Mechanics, Control and Lie algebroids

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Abstract

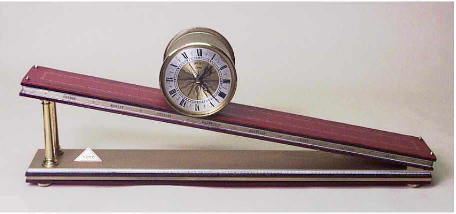
The most relevant ideas and results about mechanical systems defined on Lie algebroids are presented. This was a program originally proposed by Alan Weinstein (1996) and developed by many authors.

Several reasons for formulating Mechanics on Lie algebroids

- The inclusive nature of the Lie algebroid framework: under the same formalism one can consider standard mechanical systems, systems on Lie algebras, systems on semidirect products, systems with symmetries.
- The reduction of a mechanical system on a Lie algebroid is a mechanical system on a Lie algebroid, and this reduction procedure is done via morphisms of Lie algebroids.
- Well adapted: the geometry of the underlying Lie algebroid determines some dynamical properties as well as the geometric structures associated to it (e.g. Symplectic structure). Provides a natural way to use quasi-velocities in Mechanics.

Introduction

Lagrangian systems



Given a Lagrangian $L \in C^\infty(TQ)$, the Euler-Lagrange equations define a dynamical system

$$\dot{q}^i = v^i$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial v^i}\right) = \frac{\partial L}{\partial q^i}.$$

Variational Calculus.

Symplectic formalism

Geodesics of left invariant metrics

For instance, if k is a Riemannian metric on a manifold Q , the Euler-Lagrange equations for $L(v) = \frac{1}{2}k(v, v)$ are the equation for the geodesics $\nabla_v v = 0$

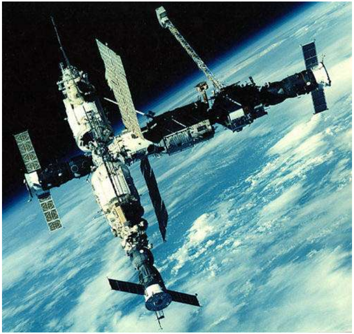
If $Q = G$ is a Lie group and k is left invariant, then L defines a function l on the Lie algebra $\mathfrak{g}$, by restriction $l(\xi) = \frac{1}{2}k(\xi, \xi)$. The equations for geodesics are the Euler-Poincaré equations (Poincaré, 1901)

$$\frac{d}{dt}k(\xi, \ ) = \mathrm{ad}^*_\xi \, k(\xi, \ ).$$

To obtain them: go back to the group and trivialize $TG = G \times \mathfrak{g}$, express the connection in terms of lefts invariant vector fields, ...

How to get the dynamic equations directly from the reduced Lagrangian? Variational Calculus? Symplectic description?

Rigid body

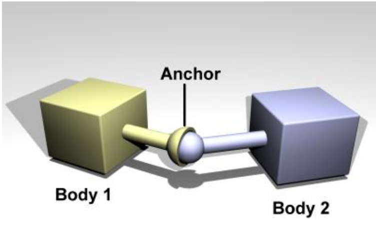


A concrete example is the rigid body: $G = SO(3)$.
Euler equations for rigid body motion are of this type.

$$I\dot{\omega} + \omega \times I\omega = 0.$$

How to get them from the Lagrangian $L = \frac{1}{2}\omega \cdot I\omega$?

Systems with symmetry



If we have a group G acting on Q and L is invariant, then the Lagrangian and the dynamics reduces to a dynamical system defined on TQ/G.
How to get the reduced dynamics from the reduced Lagrangian?

