t)$ is

$$
Var\left[\overline{\lambda}_{TWP}^*(t)\right] = Var\left[J_1\right] + 2Cov\left[J_1, J_2^*\right] + Var\left[J_2^*\right].
$$

We now study the asymptotic behaviour of $Var\left[J_2^*\right]$ and $Cov\left[J_1, J_2^*\right]$ —some expression for $Var\left[J_1\right]$ was found in (89) —. Obviously,

$$
\begin{aligned}
Var\left[J_2^*\right] &= Var\left[J_{2,1}^*\right] + Var\left[J_{2,2}^*\right] + Var\left[J_{2,3}^*\right] \\
&+ 2Cov\left[J_{2,1}^*, J_{2,2}^*\right] + 2Cov\left[J_{2,1}^*, J_{2,3}^*\right] + 2Cov\left[J_{2,2}^*, J_{2,3}^*\right].
\end{aligned}
$$

Let us introduce the notation,

$$
\begin{aligned}
M_1 &= Var\left[E\left[\frac{K_{b_1}(t-T_1)\,K_{b_2}(T_1-T_2)\,(\delta_2-p(T_1))\,\mathbf{1}(T_1\in I)}{\left(1-H(T_1)+\frac{1}{n}\right)h(T_1)}\,|Z_2\right]\right],\\
M_2 &= Var\left[\frac{K_{b_1}(t-T_1)\,K_{b_2}(0)\,(\delta_1-p(T_1))\,\mathbf{1}(T_1\in I)}{\left(1-H(T_1)+\frac{1}{n}\right)h(T_1)}\right],\\
M_3 &= Var\left[E\left[\frac{K_{b_1}(t-T_2)\,K_{b_2}(T_2-T_1)\,(\delta_1-p(T_2))\,\mathbf{1}(T_2\in I)}{\left(1-H(T_2)+\frac{1}{n}\right)h(T_2)}\,|Z_2\right]\right],
\end{aligned}
$$

then

$$
Var\left[J^*_{2,1}\right]=\frac{(n-1)^2}{n^4}\left((n-1)M_1+M_2\right), \tag{145}
$$

$$
Var\left[J^*_{2,2}\right]=\frac{(n-1)^2}{n^4}\left((n-1)M_3+M_2\right) \tag{146}
$$

and

$$
Var\left[J^*_{2,3}\right]=\frac{1}{n^3}M_2. \tag{147}
$$

We now study the terms M_1, M_2 and M_3. For M_1,

$$
\begin{aligned}
M_1 &= E\left[\left(\int_\varepsilon^{t_0}\frac{K_{b_1}(t-t_1)\,K_{b_2}(t_1-T_2)\,(\delta_2-p(t_1))}{1-H(t_1)+\frac{1}{n}}dt_1\right)^2\right]\\
&\quad-\left(E\left[\int_\varepsilon^{t_0}\frac{K_{b_1}(t-t_1)\,K_{b_2}(t_1-T_2)\,(\delta_2-p(t_1))}{1-H(t_1)+\frac{1}{n}}dt_1\right]\right)^2
\end{aligned} \tag{148}
$$

and the first of these expectations is

$$
\begin{aligned}
&E\left[\left(\delta_2\int_\varepsilon^{t_0}\frac{K_{b_1}(t-t_1)\,K_{b_2}(t_1-T_2)}{1-H(t_1)+\frac{1}{n}}dt_1-\int_\varepsilon^{t_0}\frac{K_{b_1}(t-t_1)\,K_{b_2}(t_1-T_2)\,p(t_1)}{1-H(t_1)+\frac{1}{n}}dt_1\right)^2\right]\\
=\;&E\left[p(T_2)\int_\varepsilon^{t_0}\frac{K_{b_1}(t-t_1)\,K_{b_2}(t_1-T_2)}{1-H(t_1)+\frac{1}{n}}dt_1\right.\\
&\times\left(\int_\varepsilon^{t_0}\frac{K_{b_1}(t-t_1)\,K_{b_2}(t_1-T_2)}{1-H(t_1)+\frac{1}{n}}dt_1-2\int_\varepsilon^{t_0}\frac{K_{b_1}(t-t_1)\,K_{b_2}(t_1-T_2)\,p(t_1)}{1-H(t_1)+\frac{1}{n}}dt_1\right)\\
&\left.+\left(\int_\varepsilon^{t_0}\frac{K_{b_1}(t-t_1)\,K_{b_2}(t_1-T_2)\,p(t_1)}{1-H(t_1)+\frac{1}{n}}dt_1\right)^2\right]\\
=\;&\int_0^\infty\left(p(t_2)\int_\varepsilon^{t_0}\frac{K_{b_1}(t-t_1)\,K_{b_2}(t_1-t_2)}{1-H(t_1)+\frac{1}{n}}dt_1\int_\varepsilon^{t_0}\frac{K_{b_1}(t-t_1)\,K_{b_2}(t_1-t_2)\,(1-2p(t_1))}{1-H(t_1)+\frac{1}{n}}dt_1\right.\\
&\left.+\left(\int_\varepsilon^{t_0}\frac{K_{b_1}(t-t_1)\,K_{b_2}(t_1-t_2)\,p(t_1)}{1-H(t_1)+\frac{1}{n}}dt_1\right)^2\right)h(t_2)dt_2.
\end{aligned}
$$

The change of variable $\frac{t-t_1}{b_1} = t_{11}$, $\frac{t-t_2}{b_2} = t_{21}$ and conditions $b_1 < \frac{(t_0-t)\wedge(t-\varepsilon)\wedge t/2}{L}$, $b_2 < \frac{t}{2L}$, give

$$\begin{aligned}
&\frac{1}{b_2}\int_{-L\left(1+\frac{b_1}{b_2}\right)}^{L\left(1+\frac{b_1}{b_2}\right)}\left(p(t-b_2t_{21})\int_{-L}^{L}\frac{K(t_{11})K\left(t_{21}-\frac{b_1t_{11}}{b_2}\right)}{1-H(t-b_1t_{11})+\frac{1}{n}}dt_{11}\right.\\
&\times\int_{-L}^{L}\frac{K(t_{11})K\left(t_{21}-\frac{b_1t_{11}}{b_2}\right)(1-2p(t-b_1t_{11}))}{1-H(t-b_1t_{11})+\frac{1}{n}}dt_{11}\\
&\left.+\left(\int_{-L}^{L}\frac{K(t_{11})K\left(t_{21}-\frac{b_1t_{11}}{b_2}\right)p(t-b_1t_{11})}{1-H(t-b_1t_{11})+\frac{1}{n}}dt_{11}\right)^2\right)h(t-b_2t_{21})dt_{21}\\
=\ &\frac{1}{b_1}\frac{ap(t)(1-p(t))h(t)}{(1-H(t))^2}\int_{-L(1+a)}^{L(1+a)}\left(\int_{-L}^{L}K(t_{11})K(t_{21}-at_{11})dt_{11}\right)^2dt_{21}+o\left(b_1^{-1}\right),
\end{aligned}$$

after using Taylor expansions and condition **(V.2)**. On the other hand, for the second expectation in (148), the following expression was obtained when computing the expectation of J_2 in the proof of Theorem 11

$$\int_0^{\infty}\int_{\varepsilon}^{t_0}\frac{K_{b_1}(t-t_1)K_{b_2}(t_1-t_2)(p(t_2)-p(t_1))h(t_2)}{1-H(t_1)+\frac{1}{n}}dt_1dt_2 = O\left(b_1^2\right),$$

which implies

$$M_1 = \frac{1}{b_1}\frac{ap(t)(1-p(t))h(t)}{(1-H(t))^2}\int_{-L(1+a)}^{L(1+a)}\left(\int_{-L}^{L}K(t_{11})K(t_{21}-at_{11})dt_{11}\right)^2dt_{21}+o\left(b_1^{-1}\right). \tag{149}$$

The term M_2 is

$$\begin{aligned}
M_2 &= E\left[\frac{K_{b_1}(t-T_1)^2K_{b_2}(0)^2(\delta_1-p(T_1))^2\mathbf{1}(T_1\in I)}{\left(1-H(T_1)+\frac{1}{n}\right)^2h(T_1)^2}\right]\\
&\quad-\left(E\left[\frac{K_{b_1}(t-T_1)K_{b_2}(0)(\delta_1-p(T_1))\mathbf{1}(T_1\in I)}{\left(1-H(T_1)+\frac{1}{n}\right)h(T_1)}\right]\right)^2
\end{aligned}$$

and standard arguments give

$$
\begin{aligned}
&E\left[\frac{K_{b_1}(t-T_1)^2 K_{b_2}(0)^2(\delta_1-p(T_1))^2\mathbf{1}(T_1\in I)}{\left(1-H(T_1)+\frac{1}{n}\right)^2 h(T_1)^2}\right] \\
&= \frac{K(0)^2}{b_2^2}\int_{\varepsilon}^{t_0}\frac{K_{b_1}(t-t_1)^2 p(t_1)(1-p(t_1))}{\left(1-H(t_1)+\frac{1}{n}\right)^2 h(t_1)}dt_1 \\
&= \frac{K(0)^2}{b_1 b_2^2}\int_{\frac{t-t_0}{b_1}}^{\frac{t-\varepsilon}{b_1}}\frac{K(t_{11})^2 p(t-b_1t_{11})(1-p(t-b_1t_{11}))}{\left(1-H(t-b_1t_{11})+\frac{1}{n}\right)^2 h(t-b_1t_{11})}dt_{11} \\
&= \frac{1}{b_1^3}\frac{a^2K(0)^2 p(t)(1-p(t))}{(1-H(t))^2 h(t)}c_K+o\left(b_1^{-3}\right)
\end{aligned}
$$

and

$$
E\left[\frac{K_{b_1}(t-T_1)K_{b_2}(0)(\delta_1-p(T_1))\mathbf{1}(T_1\in I)}{\left(1-H(T_1)+\frac{1}{n}\right)h(T_1)}\right]=0,
$$

which give

$$
M_2=O\left(b_1^{-3}\right). \tag{150}
$$

The term M_3, may be dealt with in the same manner than M_1,

$$
\begin{aligned}
M_3 &= E\left[\frac{K_{b_1}(t-T_2)^2\mathbf{1}(T_2\in I)}{\left(1-H(T_2)+\frac{1}{n}\right)^2 h(T_2)^2}\left(\int_0^\infty K_{b_2}(T_2-t_1)(p(t_1)-p(T_2))h(t_1)dt_1\right)^2\right] \\
&\quad -\left(E\left[\frac{K_{b_1}(t-T_2)\mathbf{1}(T_2\in I)}{\left(1-H(T_2)+\frac{1}{n}\right)h(T_2)}\int_0^\infty K_{b_2}(T_2-t_1)(p(t_1)-p(T_2))h(t_1)dt_1\right]\right)^2.
\end{aligned}
$$

Now, the first expectation is

$$
\begin{aligned}
&\int_{\varepsilon}^{t_0}\frac{K_{b_1}(t-t_2)^2}{\left(1-H(t_2)+\frac{1}{n}\right)^2 h(t_2)}\left(\int_0^\infty K_{b_2}(t_2-t_1)(p(t_1)-p(t_2))h(t_1)dt_1\right)^2 dt_2 \\
&= \frac{1}{b_2}\int_{\frac{t-t_0}{b_1}}^{\frac{t-\varepsilon}{b_1}}\frac{K(t_{21})^2}{\left(1-H(t-b_1t_{21})+\frac{1}{n}\right)^2 h(t-b_1t_{21})} \\
&\quad\times\left(\int_{-\infty}^{\frac{t}{b_2}}K\left(t_{11}-\frac{b_1t_{21}}{b_2}\right)(p(t-b_2t_{11})-p(t-b_1t_{21}))h(t-b_2t_{11})dt_{11}\right)^2 dt_{21}.
\end{aligned}
$$

Provided that $b_1<\frac{(t_0-t)\wedge(t-\varepsilon)\wedge t/2}{L}$ and $b_2<\frac{t}{2L}$, this expression is

$$
\begin{aligned}
&\frac{1}{b_2}\int_{-L}^{L}\frac{K(t_{21})^2}{\left(1-H(t-b_1t_{21})+\frac{1}{n}\right)^2 h(t-b_1t_{21})} \\
&\quad\times\left(\int_{-L\left(1+\frac{b_1}{b_2}\right)}^{L\left(1+\frac{b_1}{b_2}\right)}K\left(t_{11}-\frac{b_1t_{21}}{b_2}\right)(p(t-b_2t_{11})-p(t-b_1t_{21}))h(t-b_2t_{11})dt_{11}\right)^2 dt_{21} \\
&= o\left(b_1^{-1}\right)
\end{aligned}
$$

where we have used the dominated convergence theorem and **(V.2)**. On the other hand, as proved when dealing with the expectation of J_2,

$$E\left[\frac{K_{b_1}(t-T_2)\,\mathbf{1}(T_2\in I)}{\left(1-H(T_2)+\frac{1}{n}\right)h(T_2)}\int_0^\infty K_{b_2}(T_2-t_1)\left(p(t_1)-p(T_2)\right)dt_1\right]$$
$$= \int_\varepsilon^{t_0}\frac{K_{b_1}(t-t_2)}{1-H(t_2)+\frac{1}{n}}\int_0^\infty K_{b_2}(t_2-t_1)\left(p(t_1)-p(t_2)\right)h(t_1)dt_1dt_2 = O\left(b_1^2\right),$$

which implies

$$M_3 = o\left(b_1^{-1}\right). \tag{151}$$

Plugging (149), (150) and (151) into (145), (146) and (147) an using **(V.1)**,

$$\begin{aligned} Var\left[J_{2,1}^*\right] &= \frac{1}{nb_1}\frac{ap(t)\left(1-p(t)\right)h(t)}{\left(1-H(t)\right)^2}\int_{-L(1+a)}^{L(1+a)}\left(\int_{-L}^{L}K\left(t_{11}\right)K\left(t_{21}-at_{11}\right)dt_{11}\right)^2dt_{21} \\ &\quad +o\left(n^{-1}b_1^{-1}\right), \end{aligned}$$

$$Var\left[J_{2,2}^*\right] = o\left(n^{-1}b_1^{-1}\right)$$

and

$$Var\left[J_{2,3}^*\right] = o\left(n^{-1}b_1^{-1}\right).$$

More over, Cauchy-Schwarz inequality gives

$$Cov\left[J_{2,1}^*, J_{2,2}^*\right] = o\left(n^{-1}b_1^{-1}\right),$$

$$Cov\left[J_{2,1}^*, J_{2,3}^*\right] = o\left(n^{-1}b_1^{-1}\right)$$

and

$$Cov\left[J_{2,2}^*, J_{2,3}^*\right] = o\left(n^{-1}b_1^{-1}\right).$$

Therefore,

$$\begin{aligned} Var\left[J_2^*\right] &= \frac{1}{nb_1}\frac{ap(t)\left(1-p(t)\right)h(t)}{\left(1-H(t)\right)^2}\int_{-L(1+a)}^{L(1+a)}\left(\int_{-L}^{L}K\left(t_{11}\right)K\left(t_{21}-at_{11}\right)dt_{11}\right)^2dt_{21} \\ &\quad +o\left(n^{-1}b_1^{-1}\right). \end{aligned} \tag{152}$$

