RIGOROUS JUSTIFICATION OF THE REYNOLDS EQUATIONS FOR GAS LUBRICATION

Eduard Marušić-Paloka

Department of Mathematics, University of Zagreb, bijenička 30, Zagreb, Croatia

There are several differences in qualitative behavior of gases compared to liquids, mainly: compressibility and small viscosities. In general, gas bearing operates with higher velocity and smaller clearance ratio than the liquid one. Although the gas viscosity is small (typically of order 10^{-5}) we rarely have to consider the turbulence, due to the small typical length (gap thickness smaller than $1\mu m$ is not uncommon). The most common examples where the gas lubrication appears are computer hard discs, magnetic tapes and some high precision measuring devices. Our goal is to derive the isothermal Reynolds model for gas lubrication using the rigorous asymptotic analysis. To derive the model we start from equations of motion governing the compressible, stationary flow through a thin domain with thickness ε described by the shape function h:

$$\Omega_\varepsilon = \{x = (x', x_n) \in \mathbf{R}^n \ ; \ x' = (x_1, \cdots, x_{n-1}) \in \mathcal{O} \ , \ 0 < x_n < \varepsilon\, h(x') \ \} \ ,$$

where $\mathcal{O} \subset \mathbf{R}^{n-1}$ is a bounded smooth domain and $h : \overline{\mathcal{O}} \to \mathbf{R}$ is a smooth, positive function. Let Γ_ε be the lateral boundary. We shall also need rescaled domain $\Omega = \{(x', y_n) \in \mathbf{R}^n \ ; \ x' = (x_1, \cdots, x_{n-1}) \in \mathcal{O} \ , \ 0 < y_n < h(x') \ \}$. The unknowns in the model are u^ε - the velocity , p^ε - the pressure , ρ^ε - the density . We suppose that the fluid is viscous and compressible and that the flow is stationary and isothermal. As usual, we use the ideal gas law, which in the isothermal case reduces to the simple pressure-density relation $p^\varepsilon = a_\varepsilon\, \rho^\varepsilon$, where $a_\varepsilon = T_\varepsilon\ R > 0$ is a constant. We also neglect the inertial term, i.e. we assume that the Reynolds number $Re_\varepsilon \ll 1$. The total quantity of the fluid in the domain is prescribed and equal to $M_\varepsilon > 0$, i.e. $M_\varepsilon = \int_{\Omega_\varepsilon} \rho^\varepsilon(x)\, dx$. The velocity of the relative motion of two surfaces is denoted by $\mathbf{V}$. Our system then reads

$$-\mu\Delta u^\varepsilon - (\lambda+\mu)\nabla(\mathrm{div}\, u^\varepsilon\,) + \nabla p^\varepsilon = 0 \ , \ \mathrm{div}(\rho^\varepsilon\, u^\varepsilon) = 0 \ \text{ in } \ \Omega_\varepsilon \qquad (1)$$

$$u^\varepsilon = 0 \ \text{ for } \ x_n = \varepsilon\, h(x') \ , \ u^\varepsilon = \mathbf{V} \ \text{ for } \ x_n = 0 \ , \ u^\varepsilon = 0 \ \text{ on } \ \Gamma_\varepsilon \ . \qquad (2)$$

For our asymptotic analysis we need additional hypothesis $\lim_{\varepsilon\to 0} \varepsilon^2 a_\varepsilon \frac{M_\varepsilon}{|\Omega_\varepsilon|} = M$. Using the a priori estimates and the pressure decomposition, under some technical hypothesis, we prove:

Theorem 1 *Let $(u^\varepsilon, p^\varepsilon)$ be the solution of the equations of motion (1)-(2) and let U^ε , P^ε be defined from it by change of variables $U^\varepsilon(x_1, x_2, y) = u^\varepsilon(x_1, x_2, \varepsilon\, y)$, $P^\varepsilon(x_1, x_2, y) = p^\varepsilon(x_1, x_2, \varepsilon\, y)$. Then*

$$U^\varepsilon \to U \quad , \quad \frac{\partial U^\varepsilon}{\partial y} \rightharpoonup \frac{\partial U}{\partial y} \quad \textit{and} \quad \varepsilon^2 P^\varepsilon \to P \quad \textit{weakly in} \quad L^2(\Omega) \qquad (3)$$

where (U, P) is the unique solution of the compressible Reynolds equations

$$U = -\frac{1}{2\mu}\, y_n\, (h - y_n)\, \nabla_{x'} P + (1 - \frac{y_n}{h})\, \mathbf{V} \ , \ P \geq 0 \ , \ \int_\Omega P = M\, |\Omega| \qquad (4)$$

$$div_{x'}\, (P \int_0^h U) = 0 \quad in \ \ \mathcal{O} \ , \ \ P \int_0^h U \ \cdot \mathbf{n} = 0 \ \ on \ \ \partial\mathcal{O} \ . \qquad (5)$$