Semidirect products

In systems with parameters, when considering a moving frame, some parameters are promoted to dynamical variables. The Lagrangian is no longer a function on a tangent bundle.
How to get the dynamics form this new Lagrangian?
For instance, for the heavy top, the Lagrangian in body coordinates is

$$L = \frac{1}{2}\omega \cdot I\omega - mgl\gamma \cdot \mathrm{e}$$

anf the dynamical equations are

$$\begin{gathered}\dot{\gamma} + \omega \times \gamma = 0 \\ I\dot{\omega} + \omega \times I\omega = mgl\gamma \times \mathrm{e}.\end{gathered}$$

Holonomic Constraints

$$x^2 + y^2 - z^2 = \alpha \in \mathbb{R}$$

Consider a foliation and a Lagrangian system. Restrict the system to any of the leaves. For different α we have different topologies, and we have to perform a case by case analysis.
Is there a common Lagrangian description, independent of the value of α?
If both the Lagrangian and the foliation admit a symmetry group ... constraint and then reduce? reduce and then constraint? Can we do it directly?

General form

In all cases the dynamical equations are of the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial y^\alpha}\right) + \frac{\partial L}{\partial y^\gamma} C^\gamma_{\alpha\beta} y^\beta = \rho^i_\alpha \frac{\partial L}{\partial x^i}$$
$$\dot{x}^i = \rho^i_\alpha y^\alpha.$$

Is there a general geometric formalism that includes all this examples? What properties remains? Hamiltonian Formalism?

Weinstein (1996) propose to use Lie algebroid geometry.

P. Libermann, E. M., P. Popescu, M. Popescu, JC. Marrero, M. de León, W. Sarlet, T. Metdag, J. Cortés, JF. Cariñena, P. Santos, J. Nunes da Costa, J. Grabowski, K. Grabowska, P. Urbanski, E. Padrón, D. Martín de Diego, D. Iglesias, ...

Lie Algebroids

Lie Algebroids

A Lie algebroid structure on a vector bundle $\tau\colon E \to M$ is given by

- a Lie algebra structure $(\mathrm{Sec}(E), [\,,\,])$ on the set of sections of E,

$$\sigma, \eta \in \mathrm{Sec}(E) \qquad \Rightarrow \qquad [\sigma, \eta] \in \mathrm{Sec}(E)$$

- a morphism of vector bundles $\rho\colon E \to TM$ over the identity, such that

$$[\sigma, f\eta] = f[\sigma, \eta] + (\rho(\sigma) f)\,\eta,$$

where $\rho(\sigma)(m) = \rho(\sigma(m))$. The map ρ is said to be **the anchor**.

As a consequence of the Jacobi identity

$$\rho([\sigma, \eta]) = [\rho(\sigma), \rho(\eta)]$$

Examples

Tangent bundle.

$E = TM$,

$\rho = \mathrm{id}$,

$[\,,\,] =$ bracket of vector fields.

Tangent bundle and parameters.

$E = TM \times \Lambda \to M \times \Lambda$,

$\rho\colon TM \times \Lambda \to TM \times T\Lambda, \qquad \rho\colon (v, \lambda) \mapsto (v, 0_\lambda)$,

$[\,,\,] =$ bracket of vector fields (with parameters).

Integrable subbundle.

$E \subset TM$, integrable distribution

$\rho = i$, canonical inclusion

$[\,,\,] =$ restriction of the bracket to vector fields in E.

Lie algebra.

$E = \mathfrak{g} \to M = \{e\}$, Lie algebra (fiber bundle over a point)

$\rho = 0$, trivial map (since $TM = \{0_e\}$)

$[\,,\,] =$ the bracket in the Lie algebra.

Atiyah algebroid.

Let $\pi\colon Q \to M$ a principal G-bundle.

$E = TQ/G \to M = Q/G$, (Sections are equivariant vector fields)

$\rho([v]) = T\pi(v)$ induced projection map

$[\,,\,] =$ bracket of equivariant vector fields (is equivariant).

Transformation Lie algebroid.

Let $\Phi\colon \mathfrak{g} \to \mathfrak{X}(M)$ be an action of a Lie algebra $\mathfrak{g}$ on M.