The covariance of J_1 and J_2^* will be considered now. Obviously,

$$Cov\left[J_1, J_2^*\right] = Cov\left[J_1, J_{2,1}^*\right] + Cov\left[J_1, J_{2,2}^*\right] + Cov\left[J_1, J_{2,3}^*\right] \tag{153}$$

which, using the notation

$$\begin{aligned}
M_4 &= Cov\left[\frac{K_{b_1}(t-T_2)\,p(T_2)\mathbf{1}(T_2\in I)}{1-H(T_2)+\frac{1}{n}},\right.\\
&\quad \left. E\left[\frac{K_{b_1}(t-T_1)\,K_{b_2}(T_1-T_2)\,(\delta_2-p(T_1))\,\mathbf{1}(T_1\in I)}{\left(1-H(T_1)+\frac{1}{n}\right)h(T_1)}\,|Z_2\right]\right],\\
M_5 &= Cov\left[\frac{K_{b_1}(t-T_1)\,p(T_1)\mathbf{1}(T_1\in I)}{1-H(T_1)+\frac{1}{n}},\frac{K_{b_1}(t-T_1)\,K_{b_2}(0)\,(\delta_1-p(T_1))\,\mathbf{1}(T_1\in I)}{\left(1-H(T_1)+\frac{1}{n}\right)h(T_1)}\right],\\
M_6 &= Cov\left[\frac{K_{b_1}(t-T_2)\,p(T_2)\mathbf{1}(T_2\in I)}{1-H(T_2)+\frac{1}{n}},\right.\\
&\quad \left. E\left[\frac{K_{b_1}(t-T_2)\,K_{b_2}(T_2-T_1)\,(\delta_1-p(T_2))\,\mathbf{1}(T_2\in I)}{\left(1-H(T_2)+\frac{1}{n}\right)h(T_2)}\,|Z_2\right]\right],
\end{aligned}$$

gives

$$Cov\left[J_1,J_{2,1}^*\right]=\frac{n-1}{n^3}\left((n-1)\,M_4+M_5\right), \tag{154}$$

$$Cov\left[J_1,J_{2,2}^*\right]=\frac{n-1}{n^3}\left((n-1)\,M_6+M_5\right) \tag{155}$$

and

$$Cov\left[J_1,J_{2,3}^*\right]=\frac{1}{n^2}M_5. \tag{156}$$

The asymptotic behaviour of the terms M_4, M_5 and M_6 will be studied next. The first one is

$$\begin{aligned}
M_4 &= E\left[\frac{K_{b_1}(t-T_2)\,p(T_2)\mathbf{1}(T_2\in I)}{1-H(T_2)+\frac{1}{n}}\right.\\
&\quad \left.\times E\left[\frac{K_{b_1}(t-T_1)\,K_{b_2}(T_1-T_2)\,(\delta_2-p(T_1))\,\mathbf{1}(T_1\in I)}{\left(1-H(T_1)+\frac{1}{n}\right)h(T_1)}\,|Z_2\right]\right]\\
&\quad -E\left[\frac{K_{b_1}(t-T_1)\,p(T_1)\mathbf{1}(T_1\in I)}{1-H(T_1)+\frac{1}{n}}\right]\\
&\quad \times E\left[\frac{K_{b_1}(t-T_1)\,K_{b_2}(T_1-T_2)\,(p(T_2)-p(T_1))\,\mathbf{1}(T_1\in I)}{\left(1-H(T_1)+\frac{1}{n}\right)h(T_1)}\right],
\end{aligned}$$

whose first summand

$$
\begin{aligned}
& \int_{\varepsilon}^{t_0} \frac{K_{b_1}(t-t_2)\, p(t_2) h(t_2)}{1-H(t_2)+\frac{1}{n}} \int_{\varepsilon}^{t_0} \frac{K_{b_1}(t-t_1)\, K_{b_2}(t_1-t_2)\,(p(t_2)-p(t_1))}{1-H(t_1)+\frac{1}{n}} dt_1 dt_2 \\
= & \frac{1}{b_2} \int_{\frac{t-t_0}{b_1}}^{\frac{t-\varepsilon}{b_1}} \frac{K(t_{21})\, p(t-b_1 t_{21}) h(t-b_1 t_{21})}{1-H(t-b_1 t_{21})+\frac{1}{n}} \\
& \times \int_{\frac{t-t_0}{b_1}}^{\frac{t-\varepsilon}{b_1}} \frac{K(t_{11})\, K\left(\frac{b_1(t_1-t_2)}{b_2}\right)(p(t-b_1 t_{21})-p(t-b_1 t_{11}))}{1-H(t-b_1 t_{11})+\frac{1}{n}} dt_{11} dt_{21}
\end{aligned}
$$

after using the change of variable $\frac{t-t_1}{b_1} = t_{11}$, $\frac{t-t_2}{b_1} = t_{21}$. Condition $b_1 < \frac{(t_0-t)\wedge(t-\varepsilon)}{L}$ gives

$$
\begin{aligned}
& \frac{1}{b_2} \int_{-L}^{L} \frac{K(t_{21})\, p(t-b_1 t_{21}) h(t-b_1 t_{21})}{1-H(t-b_1 t_{21})+\frac{1}{n}} \\
& \times \int_{-L}^{L} \frac{K(t_{11})\, K\left(\frac{b_1(t_1-t_2)}{b_2}\right)(p(t-b_1 t_{21})-p(t-b_1 t_{11}))}{1-H(t-b_1 t_{11})+\frac{1}{n}} dt_{11} dt_{21} \\
= & \; o\left(b_1^{-1}\right),
\end{aligned}
$$

after using the dominated convergence theorem and condition **(V.2)**. When studying the expectations of J_1 and J_2, in the proof of Theorem 11, is has been shown that

$$
E\left[\frac{K_{b_1}(t-T_1)\, p(T_1) \mathbf{1}(T_1 \in I)}{1-H(T_1)+\frac{1}{n}}\right] = O(1)
$$

and

$$
E\left[\frac{K_{b_1}(t-T_1)\, K_{b_2}(T_1-T_2)\,(p(T_2)-p(T_1))\, \mathbf{1}(T_1 \in I)}{\left(1-H(T_1)+\frac{1}{n}\right) h(T_1)}\right] = O\left(b_1^2\right),
$$

which give

$$
M_4 = o\left(b_1^{-1}\right). \tag{157}
$$

For M_5,

$$
\begin{aligned}
M_5 = & \; E\left[\frac{K_{b_1}(t-T_1)^2 K_{b_2}(0)\, p(T_1)\,(\delta_1-p(T_1))\, \mathbf{1}(T_1 \in I)}{\left(1-H(T_1)+\frac{1}{n}\right)^2 h(T_1)}\right] \\
& -E\left[\frac{K_{b_1}(t-T_1)\, p(T_1) \mathbf{1}(T_1 \in I)}{1-H(T_1)+\frac{1}{n}}\right] E\left[\frac{K_{b_1}(t-T_1)\, K_{b_2}(0)\,(\delta_1-p(T_1))\, \mathbf{1}(T_1 \in I)}{\left(1-H(T_1)+\frac{1}{n}\right) h(T_1)}\right]
\end{aligned}
$$

and it is immediate to prove that

$$
M_5 = O\left(b_1^2\right). \tag{158}
$$

Finally, M_6 is

$$\begin{aligned} M_6 &= E\left[\frac{K_{b_1}(t-T_2)\,p(T_2)\mathbf{1}(T_2\in I)}{1-H(T_2)+\frac{1}{n}}\right. \\ &\quad \times E\left[\left.\frac{K_{b_1}(t-T_2)\,K_{b_2}(T_2-T_1)\,(\delta_1-p(T_2))\,\mathbf{1}(T_2\in I)}{\left(1-H(T_2)+\frac{1}{n}\right)h(T_2)}|Z_2\right]\right] \\ &\quad -E\left[\frac{K_{b_1}(t-T_1)\,p(T_1)\mathbf{1}(T_1\in I)}{1-H(T_1)+\frac{1}{n}}\right] \\ &\quad \times E\left[\frac{K_{b_1}(t-T_2)\,K_{b_2}(T_2-T_1)\,(\delta_1-p(T_2))\,\mathbf{1}(T_2\in I)}{\left(1-H(T_2)+\frac{1}{n}\right)h(T_2)}\right] \end{aligned}$$

whose first summand is

$$\begin{aligned} &\int_{\varepsilon}^{t_0}\frac{K_{b_1}(t-t_2)^2\,p(t_2)\int_0^{\infty}K_{b_2}(t_2-t_1)\,(p(t_1)-p(t_2))\,h(t_1)dt_1}{\left(1-H(t_2)+\frac{1}{n}\right)^2}dt_2 \\ =\;& \frac{1}{b_1}\int_{\frac{t-t_0}{b_1}}^{\frac{t-\varepsilon}{b_1}}K(t_{21})^2\,p(t-b_1t_{21})\int_{-\infty}^{\frac{t}{b_2}}K\left(t_{11}-\frac{b_1t_{21}}{b_2}\right)(p(t-b_2t_{11})-p(t-b_1t_{21})) \\ &\times\frac{h(t-b_2t_{11})}{\left(1-H(t-b_1t_{21})+\frac{1}{n}\right)^2}dt_{11}dt_{21} \\ =\;& \frac{1}{b_1}\int_{-L}^{L}K(t_{21})^2\,p(t-b_1t_{21})\int_{-L\left(1+\frac{b_1}{b_2}\right)}^{L\left(1+\frac{b_1}{b_2}\right)}K\left(t_{11}-\frac{b_1t_{21}}{b_2}\right)(p(t-b_2t_{11})-p(t-b_1t_{21})) \\ &\times\frac{h(t-b_2t_{11})}{\left(1-H(t-b_1t_{21})+\frac{1}{n}\right)^2}dt_{11}dt_{21} \\ =\;& o\left(b_1^{-1}\right), \end{aligned}$$

using standard arguments. Using, once more, the results obtained for the expectations of J_1 and J_2, we have

$$M_6=o\left(b_1^{-1}\right). \tag{159}$$

We now plug (157), (158) and (159) into (154), (155), (156) and (153), to obtain

$$Cov\left[J_1,J_2^*\right]=o\left(n^{-1}b_1^{-1}\right). \tag{160}$$

Finally, collecting (89), (152) and (160), we conclude

$$\begin{aligned} Var\left[\overline{\lambda}^*_{TWP}(t)\right] &= \frac{1}{nb_1}\frac{\lambda_F(t)}{1-H(t)}\left(p(t)c_K+(1-p(t))\,a\int_{-L(1+a)}^{L(1+a)}\left(\int_{-L}^{L}K(u)\,K(v-au)\,du\right)^2dv\right) \\ &\quad +o\left(n^{-1}b_1^{-1}\right). \end{aligned} \tag{161}$$

The asymptotic expressions of the variances of $\overline{\lambda}_{TWP}(t)$ and its Hájek projection, $\overline{\lambda}^*_{TWP}(t)$, given in Theorem 11 and (161), lead to (142). ■

This section concludes with a similar result for the estimator $\widetilde{\lambda}_{TWP}(t)$.

Theorem 21 *Let us assume conditions* ***(K.1)****,* ***(P.1)****,* ***(H.1)****,* ***(V.1)*** *and* ***(V.2)*** *and the existence of the limit* $\lim_{n\to\infty} nb_1^5$. *Then*

$$\sqrt{nb_1}\left(\widetilde{\lambda}_{TWP}(t)-\lambda_F(t)\right)\xrightarrow{d} N(l_0\xi_6(t),\xi_7(t)) \tag{162}$$

where l_0 *has been defined in (111),*

$$\xi_6(t)=\frac{1}{2}\mu_K\left(\lambda_F''(t)+\frac{1}{a^2}\left(\lambda_H(t)p''(t)+2\left(\lambda_H'(t)-\lambda_H(t)^2\right)p'(t)\right)\right) \tag{163}$$

and

$$\xi_7(t)=\sqrt{\frac{\lambda_F(t)}{1-H(t)}\left(p(t)c_K+(1-p(t))a\int_{-L(1+a)}^{L(1+a)}\left(\int_{-L}^{L}K(u)K(v-ua)\,du\right)^2dv\right)}. \tag{164}$$

Proof. In the proof of Theorem 11 the following asymptotic expression has been obtained

$$\widetilde{\lambda}_{TWP}(t)-\lambda_F(t)=\overline{\lambda}_{TWP}(t)-\lambda_F(t)+o_P\left(b_1^2+n^{-1/2}b_1^{-1/2}\right) \tag{165}$$

where $\overline{\lambda}_{TWP}(t)$ has been defined in (83).

Along the proof of Lemma 20 the asymptotic equivalence of $\overline{\lambda}_{TWP}(t)$ and its Hájek projection, $\overline{\lambda}_{TWP}^*(t)$, has been obtained. Therefore, Theorem 14, gives

$$\frac{\overline{\lambda}_{TWP}(t)-E\left[\overline{\lambda}_{TWP}(t)\right]}{\sqrt{Var\left[\overline{\lambda}_{TWP}(t)\right]}}-\frac{\overline{\lambda}_{TWP}^*(t)-E\left[\overline{\lambda}_{TWP}^*(t)\right]}{\sqrt{Var\left[\overline{\lambda}_{TWP}^*(t)\right]}}=o_P(1).$$

We now write

$$\begin{aligned}\frac{\overline{\lambda}_{TWP}(t)-E\left[\overline{\lambda}_{TWP}(t)\right]}{\sqrt{Var\left[\overline{\lambda}_{TWP}(t)\right]}} &= \frac{\overline{\lambda}_{TWP}^*(t)-E\left[\overline{\lambda}_{TWP}^*(t)\right]}{\sqrt{Var\left[\overline{\lambda}_{TWP}^*(t)\right]}}\\ &\quad+\frac{\overline{\lambda}_{TWP}(t)-E\left[\overline{\lambda}_{TWP}(t)\right]}{\sqrt{Var\left[\overline{\lambda}_{TWP}(t)\right]}}-\frac{\overline{\lambda}_{TWP}^*(t)-E\left[\overline{\lambda}_{TWP}^*(t)\right]}{\sqrt{Var\left[\overline{\lambda}_{TWP}^*(t)\right]}}.\end{aligned}$$

It is then clear that the asymptotic normality of $\frac{\overline{\lambda}_{TWP}(t)-E\left[\overline{\lambda}_{TWP}(t)\right]}{\sqrt{Var\left[\overline{\lambda}_{TWP}(t)\right]}}$ is a consequence of that of $\frac{\overline{\lambda}_{TWP}^*(t)-E\left[\overline{\lambda}_{TWP}^*(t)\right]}{\sqrt{Var\left[\overline{\lambda}_{TWP}^*(t)\right]}}$ by using Slutsky's theorem.

We now prove that $\frac{\overline{\lambda}^*_{TWP}(t)-E\left[\overline{\lambda}^*_{TWP}(t)\right]}{\sqrt{Var\left[\overline{\lambda}^*_{TWP}(t)\right]}}$ has an asymptotically normal distribution. Let's define

$$
\begin{aligned}
W^*_{n,i} &= \frac{1}{n}\frac{K_{b_1}(t-T_i)\,p(T_i)\mathbf{1}(T_i\in I)}{1-H(T_i)+\frac{1}{n}} \\
&\quad +\frac{n-1}{n^2}E\left[\frac{K_{b_1}(t-T_j)\,K_{b_2}(T_j-T_i)\,(\delta_i-p(T_j))\,\mathbf{1}(T_j\in I)}{\left(1-H(T_j)+\frac{1}{n}\right)h(T_j)}|Z_i\right] \\
&\quad +\frac{n-1}{n^2}E\left[\frac{K_{b_1}(t-T_i)\,K_{b_2}(T_i-T_j)\,(\delta_j-p(T_i))\,\mathbf{1}(T_i\in I)}{\left(1-H(T_i)+\frac{1}{n}\right)h(T_i)}|Z_i\right] \\
&\quad +\frac{1}{n^2}\frac{K_{b_1}(t-T_i)\,K_{b_2}(0)\,(\delta_i-p(T_i))\,\mathbf{1}(T_i\in I)}{\left(1-H(T_i)+\frac{1}{n}\right)h(T_i)}-\frac{1}{n}E\left[J_2\right] \qquad (166)
\end{aligned}
$$

where j is an arbitrary index not equal to i. Then

$$
\overline{\lambda}^*_{TWP}(t)=\sum_{i=1}^{n}W^*_{n,i}
$$

where $W^*_{n,i}, i=1,2,...,n;\ n=1,2,...,$ is a triangular array of iid random variables. Using Liapunov Theorem, in order to prove that $\frac{\overline{\lambda}^*_{TWP}(t)-E\left[\overline{\lambda}^*_{TWP}(t)\right]}{\sqrt{Var\left[\overline{\lambda}^*_{TWP}(t)\right]}}$ is asymptotically normal it suffices to prove that

$$
\lim_{n\to\infty}\frac{nE\left[\left|W^*_{n,1}-E(W^*_{n,1})\right|^3\right]}{\left(Var\left[\overline{\lambda}^*_{TWP}(t)\right]\right)^{3/2}}=0. \qquad (167)
$$

The arguments already used to prove (113) also lead to

$$
\frac{nE\left[\left|W^*_{n,1}-E(W^*_{n,1})\right|^3\right]}{\left(Var\left[\overline{\lambda}^*_{TWP}(t)\right]\right)^{3/2}}\leq\frac{8nE\left[\left|W^*_{n,1}\right|^3\right]}{\left(Var\left[\overline{\lambda}^*_{TWP}(t)\right]\right)^{3/2}}. \qquad (168)
$$

For the second summand in (166),

$$
\begin{aligned}
&E\left[\frac{K_{b_1}(t-T_2)\,K_{b_2}(T_2-T_1)\,(\delta_1-p(T_2))\,\mathbf{1}(T_2\in I)}{\left(1-H(T_2)+\frac{1}{n}\right)h(T_2)}|Z_1\right] \\
&= \int_{\varepsilon}^{t_0}\frac{K_{b_1}(t-t_2)\,K_{b_2}(t_2-T_1)\,(\delta_1-p(t_2))}{1-H(t_2)+\frac{1}{n}}dt_2 \\
&= \frac{1}{b_2}\int_{\frac{t-t_0}{b_1}}^{\frac{t-\varepsilon}{b_1}}\frac{K(t_{21})\,K\left(\frac{t-b_1t_{21}-T_1}{b_2}\right)(\delta_1-p(t-b_1t_{21}))}{1-H(t-b_1t_{21})+\frac{1}{n}}dt_{21}
\end{aligned}
$$

and consequently,

$$\begin{aligned}
&\left|E\left[\frac{K_{b_1}(t-T_2)\,K_{b_2}(T_2-T_1)\,(\delta_1-p(T_2))\,\mathbf{1}(T_2\in I)}{\left(1-H(T_2)+\frac{1}{n}\right)h(T_2)}\,|Z_1\right]\right| \\
\le\;& \frac{1}{b_2}\frac{\|K\|_\infty\int_{-l}^{L}K(u)\,du}{1-H(t_0)}. \quad a.s.
\end{aligned}$$

Similarly, the third summand in (166),

$$\begin{aligned}
&E\left[\frac{K_{b_1}(t-T_1)\,K_{b_2}(T_1-T_2)\,(\delta_2-p(T_1))\,\mathbf{1}(T_1\in I)}{\left(1-H(T_1)+\frac{1}{n}\right)h(T_1)}\,|Z_1\right] \\
=\;& \frac{K_{b_1}(t-T_1)\,\mathbf{1}(T_1\in I)}{\left(1-H(T_1)+\frac{1}{n}\right)h(T_1)}E\left[K_{b_2}(T_1-T_2)\,(\delta_2-p(T_1))\,|Z_1\right] \\
=\;& \frac{K_{b_1}(t-T_1)\,\mathbf{1}(T_1\in I)}{\left(1-H(T_1)+\frac{1}{n}\right)h(T_1)}\int_0^\infty K_{b_2}(T_1-t_2)\,(p(t_2)-p(T_1))\,dt_2 \\
=\;& \frac{K_{b_1}(t-T_1)\,\mathbf{1}(T_1\in I)}{\left(1-H(T_1)+\frac{1}{n}\right)h(T_1)}\int_{-\infty}^{\frac{T_1}{b_2}} K(t_{21})\,(p(t-b_2t_{21})-p(T_1))\,dt_{21}
\end{aligned}$$

and, therefore,

$$\begin{aligned}
&\left|E\left[\frac{K_{b_1}(t-T_1)\,K_{b_2}(T_1-T_2)\,(\delta_2-p(T_1))\,\mathbf{1}(T_1\in I)}{\left(1-H(T_1)+\frac{1}{n}\right)h(T_1)}\,|Z_1\right]\right| \\
\le\;& \frac{1}{b_1}\frac{\|K\|_\infty\int_{-L}^{L}K(u)\,du}{(1-H(t_0))\,\delta} \quad a.s.
\end{aligned}$$

where $\delta>0$ is such that $h(t)>\delta$ for all $t\in I$.

Thus

$$\begin{aligned}
\left|W_{n,1}^*\right| \;\le\;& \frac{1}{nb_1}\frac{\|K\|_\infty}{1-H(t_0)}+\frac{n-1}{n^2b_2}\frac{\|K\|_\infty}{1-H(t_0)}+\frac{n-1}{n^2b_1}\frac{\|K\|_\infty}{(1-H(t_0))\,\delta} \\
&+\frac{1}{n^2b_1b_2}\frac{\|K\|_\infty^2}{(1-H(t_0))\,\delta}+\frac{1}{n}\left|E\left[J_2\right]\right| \quad a.s..
\end{aligned}$$

Now **(V.2)** and (96), give

$$E\left[\left|W_{n,1}^*\right|^3\right]=O\left(n^{-1}b_1^{-1}+n^{-2}b_1^{-2}+n^{-1}b_1^2\right)=O\left(n^{-1}b_1^{-1}\right)$$

and, using (161) and (168),

$$\frac{nE\left[\left|W_{n,1}^*-E(W_{n,1}^*)\right|^3\right]}{\left(Var\left[\overline{\lambda}_{TWP}^*(t)\right]\right)^{3/2}}=O(n)O\left(n^{-3}b_1^{-3}\right)O\left(n^{3/2}b_1^{3/2}\right)=O\left(n^{-1/2}b_1^{-3/2}\right),$$

which, using **(V.1)**, implies Liapunov condition (167).

Therefore, it has been proven

$$\frac{\overline{\lambda}^*_{TWP}(t) - E\left[\overline{\lambda}^*_{TWP}(t)\right]}{\sqrt{Var\left[\overline{\lambda}^*_{TWP}(t)\right]}} \xrightarrow{d} N(0,1)$$

and, as a consequence,

$$\frac{\overline{\lambda}_{TWP}(t) - \lambda_F(t) - E\left[\overline{\lambda}_{TWP}(t) - \lambda_F(t)\right]}{\sqrt{Var\left[\overline{\lambda}_{TWP}(t)\right]}} = \frac{\overline{\lambda}_{TWP}(t) - E\left[\overline{\lambda}_{TWP}(t)\right]}{\sqrt{Var\left[\overline{\lambda}_{TWP}(t)\right]}} \xrightarrow{d} N(0,1).$$

Using definitions (163) and (164),

$$\begin{aligned}
&\frac{\overline{\lambda}_{TWP}(t) - \lambda_F(t) - E\left[\overline{\lambda}_{TWP}(t) - \lambda_F(t)\right]}{\sqrt{Var\left[\overline{\lambda}_{TWP}(t)\right]}} \\
= \; & \sqrt{nb_1}\frac{\overline{\lambda}_{TWP}(t) - \lambda_F(t) - b_1^2\xi_6(t) + o\left(b_1^2\right)}{\xi_7(t) + o(1)} \\
= \; & \sqrt{nb_1}\frac{\overline{\lambda}_{TWP}(t) - \lambda_F(t) - b_1^2\xi_6(t)}{\xi_7(t) + o(1)} + o\left(n^{1/2}b_1^{5/2}\right) \\
= \; & \sqrt{nb_1}\frac{\overline{\lambda}_{TWP}(t) - \lambda_F(t) - b_1^2\xi_6(t)}{\xi_7(t)}\frac{\xi_7(t)}{\xi_7(t) + o(1)} \\
& + o\left(n^{1/2}b_1^{5/2}\right),
\end{aligned}$$

or, equivalently,

$$\begin{aligned}
&\sqrt{nb_1}\frac{\overline{\lambda}_{TWP}(t) - \lambda_F(t) - b_1^2\xi_6(t)}{\xi_7(t)} \\
= \; & \frac{\overline{\lambda}_{TWP}(t) - \lambda_F(t) - E\left[\overline{\lambda}_{TWP}(t) - \lambda_F(t)\right]}{\sqrt{Var\left[\overline{\lambda}_{TWP}(t)\right]}}\frac{\xi_7(t) + o(1)}{\xi_7(t)} \\
& + o\left(n^{1/2}b_1^{5/2}\right)O(1).
\end{aligned}$$

Now condition **(V.1),** $\frac{\xi_7(t)+o(1)}{\xi_7(t)} \to 1$ and Slutsky's Theorem lead to

$$\sqrt{nb_1}\frac{\overline{\lambda}_{TWP}(t) - \lambda_F(t) - b_1^2\xi_6(t)}{\xi_7(t)} \xrightarrow{d} N(0,1).$$

A further application of Slutsky's Theorem to (165) gives

$$\sqrt{nb_1}\frac{\widetilde{\lambda}_{TWP}(t) - \lambda_F(t) - b_1^2\xi_6(t)}{\xi_7(t)} \xrightarrow{d} N(0,1),$$

which is equivalent to (162). ■

Remark 22 *As in Remark 12 for a concave kerner function, we have*

$$\begin{aligned}\xi_7^2(t) &= \frac{\lambda_F(t)}{1-H(t)}\left(p(t)c_K + (1-p(t))a\int_{-L(1+a)}^{L(1+a)}\left(\int_{-L}^{L} K(u)K(v-ua)\,du\right)^2 dv\right) \\ &\leq \frac{c_K\lambda_F(t)\,(p(t)+(1-p(t))a)}{1-H(t)} = \xi_2^2(t).\end{aligned}$$

In other terms, the variance of the normal limit distribution of $\sqrt{nb_1}\left(\widetilde{\lambda}_{TWP}(t)-\lambda_F(t)\right)$ *is not larger than that of* $\sqrt{nb_1}\left(\widehat{\lambda}(t)-\lambda_F(t)\right)$ *where* $\widehat{\lambda}(t)$ *is any of the estimators* $\widehat{\lambda}_{WLNW}(t)$, $\widehat{\lambda}_{WLLL}(t)$, $\widehat{\lambda}_{RRNW}(t)$ *or* $\widehat{\lambda}_{RRLL}(t)$.

4 Asymptotically optimal bandwidths

Theorems 7-11 in Section 2 lead to asymptotic expressions for the mean integrated squared error of the leading part in the representations of the estimators. Under certain conditions on the bandwidths, it is possible to obtain explicit expressions for the asymptotically optimal bandwidths, in the sense of minimizing the mean integrated squared error.

If $\widehat{\lambda}$ denotes a general estimator of the hazard rate and $b=(b_1,b_2)$ is the bandwidth vector, the mean integrated squared error of the estimator, $MISE_w(\widehat{\lambda};b)$, is defined by

$$MISE_w(\widehat{\lambda};b) = E\left[\int_0^{\infty}\left(\widehat{\lambda}(t)-\lambda(t)\right)^2 w(t)\,dt\right]$$

where w is a weight function.

In the rest of the paper we will assume some new condition on the function w:

(W.1) w is a nonnegative bounded function, with compact support included in the interval (ε, t_0).

The first result concerns the dominant part of $\widehat{\lambda}_{WLNW}(t)$.

Theorem 23 *Let us assume* ***(K.1)****,* ***(P.1)****,* ***(H.1)****,* ***(V.1)****,* ***(W.1)*** *and* $\frac{b_1}{b_2}=a\in(0,\infty)$*. Then*

$$MISE_w(\overline{\lambda}_{WLNW};b) = AMISE_w(\overline{\lambda}_{WLNW};b) + o\left(AMISE_w(\overline{\lambda}_{WLNW};b)\right) \tag{169}$$

where

$$AMISE_w(\overline{\lambda}_{WLNW};b) = \frac{b_1^4\mu_K^2 I_1(a)}{4} + \frac{c_K I_2(a)}{nb_1} \tag{170}$$

and

$$I_1(a) = \int_0^\infty \left(\lambda_H''(t)p(t) + \frac{1}{a^2} \left(\lambda_H(t)p''(t) + 2 \left(\lambda_H'(t) - \lambda_H(t)^2 \right) p'(t) \right) \right)^2 w(t)dt,$$

$$I_2(a) = \int_0^\infty \frac{\lambda_H(t)p(t) \left(p(t) + a(1-p(t)) \right) w(t)}{1-H(t)} dt.$$

Moreover, the bandwidth pair that minimizes $AMISE_w(\overline{\lambda}_{WLNW}; b)$ *is*

$$b_{AMISE_w}(a) = (b_{1,AMISE_w}(a), b_{2,AMISE_w}(a))$$

where

$$b_{1,AMISE_w}(a) = \left(\frac{c_K I_2(a)}{n\mu_K^2 I_1(a)} \right)^{1/5} \tag{171}$$

and

$$b_{2,AMISE_w}(a) = \frac{b_{1,AMISE}(a)}{a}, \tag{172}$$

that satisfy

$$AMISE_w(\overline{\lambda}_{WLNW}; b_{AMISE_w}(a)) = \frac{5}{4} n^{-4/5} \mu_K^{2/5} c_K^{4/5} I_1(a)^{1/5} I_2(a)^{4/5}. \tag{173}$$

Proof. Expressions (59) and (60) in Theorem 7 for the expected value and the variance of $\overline{\lambda}_{WLNW}(t)$, give

$$\begin{aligned} MSE(\overline{\lambda}_{WLNW}(t)) &= \left(E\left[\overline{\lambda}_{WLNW}(t) - \lambda_F(t) \right] \right)^2 + Var\left[\overline{\lambda}_{WLNW}(t) \right] \\ &= \frac{1}{4} b_1^4 \mu_K^2 \left(\lambda_H''(t)p(t) + \frac{1}{a^2} \left(\lambda_H(t)p''(t) + 2 \left(\lambda_H'(t) - \lambda_H(t)^2 \right) p'(t) \right) \right)^2 \\ &\quad + \frac{1}{nb_1} \frac{c_K \lambda_H(t)p(t) \left(p(t) + a(1-p(t)) \right)}{1-H(t)} + o\left(b_1^4 + n^{-1} b_1^{-1} \right). \end{aligned} \tag{174}$$

Now (170) holds defining $AMISE_w(\overline{\lambda}_{WLNW}; b)$ as the weighted integral, with respect to w, of the first two summands in the previous expression.

The negligible term in (169) comes from the integration of the negligible terms in (174). This term can be proved to be negligible with respect to $AMISE_w(\overline{\lambda}_{WLNW}; b)$ by applying the bounded convergence theorem, using condition **(W.1)** and the fact that the terms of order $o\left(b_1^4 + n^{-1} b_1^{-1} \right)$ in (174) come from remainders of Taylor expansions where bounded functions appear, due to **(K.1)**, **(P.1)** and **(H.1)**.

Using the classical bias-variance balance, it is immediate to obtain from (170) the optimal bandwidths (171) and (172). Finally, (173) holds by plugging (171) into (170). ■

The following four theorems give similar results for the leading parts of $\widehat{\lambda}_{RRNW}(t)$, $\widehat{\lambda}_{WLLL}(t)$, $\widehat{\lambda}_{RRLL}(t)$ and $\widetilde{\lambda}_{TWP}(t)$.

Theorem 24 *Let us assume conditions* ***(K.1)***, ***(P.1)***, ***(H.1)***, ***(V.1)***, ***(W.1)*** *and* $\frac{b_1}{b_2} = a \in (0, \infty)$. *Then*

$$MISE_w(\overline{\lambda}_{RRNW}; b) = AMISE_w(\overline{\lambda}_{RRNW}; b) + o\left(AMISE_w(\overline{\lambda}_{RRNW}; b)\right) \tag{175}$$

where

$$AMISE_w(\overline{\lambda}_{RRNW}; b) = \frac{b_1^4 \mu_K^2 I_3(a)}{4} + \frac{c_K I_2(a)}{nb_1} \tag{176}$$

and

$$\begin{aligned} I_3(a) &= \int_0^\infty \Bigg(\left(\lambda''_H(t) + \lambda_H(t)^3 - 3\lambda'_H(t)\lambda_H(t)\right) p(t) \\ &\quad + \frac{1}{a^2}\left(\lambda_H(t)p''(t) + 2\left(\lambda'_H(t) - \lambda_H(t)^2\right)p'(t)\right)\Bigg)^2 w(t)dt, \end{aligned}$$

$$I_2(a) = \int_0^\infty \frac{\lambda_H(t)p(t)\left(p(t) + a(1 - p(t))\right)w(t)}{1 - H(t)} dt.$$

Moreover, the pair of bandwidths that minimizes $AMISE_w(\overline{\lambda}_{RRNW}; b)$ *is*

$$b_{AMISE_w}(a) = (b_{1,AMISE_w}(a), b_{2,AMISE_w}(a))$$

where

$$b_{1,AMISE_w}(a) = \left(\frac{c_K I_2(a)}{n\mu_K^2 I_3(a)}\right)^{1/5} \tag{177}$$

and

$$b_{2,AMISE_w}(a) = \frac{b_{1,AMISE_w}(a)}{a}, \tag{178}$$

that satisfy

$$AMISE_w(\overline{\lambda}_{RRNW}; b_{AMISE_w}(a)) = \frac{5}{4} n^{-4/5} \mu_K^{2/5} c_K^{4/5} I_3(a)^{1/5} I_2(a)^{4/5}. \tag{179}$$

Proof. Expression (62) in Theorem 8 gives the definition of $\overline{\lambda}_{RRNW}(t)$ in terms of $\overline{\lambda}_{WLNW}(t)$. It is straight forward to conclude that

$$\begin{aligned} E\left[\overline{\lambda}_{RRNW}(t) - \lambda_F(t)\right] &= E\left[\overline{\lambda}_{WLNW}(t) - \lambda_F(t)\right] \\ &\quad + \frac{1}{2} b_1^2 \mu_K \left(\lambda_H(t)^2 - 3\lambda'_H(t)\right)\lambda_H(t)p(t) \end{aligned}$$

and

$$Var\left[\overline{\lambda}_{RRNW}(t) - \lambda_F(t)\right] = Var\left[\overline{\lambda}_{WLNW}(t) - \lambda_F(t)\right].$$

Therefore, using the results in Theorem 7, we have

$$\begin{aligned} MSE(\overline{\lambda}_{RRNW}(t)) &= \frac{1}{4}b_1^4\mu_K^2\left(\left(\lambda_H''(t)+\lambda_H(t)^3-3\lambda_H'(t)\lambda_H(t)\right)p(t)\right. \\ &\left.+\frac{1}{a^2}\left(\lambda_H(t)p''(t)+2\left(\lambda_H'(t)-\lambda_H(t)^2\right)p'(t)\right)\right)^2 \\ &+\frac{1}{nb_1}\frac{c_K\lambda_H(t)p(t)\left(p(t)+a(1-p(t))\right)}{1-H(t)}+o\left(b_1^4+n^{-1}b_1^{-1}\right). \end{aligned} \tag{180}$$

Let us define $AMISE_w(\overline{\lambda}_{RRNW};b)$ as the integral, with respect to the weight w, of the sum of the first two terms in (180). Then, (176) holds. The negligibility, with respect to $AMISE_w(\overline{\lambda}_{RRNW};b)$, of the second summand in (175) can be proved by similar arguments as those in the proof of the previous theorem.

Finally, (177), (178) and (179) can be obtained by minimizing in b_1 the function $AMISE_w(\overline{\lambda}_{RRNW};b)$. ■

Theorem 25 *Conditions* ***(K.1)**, **(P.1)**, **(H.1)**, **(V.1)**, **(W.1)*** *and* $\frac{b_1}{b_2}=a\in(0,\infty)$ *imply*

$$MISE_w(\overline{\lambda}_{WLLL};b)=AMISE_w(\overline{\lambda}_{WLLL};b)+o\left(AMISE_w(\overline{\lambda}_{WLLL};b)\right)$$

where

$$AMISE_w(\overline{\lambda}_{WLLL};b)=\frac{b_1^4\mu_K^2I_4(a)}{4}+\frac{c_KI_2(a)}{nb_1} \tag{181}$$

and

$$I_4(a)=\int_0^\infty\left(\lambda_H''(t)p(t)+\frac{\lambda_H(t)p''(t)}{a^2}\right)^2w(t)dt,$$

$$I_2(a)=\int_0^\infty\frac{\lambda_H(t)p(t)\left(p(t)+a(1-p(t))\right)w(t)}{1-H(t)}dt.$$

Moreover, the pair of bandwidths that minimizes $AMISE_w(\overline{\lambda}_{WLLL};b)$ *is*

$$b_{AMISE_w}(a)=(b_{1,AMISE_w}(a),b_{2,AMISE_w}(a))$$

where

$$b_{1,AMISE_w}(a)=\left(\frac{c_KI_2(a)}{n\mu_K^2I_4(a)}\right)^{1/5}$$

and

$$b_{2,AMISE_w}(a)=\frac{b_{1,AMISE}(a)}{a},$$

that satisfy

$$AMISE_w(\overline{\lambda}_{WLLL};b_{AMISE_w}(a))=\frac{5}{4}n^{-4/5}\mu_K^{2/5}c_K^{4/5}I_4(a)^{1/5}I_2(a)^{4/5}.$$

Proof. The proof is completely parallel to that of Theorem 23, starting from the results in Theorem 9. ■

Theorem 26 *Assume conditions* ***(K.1)***, ***(P.1)***, ***(H.1)***, ***(V.1)***, ***(W.1)*** *and* $\frac{b_1}{b_2} = a \in (0, \infty)$. *Then*

$$MISE_w(\overline{\lambda}_{RRLL}; b) = AMISE_w(\overline{\lambda}_{RRLL}; b) + o\left(AMISE_w(\overline{\lambda}_{RRLL}; b)\right)$$

where

$$AMISE_w(\overline{\lambda}_{RRLL}; b) = \frac{b_1^4 \mu_K^2 I_5(a)}{4} + \frac{c_K I_2(a)}{n b_1} \tag{182}$$

and

$$I_5(a) = \int_0^\infty \left(\left(\lambda_H''(t) + \lambda_H(t)^3 - 3\lambda_H'(t)\lambda_H(t)\right) p(t) + \frac{\lambda_H(t) p''(t)}{a^2} \right)^2 w(t) dt,$$

$$I_2(a) = \int_0^\infty \frac{\lambda_H(t) p(t) \left(p(t) + a(1 - p(t))\right) w(t)}{1 - H(t)} dt.$$

The pair of bandwidths that minimizes $AMISE_w(\overline{\lambda}_{RRLL}; b)$ *is*

$$b_{AMISE_w}(a) = \left(b_{1,AMISE_w}(a), b_{2,AMISE_w}(a)\right)$$

where

$$b_{1,AMISE_w}(a) = \left(\frac{c_K I_2(a)}{n \mu_K^2 I_5(a)} \right)^{1/5}$$

and

$$b_{2,AMISE_w}(a) = \frac{b_{1,AMISE_w}(a)}{a},$$

that satisfy

$$AMISE_w(\overline{\lambda}_{RRLL}; b_{AMISE_w}(a)) = \frac{5}{4} n^{-4/5} \mu_K^{2/5} c_K^{4/5} I_5(a)^{1/5} I_2(a)^{4/5}.$$

Proof. The proof is, once more, completely parallel to that of Theorem 24, starting from Theorems 9 and 10. ■

Theorem 27 *Let us assume conditions* ***(K.1)***, ***(P.1)***, ***(H.1)***, ***(V.1)***, ***(W.1)*** *and* $\frac{b_1}{b_2} = a \in (0, \infty)$. *Then*

$$MISE_w(\overline{\lambda}_{TWP}; b) = AMISE_w(\overline{\lambda}_{TWP}; b) + o\left(AMISE_w(\overline{\lambda}_{TWP}; b)\right)$$

where

$$AMISE_w(\overline{\lambda}_{TWP};b) = \frac{b_1^4 \mu_K^2 I_6(a)}{4} + \frac{I_7(a)}{nb_1} \tag{183}$$

and

$$\begin{aligned} I_6(a) &= \int_0^\infty \Big(\lambda_H''(t)p(t) + 2\lambda_H'(t)p'(t) + \lambda_H(t)p''(t) \\ &\quad + \frac{1}{a^2}\left(\lambda_H(t)p''(t) + 2\left(\lambda_H'(t) - \lambda_H(t)^2\right)p'(t)\right)\Big)^2 w(t)dt, \end{aligned}$$

$$\begin{aligned} I_7(a) &= \int_0^\infty \frac{\lambda_H(t)p(t)}{1-H(t)} \\ &\quad \times \left(p(t)c_K + (1-p(t))a \int_{-L(1+a)}^{L(1+a)} \left(\int_{-L}^{L} K(u)K(v-ua)\,du \right)^2 dv \right) w(t)dt. \end{aligned}$$

Moreover, the pair of bandwidths that minimizes $AMISE_w(\overline{\lambda}_{TWP};b)$ *is*

$$b_{AMISE_w}(a) = (b_{1,AMISE_w}(a), b_{2,AMISE_w}(a))$$

where

$$b_{1,AMISE_w}(a) = \left(\frac{I_7(a)}{n\mu_K^2 I_6(a)} \right)^{1/5}$$

and

$$b_{2,AMISE_w}(a) = \frac{b_{1,AMISE_w}(a)}{a},$$

that satisfy

$$AMISE_w(\overline{\lambda}_{TWP};b_{AMISE_w}(a)) = \frac{5}{4} n^{-4/5} \mu_K^{2/5} I_6(a)^{1/5} I_7(a)^{4/5}.$$

Proof. Again, the proof is completely parallel to that of theorems 23 or 25, starting from Theorem 11. ■

Remark 28 *Theorems 23-27 give explicit expressions for the asymptotically optimal bandwidths assuming that* $b_1/b_2 = a \in (0,\infty)$. *These expressions depend therefore on* a. *Thus, denoting by* $\overline{\lambda}$ *any of the previous five estimators, the value* a *that minimizes* $AMISE_w(\overline{\lambda};b_{AMISE}(a))$ *(to be denoted by* a_{OPT}*) has to be computed,*

$$a_{OPT} = \arg\min_{a>0} AMISE_w(\overline{\lambda};b_{AMISE}(a)).$$

For the presmoothed Tanner-Wong estimator, it may well be that

$$
\begin{aligned}
& \inf\left\{AMISE_w(\overline{\lambda}_{TWP}; b_{AMISE}(a)) : a \in (0,\infty)\right\} \\
= & \lim_{a\to\infty} AMISE_w(\overline{\lambda}_{TWP}; b_{AMISE}(a)).
\end{aligned}
$$

In this case, the minimum of the function $AMISE_w(\overline{\lambda}_{TWP}; b_{AMISE}(a))$ is not a local minimum. Recalling Remark 13, the previous fact implies that the estimator $\overline{\lambda}_{TWP}$ only may exhibit, at most, second order asymptotic efficiency with respect to Tanner-Wong estimator.

Remark 29 *A sufficient condition for the existence of a local minimum of $AMISE_w(\overline{\lambda}_{TWP}; b_{AMISE}(a))$ is that*

$$
\int_0^\infty \lambda_F''(t)\left(\lambda_H(t)p''(t) + 2\left(\lambda_H'(t) - \lambda_H(t)^2\right)p'(t)\right)w(t)dt < 0. \tag{184}
$$

Indeed,

$$
\begin{aligned}
I_6'(a) &= -\frac{4}{a^3}\int_0^\infty \lambda_F''(t)\left(\lambda_H(t)p''(t) + 2\left(\lambda_H'(t) - \lambda_H(t)^2\right)p'(t)\right)w(t)dt \\
&\quad -\frac{4}{a^5}\int_0^\infty \left(\lambda_H(t)p''(t) + 2\left(\lambda_H'(t) - \lambda_H(t)^2\right)p'(t)\right)^2 w(t)dt,
\end{aligned}
$$

which is zero for

$$
a_0 = \sqrt{-\frac{\int_0^\infty \left(\lambda_H(t)p''(t) + 2\left(\lambda_H'(t) - \lambda_H(t)^2\right)p'(t)\right)^2 w(t)dt}{\int_0^\infty \lambda_F''(t)\left(\lambda_H(t)p''(t) + 2\left(\lambda_H'(t) - \lambda_H(t)^2\right)p'(t)\right)w(t)dt}},
$$

that belongs to the interval $(0,\infty)$ under condition (184). On the other hand, denoting by $\widetilde{K}$ the Fourier transform of K,

$$
\begin{aligned}
a\int_{-L(1+a)}^{L(1+a)}\left(\int_{-L}^{L} K(u)K(v-ua)\,du\right)^2 dv &= \sqrt{2\pi}\int_{-\infty}^{\infty}\widetilde{K}(v_1)^2\widetilde{K}\left(\frac{v_1}{a}\right)^2 dv_1 \\
&\le \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\widetilde{K}(v_1)^2 dv_1 = c_K,
\end{aligned}
$$

which implies

$$
I_7(a) \le \lim_{x\to\infty} I_7(x).
$$

Thus,

$$
AMISE_w(\overline{\lambda}_{TWP}; b_{AMISE}(a_0)) < \lim_{a\to\infty} AMISE_w(\overline{\lambda}_{TWP}; b_{AMISE}(a))
$$

and there exist

$$
a_{OPT} = \arg\min_{a>0} AMISE_w(\overline{\lambda}; b_{AMISE}(a)) < \infty.
$$

5 Simulations

In order to show the practical performance of the estimators presented in this paper and to compare to well-known hazard rate estimators some Montecarlo simulation study has been carried out.

For the first four models considered here we have assumed that the life time distribution and the censoring distribution belong to the same parametric family. Weibull and loglogistic distributions have been used.

More specifically, in the first two models (I and II) $Y \stackrel{d}{=} W(\alpha_F, \beta_F)$, $C \stackrel{d}{=} W(\alpha_G, \beta_G)$, where $W(\alpha, \beta)$ denotes the Weibull distribution with shape parameter α and scale parameter β, whose density is given by

$$\beta \alpha x^{\alpha - 1} \exp(-\beta x^{\alpha}), \quad x > 0.$$

The combination of the values $\alpha_F, \beta_F, \alpha_G$ and β_G that characterizes every model is presented in Table 2. Figure 1 plots the interest hazard rate, the observed life time hazard rate and the conditional probability of uncensoring for models I and II.

Table 2. Parameters for models I-IV.

Model	F	G
I	$W(3,1)$	$W(5,1)$
II	$W(5,1)$	$W(3,1)$
III	$LL(0,0.2)$	$LL(0,0.4)$
IV	$LL(0,0.4)$	$LL(0,0.2)$

For the last two models, III and IV, $Y \stackrel{d}{=} LL(\alpha_F, \beta_F)$, $C \stackrel{d}{=} LL(\alpha_G, \beta_G)$, where $LL(\alpha, \beta)$ denotes loglogistic distribution with location parameter α and scale parameter β, whose density is

$$\frac{1}{\beta x} \gamma \left(\frac{\log(x) - \alpha}{\beta} \right), \quad x > 0$$

where $\gamma(\cdot)$ denotes the logistic density function $\gamma(x) = \exp(x) / (1 + \exp(x))^2$. Specific values for the parameter in these two models can be found in Table 2. Figure 2 shows the interest hazard rate, the observed life time hazard rate and the conditional probability of uncensoring for models V and VI.

The unconditional probability of censoring for models I-IV is in the range between 0.47 and 0.53.

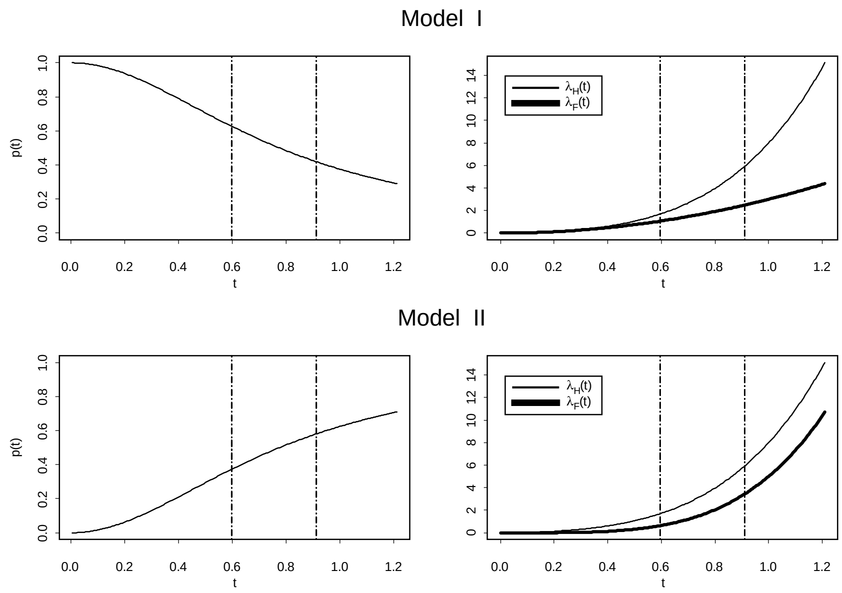

Figure 1. Functions p, λ_H and λ_F for models I and II. Dashed lines indicate the endpoints of the support of w.

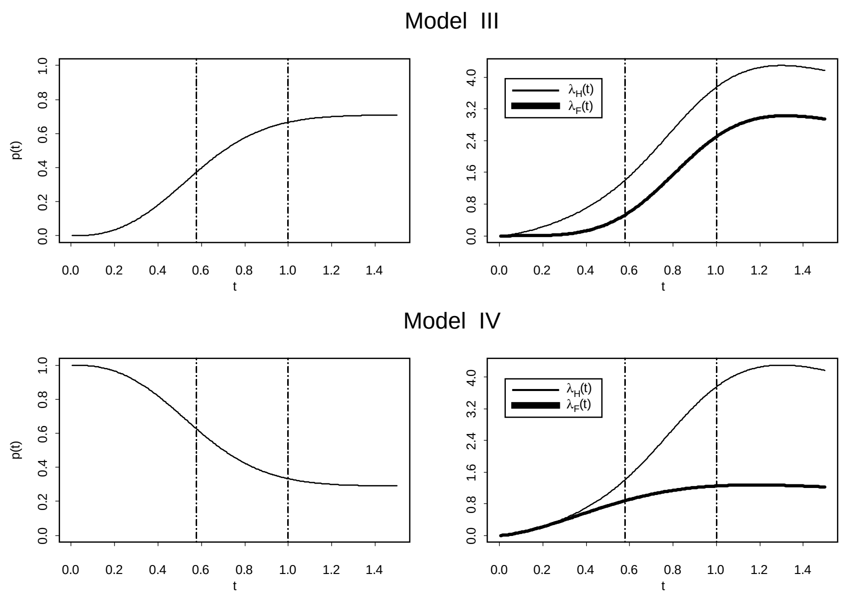

Figure 2. Functions p, λ_H and λ_F for models III and IV. Dashed lines indicate the endpoints of the support of w.

For the four models moderate ($n = 30$) and medium size ($n = 200$) samples have been drawn. For a general hazard rate estimator, $\widehat{\lambda}$, and a general bandwidth vector, b, the value of $MISE_w(\widehat{\lambda}; b)$—already defined at the beginning of section 4— has been approximated using 500 simulated sampled for every model and every sample size along a grid of values for every bandwidth. These grids have been taken equispaced in logarithmic scale, between 0.08 and $1.28 \times 2^{5/6} \simeq 2.28$. For every simulated sample ISE_w has been approximated numerically by Gaussian quadrature of 15 points. The weight function, w, has been chosen as the indicator of the interval with endpoint in the first and third quartiles of the observed lifetime distribution. The Epanechnikov kernel has been used throughout the simulations.

Tables 3-6 collect, for models *I*-*IV*, the minimal value of the approximated $MISE_w(\widehat{\lambda}; b)$ and the bandwidths, b_{MISE_w}, at which these minima are attained, for every estimator and every sample size $n = 30{,}200$. For those estimators depending on two bandwidths, we use the notation $b_{MISE_w} = (b_{1,MISE_w}, b_{2,MISE_w})$, where $b_{1,MISE_w}$ and $b_{2,MISE_w}$ denote the value of the b_1 and b_2 component of b_{MISE_w}. These tables also include the value of the $AMISE_w$-optimal bandwidths, $b_{AMISE_w} = (b_{1,AMISE_w}, b_{2,AMISE_w})$, computed by minimizing numerically the expressions for $AMISE_w(\widehat{\lambda}; b)$ in Theorems 23-27.

Figures 3-10 show, for models *I*-*IV* and sample size $n = 30$, the Montecarlo approximation of $MISE_w(\widehat{\lambda}; b)$, as a function of the bandwidths, for every estimator. Although, for brevity, the same information is also shown for $n = 200$ in Figures 11-14 but only for the estimators $\widehat{\lambda}_{TW}(t)$, $\widetilde{\lambda}_{TWP}(t)$, $\widehat{\lambda}_{WLNW}(t)$ and $\widehat{\lambda}_{WLLL}(t)$, which seem to be representative of the whole collection of estimators.

For $n = 30$ it is often observed that the minimum of the approximated $MISE_w(\widehat{\lambda}; b)$ is attained outside the grid. This suggest that it has not be attained a local minimum or that it doesn't even exist. Generally speaking, this occurs because the bandwidth b_2, that controls the amount of smoothing in the estimation of the conditional probability of uncensoring, takes the largest value in the grid. It should be pointed out that the largest value in the grid of any of the two bandwidths was 2.28, which is always greater than the 99.8% percentile of the observed life time distribution for any of the models. In all these cases, but also when the minimum is located in the interior of the searching region, the level curves in Figures 3-10 indicate that, when b_1 takes suboptimal values, the effect of b_2 on $MISE_w(\widehat{\lambda}; b)$ is —except for relative low values of b_2— much less important than the value of b_1. For $n = 200$, the minimum is located most of the times in the interior, but the previous comment on the effect of b_1 and b_2 in $MISE_w(\widehat{\lambda}; b)$ remains true in general.

Direct inspection of Tables 3-6 shows a better performance of Tanner-Wong type or Watson-Leadbetter type estimators ($\widehat{\lambda}_{TW}(t)$, $\widetilde{\lambda}_{TWP}(t)$, $\widehat{\lambda}_{WLNW}(t)$ and $\widehat{\lambda}_{WLLL}(t)$) than Blum-Susarla type or Rice-Rosenblatt type estimators ($\widehat{\lambda}_{BS}(t)$, $\widetilde{\lambda}_{BSP}(t)$, $\widehat{\lambda}_{RRNW}(t)$ and $\widehat{\lambda}_{RRLL}(t)$).

Within the estimators $\widehat{\lambda}_{TW}(t)$, $\widetilde{\lambda}_{TWP}(t)$, $\widehat{\lambda}_{WLNW}(t)$ and $\widehat{\lambda}_{WLLL}(t)$, the last one, $\widehat{\lambda}_{WLLL}(t)$, is, in general, the most efficient (in four of the eight cases it has a smaller $MISE_w$). On the other hand there seems to be an increasing effect in favour of the estimators $\widetilde{\lambda}_{TWP}(t)$ and $\widehat{\lambda}_{TW}(t)$ as the sample size grows. This could be explained from some possible second order effect in the relative efficiency of these estimators.

Although $\widehat{\lambda}_{WLLL}(t)$ exhibits a good performance, other estimators can be even 42% more efficient than this for $n = 30$ and up to 27% for $n = 200$. The estimator $\widehat{\lambda}_{WLLL}(t)$ is not even uniformly better than $\widehat{\lambda}_{WLNW}(t)$, although it is also the case that the choice of the weight function may hide possible differences between these two.

Finally, although Tanner-Wong estimator, $\widehat{\lambda}_{TW}(t)$, can be more efficient than the product-type estimators, it is always beaten by its presmoothed version, $\widetilde{\lambda}_{TWP}(t)$. The gain in efficiency is sometimes marginal (0.7%), but often very important (40%). All these features make of $\widetilde{\lambda}_{TWP}(t)$ a very advisable choice for a nonparametric hazard rate estimator.

Table 3. Simulation results for model I. Bandwidths b_{AMISE_w}, b_{MISE_w} and $MISE_w(\widehat{\lambda}; b_{MISE_w})$ for the hazard rate estimators.

n	Estimator	b_{AMISE_w}	b_{MISE_w}	$MISE_w(\widehat{\lambda}; b_{MISE_w}) \times 10^2$
30	$\widehat{\lambda}_{TW}(t)$	0.56	0.51	11.404
	$\widehat{\lambda}_{WLNW}(t)$	$(0.28, 0.41)$	$(0.81, 2.28)$	6.221
	$\widehat{\lambda}_{WLLL}(t)$	$(0.27, 0.53)$	$(0.72, 2.28)$	10.810
	$\widetilde{\lambda}_{TWP}(t)$	$(0.56, 0.12)$	$(0.72, 2.28)$	7.060
	$\widehat{\lambda}_{BS}(t)$	0.46	0.36	11.400
	$\widehat{\lambda}_{RRNW}(t)$	$(0.47, 0.45)$	$(0.36, 0.91)$	10.456
	$\widehat{\lambda}_{RRLL}(t)$	$(0.62, 0.94)$	$(0.32, 2.28)$	9.831
	$\widetilde{\lambda}_{BSP}(t)$	-	$(0.36, 2.28)$	10.634
200	$\widehat{\lambda}_{TW}(t)$	0.39	0.29	2.056
	$\widehat{\lambda}_{WLNW}(t)$	$(0.19, 0.28)$	$(0.18, 0.29)$	2.625
	$\widehat{\lambda}_{WLLL}(t)$	$(0.19, 0.36)$	$(0.18, 0.51)$	2.281
	$\widetilde{\lambda}_{TWP}(t)$	$(0.38, 0.08)$	$(0.29, 0.08)$	2.021
	$\widehat{\lambda}_{BS}(t)$	0.31	0.23	2.430
	$\widehat{\lambda}_{RRNW}(t)$	$(0.32, 0.31)$	$(0.23, 0.45)$	2.028
	$\widehat{\lambda}_{RRLL}(t)$	$(0.42, 0.64)$	$(0.20, 2.28)$	1.703
	$\widetilde{\lambda}_{BSP}(t)$	-	$(0.23, 0.45)$	2.028

Table 4. Simulation results for model II. Bandwidths b_{AMISE_w}, b_{MISE_w} and $MISE_w(\widehat{\lambda}; b_{MISE_w})$ for the hazard rate estimators.

n	Estimator	b_{AMISE_w}	b_{MISE_w}	$MISE_w(\widehat{\lambda}; b_{MISE_w}) \times 10^2$
30	$\widehat{\lambda}_{TW}(t)$	0.28	0.36	18.829
	$\widehat{\lambda}_{WLNW}(t)$	$(0.35, 0.53)$	$(0.36, 0.64)$	10.996
	$\widehat{\lambda}_{WLLL}(t)$	$(1.10, 2.92)$	$(0.51, 2.28)$	9.894
	$\widetilde{\lambda}_{TWP}(t)$	$(0.28, 0.50)$	$(0.36, 2.28)$	11.241
	$\widehat{\lambda}_{BS}(t)$	0.38	0.29	12.980
	$\widehat{\lambda}_{RRNW}(t)$	$(0.39, 0.38)$	$(0.25, 0.40)$	12.449
	$\widehat{\lambda}_{RRLL}(t)$	$(0.37, 0.47)$	$(0.29, 2.28)$	13.463
	$\widetilde{\lambda}_{BSP}(t)$	-	$(0.25, 0.36)$	12.301
200	$\widehat{\lambda}_{TW}(t)$	0.19	0.18	3.092
	$\widehat{\lambda}_{WLNW}(t)$	$(0.24, 0.36)$	$(0.23, 0.40)$	1.921
	$\widehat{\lambda}_{WLLL}(t)$	$(0.75, 2.00)$	$(0.23, 2.28)$	1.887
	$\widetilde{\lambda}_{TWP}(t)$	$(0.19, 0.34)$	$(0.20, 0.45)$	2.055
	$\widehat{\lambda}_{BS}(t)$	0.26	0.18	2.752
	$\widehat{\lambda}_{RRNW}(t)$	$(0.27, 0.26)$	$(0.16, 0.25)$	2.612
	$\widehat{\lambda}_{RRLL}(t)$	$(0.26, 0.32)$	$(0.16, 0.40)$	2.532
	$\widetilde{\lambda}_{BSP}(t)$	-	$(0.16, 0.23)$	2.593

Table 5. Simulation results for model III. Bandwidths b_{AMISE_w}, b_{MISE_w} and $MISE_w(\widehat{\lambda}; b_{MISE_w})$ for the hazard rate estimators.

n	Estimator	b_{AMISE_w}	b_{MISE_w}	$MISE_w(\widehat{\lambda}; b_{MISE_w}) \times 10^2$
30	$\widehat{\lambda}_{TW}(t)$	0.46	0.57	11.007
	$\widehat{\lambda}_{WLNW}(t)$	$(0.47, 0.36)$	$(0.51, 0.40)$	10.088
	$\widehat{\lambda}_{WLLL}(t)$	$(0.49, 0.44)$	$(0.57, 0.57)$	9.382
	$\widetilde{\lambda}_{TWP}(t)$	$(0.46, 0.00)$	$(0.51, 0.45)$	10.235
	$\widehat{\lambda}_{BS}(t)$	0.41	0.36	12.188
	$\widehat{\lambda}_{RRNW}(t)$	$(0.44, 0.39)$	$(0.36, 0.40)$	12.151
	$\widehat{\lambda}_{RRLL}(t)$	$(0.43, 0.43)$	$(0.36, 0.51)$	13.024
	$\widetilde{\lambda}_{BSP}(t)$	-	$(0.36, 0.23)$	12.117
200	$\widehat{\lambda}_{TW}(t)$	0.32	0.32	2.192
	$\widehat{\lambda}_{WLNW}(t)$	$(0.32, 0.25)$	$(0.36, 0.25)$	2.209
	$\widehat{\lambda}_{WLLL}(t)$	$(0.33, 0.30)$	$(0.36, 0.32)$	2.030
	$\widetilde{\lambda}_{TWP}(t)$	$(0.32, 0.00)$	$(0.32, 0.09)$	2.176
	$\widehat{\lambda}_{BS}(t)$	0.28	0.23	2.788
	$\widehat{\lambda}_{RRNW}(t)$	$(0.30, 0.27)$	$(0.23, 0.25)$	2.772
	$\widehat{\lambda}_{RRLL}(t)$	$(0.29, 0.30)$	$(0.23, 0.29)$	2.785
	$\widetilde{\lambda}_{BSP}(t)$	-	$(0.23, 0.11)$	2.774

Table 6. Simulation results for model IV. Bandwidths b_{AMISE_w}, b_{MISE_w} and $MISE_w(\widehat{\lambda}; b_{MISE_w})$ for the hazard rate estimators.

n	Estimator	b_{AMISE_w}	b_{MISE_w}	$MISE_w(\widehat{\lambda}; b_{MISE_w}) \times 10^2$
30	$\widehat{\lambda}_{TW}(t)$	0.69	0.64	5.548
	$\widehat{\lambda}_{WLNW}(t)$	$(0.69, 0.52)$	$(1.14, 2.28)$	2.942
	$\widehat{\lambda}_{WLLL}(t)$	$(0.54, 0.51)$	$(1.02, 2.28)$	3.984
	$\widetilde{\lambda}_{TWP}(t)$	$(0.76, 0.40)$	$(1.14, 2.28)$	3.083
	$\widehat{\lambda}_{BS}(t)$	0.46	0.57	7.038
	$\widehat{\lambda}_{RRNW}(t)$	$(0.51, 0.44)$	$(0.51, 0.57)$	7.034
	$\widehat{\lambda}_{RRLL}(t)$	$(0.65, 0.64)$	$(0.51, 2.28)$	4.880
	$\widetilde{\lambda}_{BSP}(t)$	-	$(0.57, 0.09)$	7.044
200	$\widehat{\lambda}_{TW}(t)$	0.47	0.45	1.088
	$\widehat{\lambda}_{WLNW}(t)$	$(0.47, 0.35)$	$(0.64, 0.40)$	1.136
	$\widehat{\lambda}_{WLLL}(t)$	$(0.37, 0.35)$	$(0.45, 0.45)$	1.219
	$\widetilde{\lambda}_{TWP}(t)$	$(0.52, 0.27)$	$(0.72, 0.64)$	0.891
	$\widehat{\lambda}_{BS}(t)$	0.31	0.32	1.415
	$\widehat{\lambda}_{RRNW}(t)$	$(0.35, 0.30)$	$(0.32, 0.40)$	1.371
	$\widehat{\lambda}_{RRLL}(t)$	$(0.44, 0.44)$	$(0.40, 1.02)$	0.901
	$\widetilde{\lambda}_{BSP}(t)$	-	$(0.32, 0.18)$	1.399

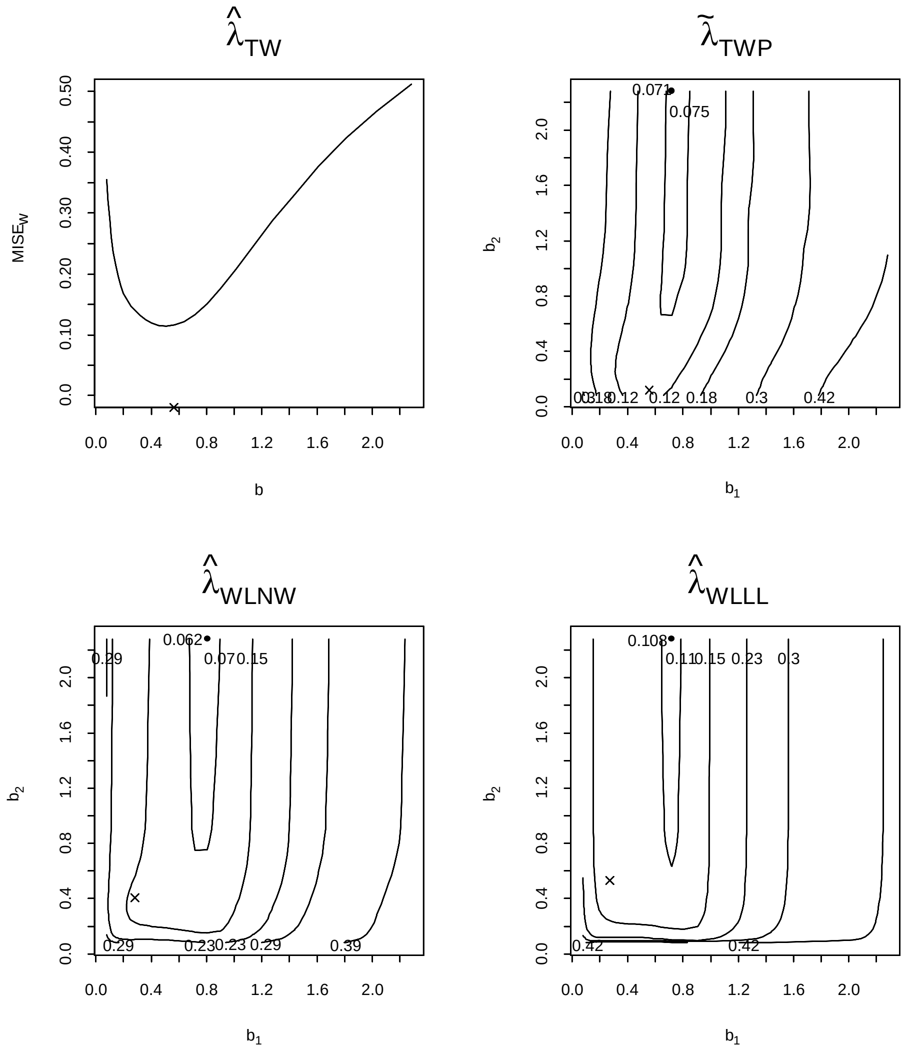

Figure 3. Approximation of $MISE_w(\widehat{\lambda}; b)$ for model I, $n = 30$ and the estimators $\widehat{\lambda}_{TW}(t)$, $\widetilde{\lambda}_{TWP}(t)$, $\widehat{\lambda}_{WLNW}(t)$ and $\widehat{\lambda}_{WLLL}(t)$. The cross ($\times$) points out the b_{AMISE_w} value.

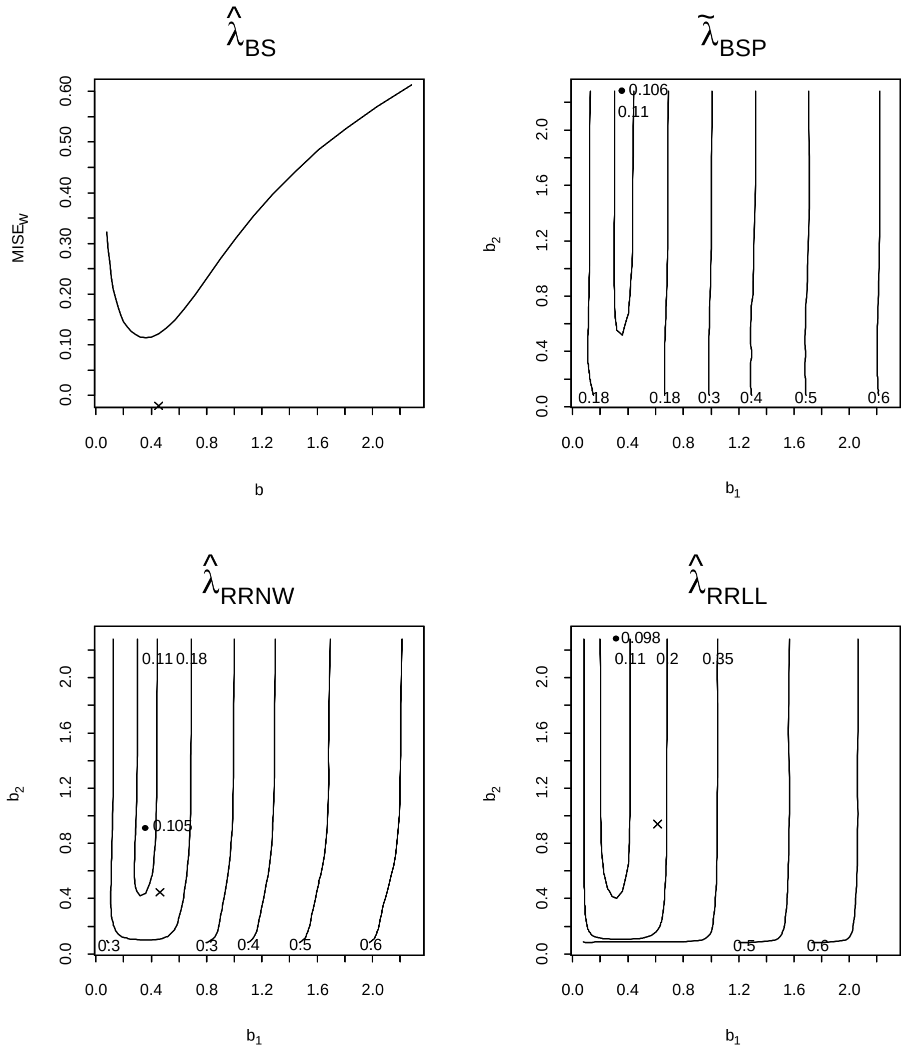

Figure 4. Approximation of $MISE_w(\widehat{\lambda}; b)$ for model I, $n = 30$ and the estimators $\widehat{\lambda}_{BS}(t)$, $\widetilde{\lambda}_{BSP}(t)$, $\widehat{\lambda}_{RRNW}(t)$ and $\widehat{\lambda}_{RRLL}(t)$. The cross $(\times)$ points out the b_{AMISE_w} value.

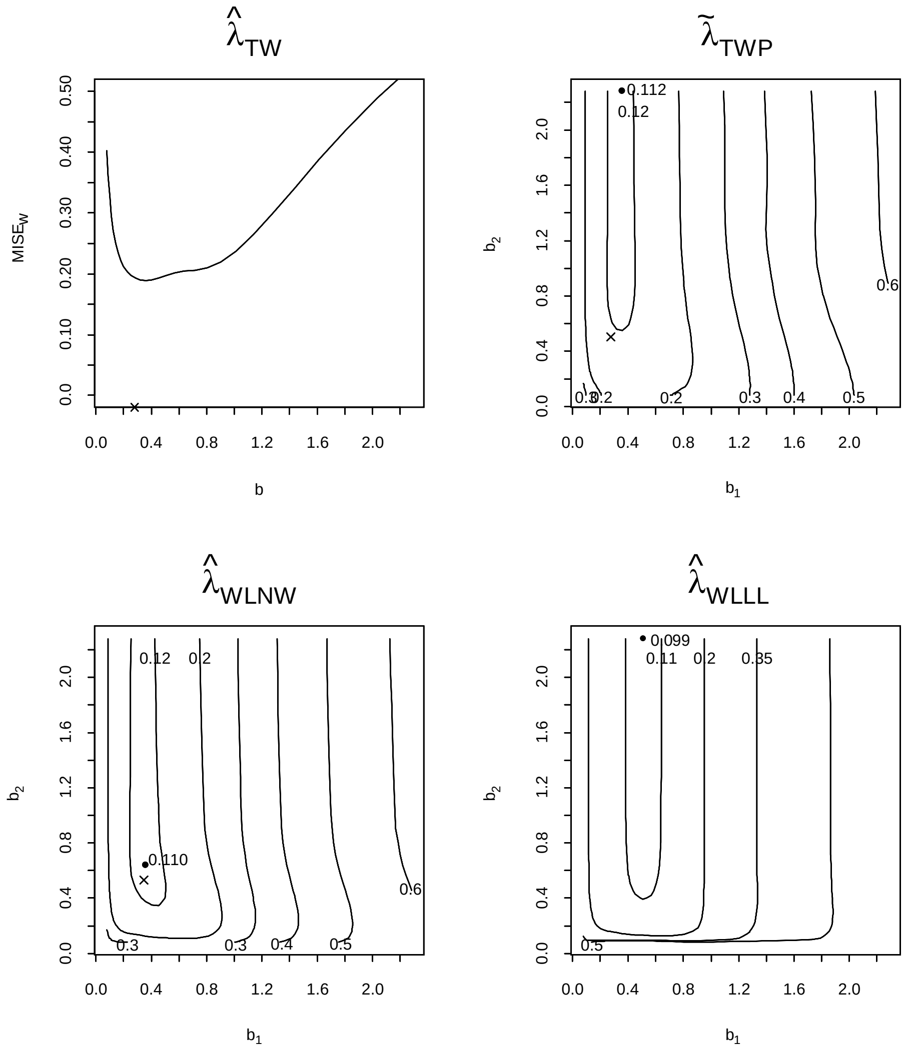

Figure 5. Approximation of $MISE_w(\widehat{\lambda}; b)$ for model II, $n = 30$ and the estimators $\widehat{\lambda}_{TW}(t)$, $\widetilde{\lambda}_{TWP}(t)$, $\widehat{\lambda}_{WLNW}(t)$ and $\widehat{\lambda}_{WLLL}(t)$. The cross $(\times)$ points out the b_{AMISE_w} value (for $\widehat{\lambda}_{WLLL}(t)$ it is outside of the plot; see Table 4).

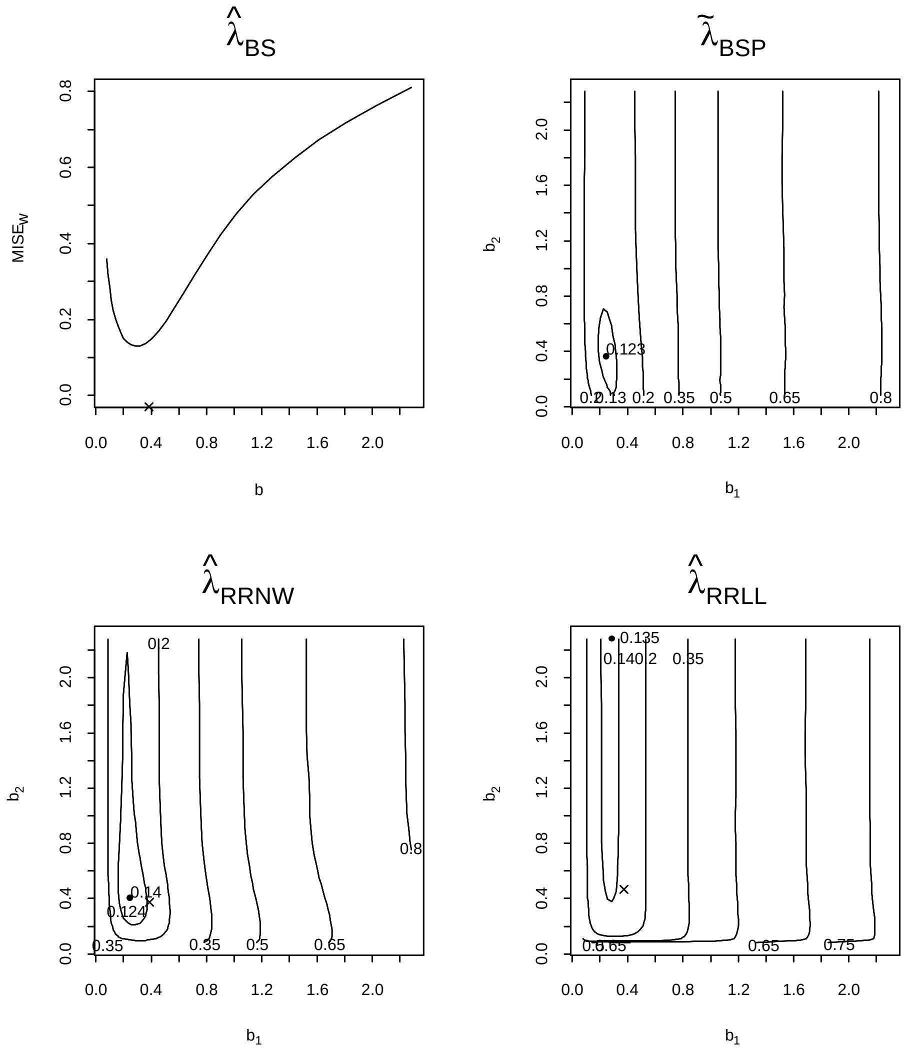

Figure 6. Approximation of $MISE_w(\widehat{\lambda}; b)$ for model II, $n = 30$ and the estimators $\widehat{\lambda}_{BS}(t)$, $\widetilde{\lambda}_{BSP}(t)$, $\widehat{\lambda}_{RRNW}(t)$ and $\widehat{\lambda}_{RRLL}(t)$. The cross ($\times$) points out the b_{AMISE_w} value.

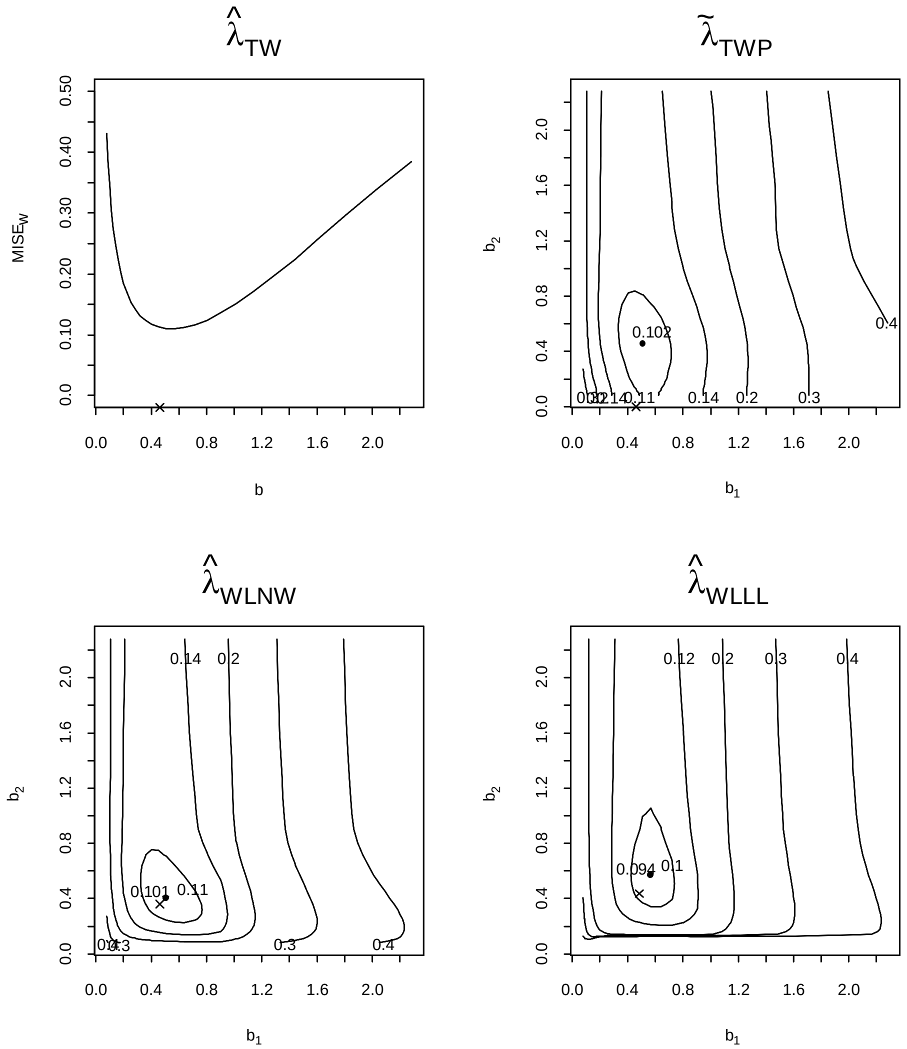

Figure 7. Approximation of $MISE_w(\widehat{\lambda}; b)$ for model III, $n = 30$ and the estimators $\widehat{\lambda}_{TW}(t)$, $\widetilde{\lambda}_{TWP}(t)$, $\widehat{\lambda}_{WLNW}(t)$ and $\widehat{\lambda}_{WLLL}(t)$. The cross ($\times$) points out the b_{AMISE_w} value.

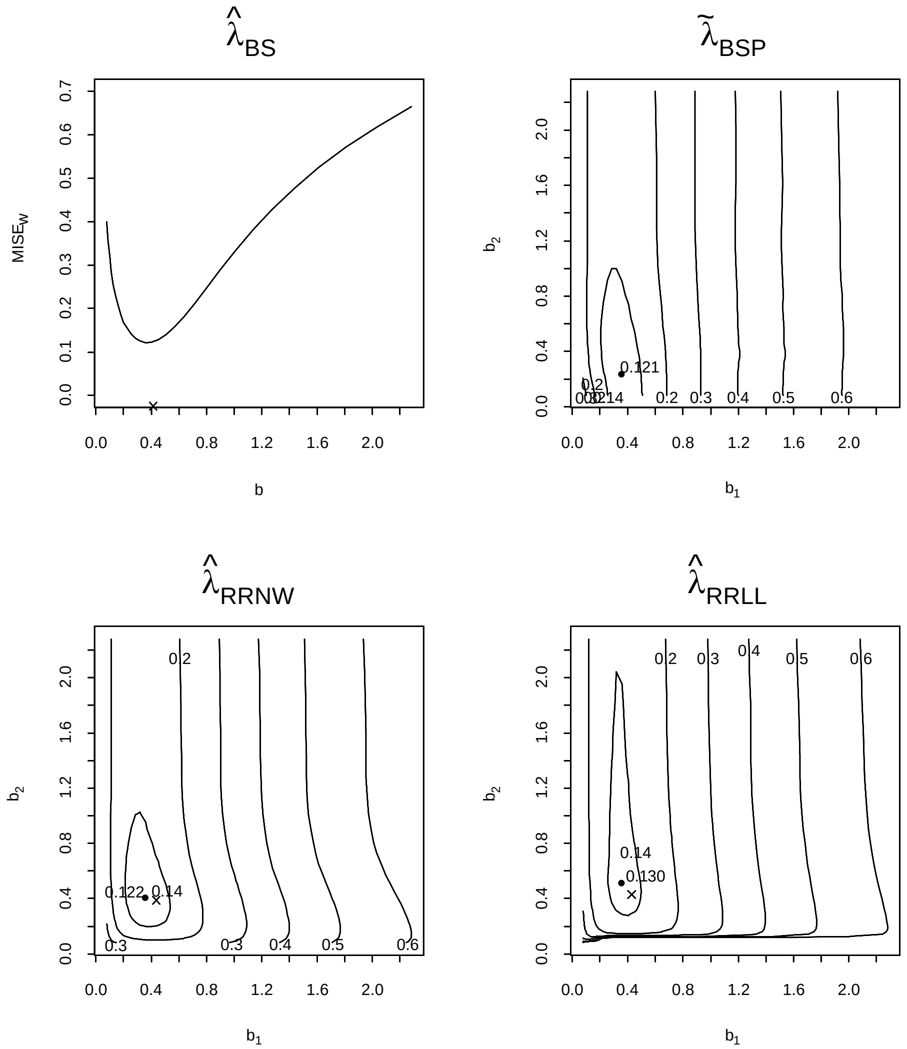

Figure 8. Approximation of $MISE_w(\widehat{\lambda}; b)$ for model III, $n = 30$ and the estimators $\widehat{\lambda}_{BS}(t)$, $\widetilde{\lambda}_{BSP}(t)$, $\widehat{\lambda}_{RRNW}(t)$ and $\widehat{\lambda}_{RRLL}(t)$. The cross ($\times$) points out the b_{AMISE_w} value.

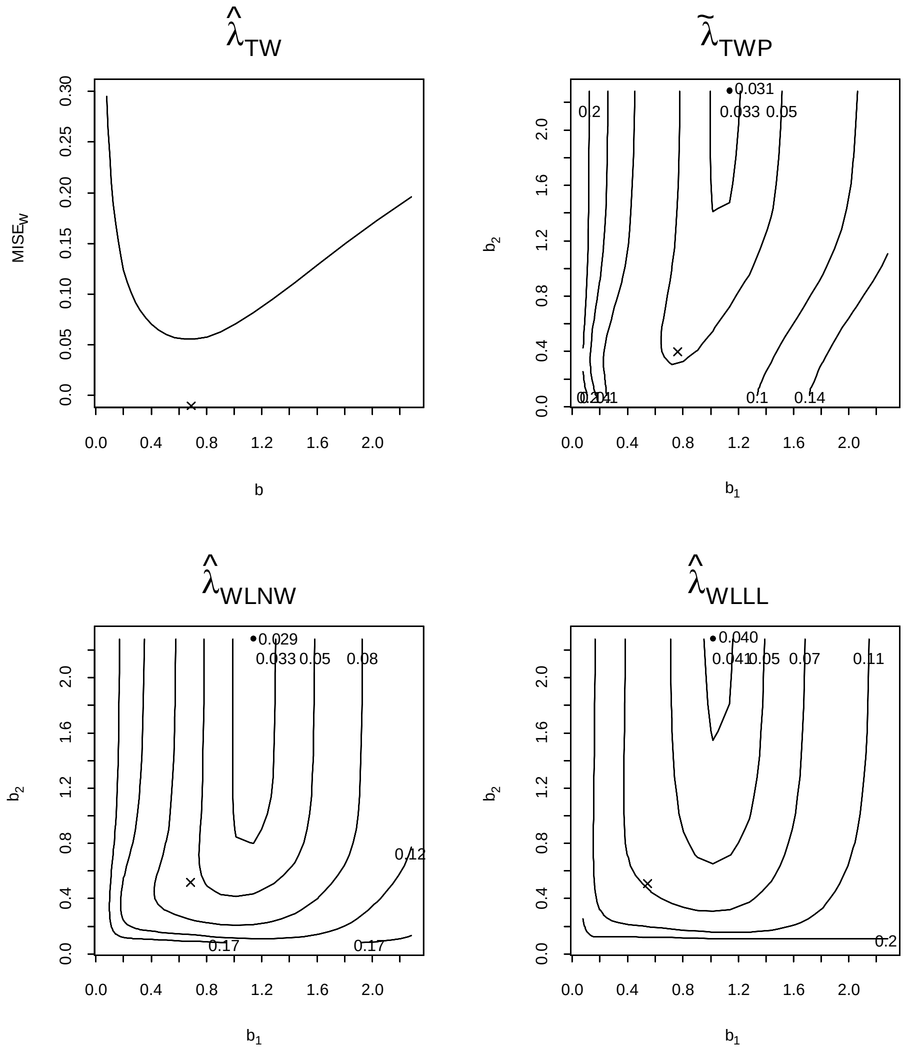

Figure 9. Approximation of $MISE_w(\widehat{\lambda}; b)$ for model IV, $n = 30$ and the estimators $\widehat{\lambda}_{TW}(t)$, $\widetilde{\lambda}_{TWP}(t)$, $\widehat{\lambda}_{WLNW}(t)$ and $\widehat{\lambda}_{WLLL}(t)$. The cross ($\times$) points out the b_{AMISE_w} value.

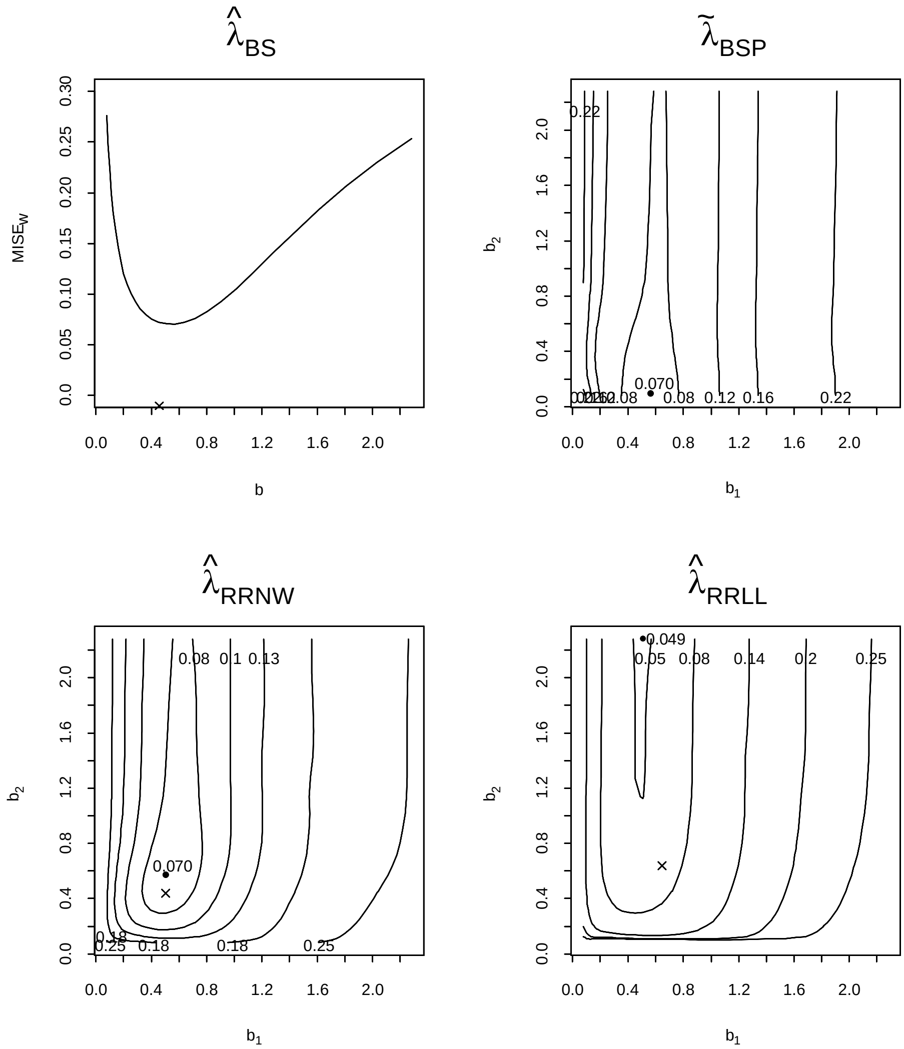

Figure 10. Approximation of $MISE_w(\widehat{\lambda}; b)$ for model IV, $n = 30$ and the estimators $\widehat{\lambda}_{BS}(t)$, $\widetilde{\lambda}_{BSP}(t)$, $\widehat{\lambda}_{RRNW}(t)$ and $\widehat{\lambda}_{RRLL}(t)$. The cross $(\times)$ points out the b_{AMISE_w} value.

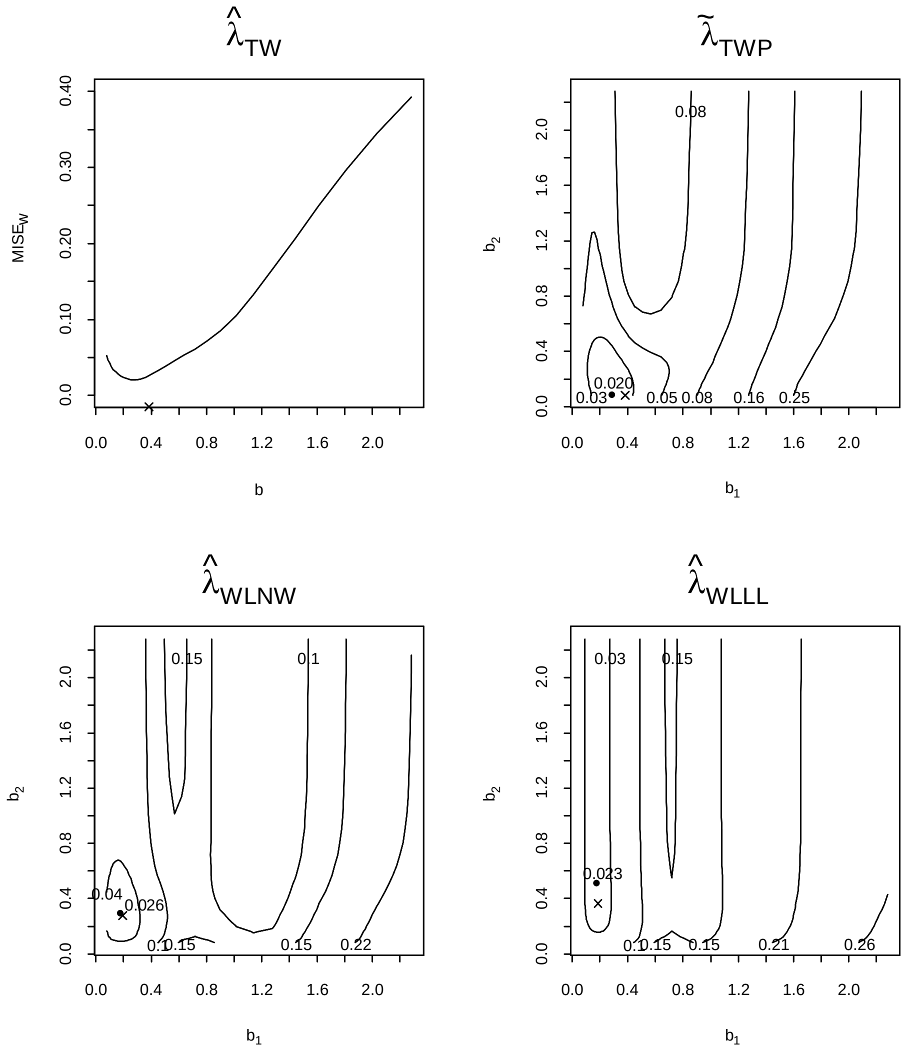

Figure 11. Approximation of $MISE_w(\widehat{\lambda}; b)$ for model I, $n = 200$ and the estimators $\widehat{\lambda}_{TW}(t)$, $\widetilde{\lambda}_{TWP}(t)$, $\widehat{\lambda}_{WLNW}(t)$ and $\widehat{\lambda}_{WLLL}(t)$. The cross ($\times$) points out the b_{AMISE_w} value.

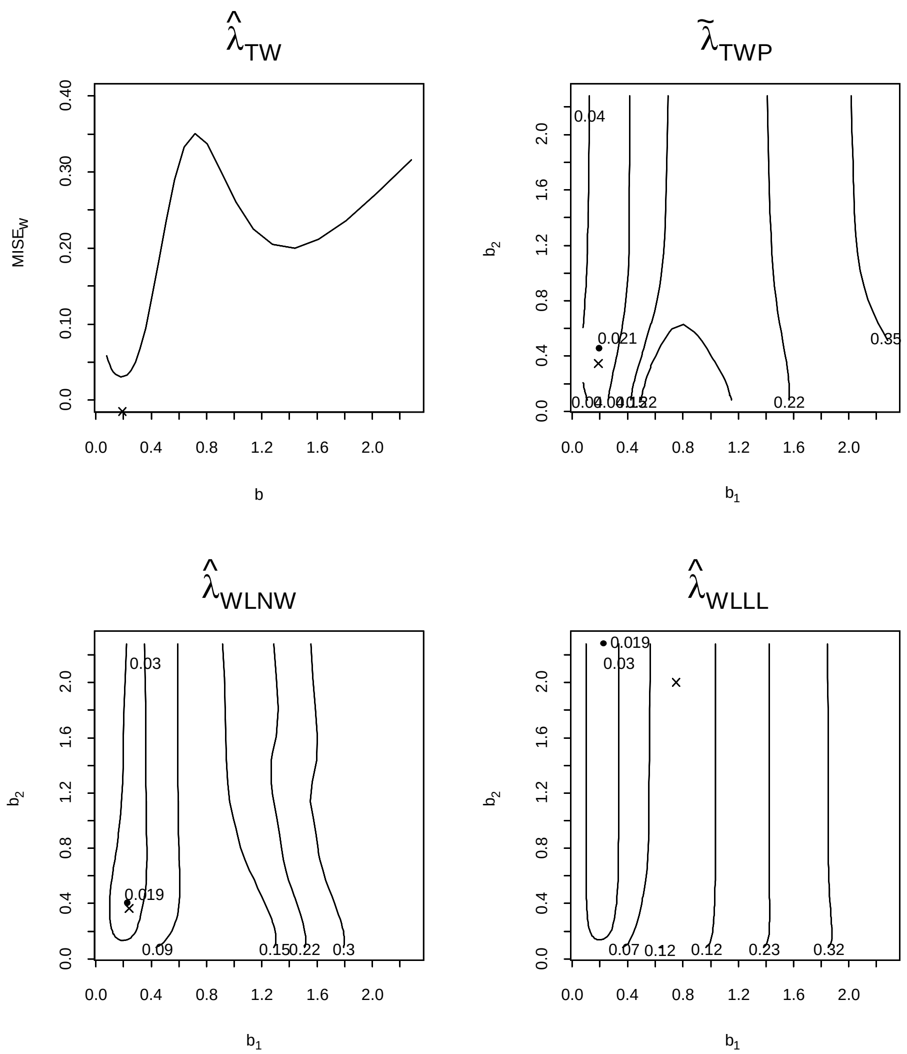

Figure 12. Approximation of $MISE_w(\widehat{\lambda}; b)$ for model II, $n = 200$ and the estimators $\widehat{\lambda}_{TW}(t)$, $\widetilde{\lambda}_{TWP}(t)$, $\widehat{\lambda}_{WLNW}(t)$ and $\widehat{\lambda}_{WLLL}(t)$. The cross ($\times$) points out the b_{AMISE_w} value.

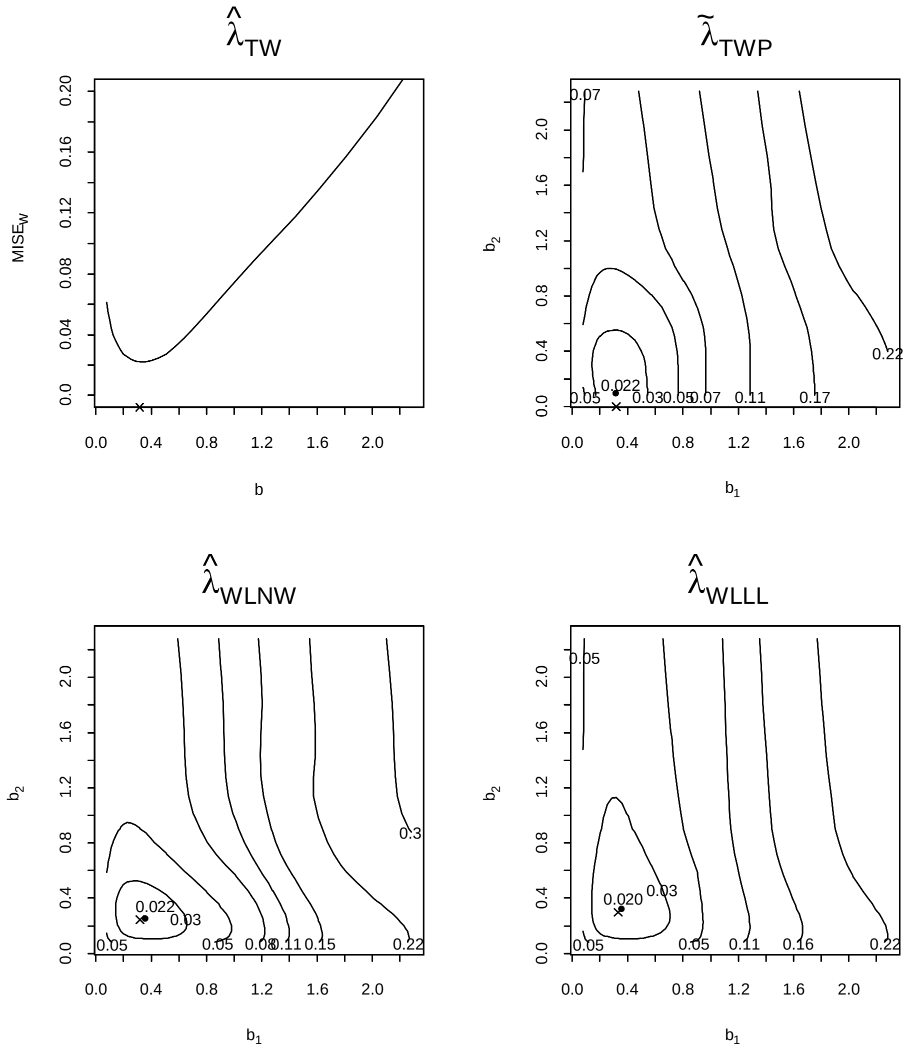

Figure 13. Approximation of $MISE_w(\widehat{\lambda}; b)$ for model III, $n = 200$ and the estimators $\widehat{\lambda}_{TW}(t)$, $\widetilde{\lambda}_{TWP}(t)$, $\widehat{\lambda}_{WLNW}(t)$ and $\widehat{\lambda}_{WLLL}(t)$. The cross $(\times)$ points out the b_{AMISE_w} value.

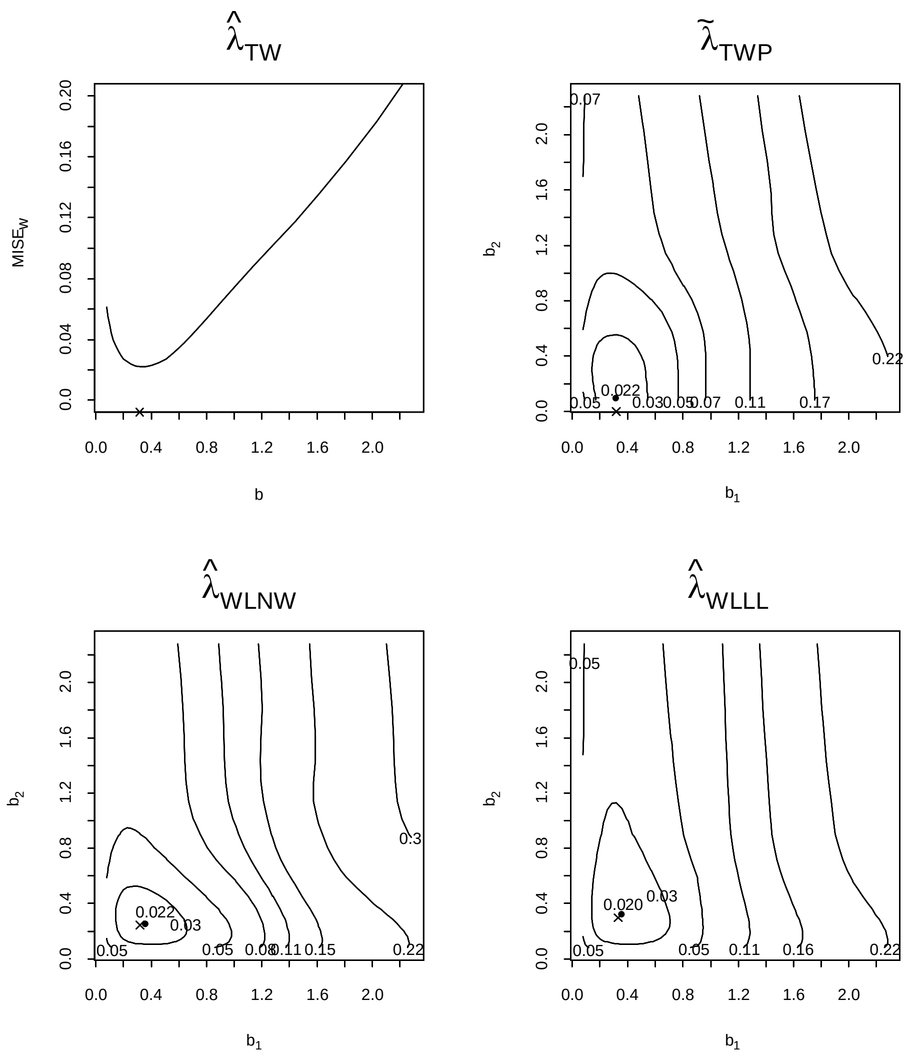

Figure 14. Approximation of $MISE_w(\widehat{\lambda}; b)$ for model IV, $n = 200$ and the estimators $\widehat{\lambda}_{TW}(t)$, $\widetilde{\lambda}_{TWP}(t)$, $\widehat{\lambda}_{WLNW}(t)$ and $\widehat{\lambda}_{WLLL}(t)$. The cross ($\times$) points out the b_{AMISE_w} value.

6 Acknowledgments

Research partly supported by the MCyT Grant BFM2002-00265 (European FEDER support included) and XUGA Grant PGIDIT03PXIC10505PN.

References

[1] Aalen, O. O. (1978). Nonparametric inference for a family of counting processes. *The Annals of Statistics*, **6**, 701-726.

[2] Blum, J. R. and Susarla, V. (1980). Maximal deviation theory of density and failure rate function estimates based on censored data. En *Multivariate Analysis-V,* ed. P. R. Krishnaiah. North-Holland Publishing Company, Amsterdam, 213-222.

[3] Cao, R., López-de-Ullibarri, I., Janssen, P. and Veraverbeke, N. (2003). Presmoothed estimators for censored data. Unpublished manuscript.

[4] Cao, R. and Jácome, M. A. (2004). Presmoothed kernel density estimation for censored data. *Journal of Nonparametric Statistics,* **16**, *289-309.*

[5] Fan, J. and Gijbels, I. (1996). *Local Polynomial Modelling and its Applications.* Chapman & Hall, London.

[6] Mack, Y. P. and Silverman, B. W. (1982). Weak and strong uniform consistency of kernel regression estimates. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, **61**, 405-415.

[7] Murthy, V. K. (1965). Estimation of jumps, reliability and hazard rate. *Annals of Mathematical Statistics*, **36**, 1032-1040.

[8] Nadaraya, E. A. (1964). On estimating regression. *Theory of Probability and its Applications*, **10**, 186-190.

[9] Nelson, W. (1972). Theory and applications of hazard plotting for censored failure data. *Technometrics*, **14**, 945-965.

[10] Rice, J. and Rosenblatt, M. (1976). Estimation of the log survivor function and hazard function. *Sankhya Series A*, **38**, 60-78.

[11] Silverman, B. W. (1978). Weak and strong uniform consistency of the kernel estimate of a density and its derivatives. *The Annals of Statistics*, **6**, 177-184.

[12] Tanner, M. A. and Wong W. H. (1983). The estimation of the hazard function from randomly censored data by the kernel method. *The Annals of Statistics*, **11**, 989-993.

[13] Vaart, A. W. van der (1998). *Asymptotic Statistics.* Cambridge University Press, Cambridge.

[14] Watson, G. S. and Leadbetter, M. R. (1964) Hazard analysis II. *Sankhya Series A*, **26**, 110-116.

[15] Watson, G. S. (1964). Smooth Regression Analysis. *Sankhya Series A*, **26**, 359-372.