$E = M \times \mathfrak{g} \to M$,

$\rho(m,\xi) = \Phi(\xi)(m)$ value of the fundamental vector field

$[\,,\,] =$ induced by the bracket on $\mathfrak{g}$.

Mechanics on Lie algebroids

Lie algebroid $E \to M$.
$L \in C^\infty(E)$ or $H \in C^\infty(E^*)$

- $E = TM \to M$ Standard classical Mechanics
- $E = \mathcal{D} \subset TM \to M$ (integrable) System with holonomic constraints
- $E = TQ/G \to M = Q/G$ System with symmetry (eg. Classical particle on a Yang-Mills field)
- $E = \mathfrak{g} \to \{e\}$ System on a Lie algebra (eg. Rigid body)
- $E = M \times \mathfrak{g} \to M$ System on a semidirect product (eg. heavy top)

Structure functions

A local coordinate system (x^i) in the base manifold M and a local basis of sections (e_α) of E, determine a local coordinate system (x^i, y^α) on E.

The anchor and the bracket are locally determined by the local functions $\rho^i_\alpha(x)$ and $C^\alpha_{\beta\gamma}(x)$ on M given by

$$\rho(e_\alpha) = \rho^i_\alpha \frac{\partial}{\partial x^i}$$

$$[e_\alpha, e_\beta] = C^\gamma_{\alpha\beta}\ e_\gamma.$$

The function ρ^i_α and $C^\alpha_{\beta\gamma}$ satisfy some relations due to the compatibility condition and the Jacobi identity which are called the structure equations:

$$\rho^j_\alpha \frac{\partial \rho^i_\beta}{\partial x^j} - \rho^j_\beta \frac{\partial \rho^i_\alpha}{\partial x^j} = \rho^i_\gamma C^\gamma_{\alpha\beta}$$

$$\sum_{\mathrm{cyclic}(\alpha,\beta,\gamma)} \left[\rho^i_\alpha \frac{\partial C^\nu_{\beta\gamma}}{\partial x^i} + C^\mu_{\beta\gamma} C^\nu_{\alpha\mu} \right] = 0.$$

Lagrange equations

Given a function $L \in C^\infty(E)$, we define a dynamical system on E by means of a system of differential equations, which in local coordinates reads

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{\partial L}{\partial y^\alpha}\right) + \frac{\partial L}{\partial y^\gamma} C^\gamma_{\alpha\beta} y^\beta = \rho^i_\alpha \frac{\partial L}{\partial x^i} \\
&\dot{x}^i = \rho^i_\alpha y^\alpha .
\end{aligned}$$

(Weinstein 1996)

The equation $\dot{x}^i = \rho^i_\alpha y^\alpha$ is the local expression of the admissibility condition: A curve $a\colon \mathbb{R} \to E$ is said to be **admissible** if

$$\rho \circ a = \frac{d}{dt}(\tau \circ a).$$

Admissible curves are also called **E-paths**.

Exterior differential

On 0-forms

$$df(\sigma) = \rho(\sigma)f$$

On p-forms ($p > 0$)

$$\begin{aligned} d\omega(\sigma_1, \ldots, \sigma_{p+1}) = \\ &= \sum_{i=1}^{p+1} (-1)^{i+1} \rho(\sigma_i)\omega(\sigma_1, \ldots, \widehat{\sigma_i}, \ldots, \sigma_{p+1}) \\ &- \sum_{i<j} (-1)^{i+j} \omega([\sigma_i, \sigma_j], \sigma_1, \ldots, \widehat{\sigma_i}, \ldots, \widehat{\sigma_j}, \ldots, \sigma_{p+1}). \end{aligned}$$

d is a cohomology operator $d^2 = 0$.

Exterior differential-local

Locally determined by

$$dx^i = \rho^i_\alpha e^\alpha$$

and

$$de^\alpha = -\frac{1}{2} C^\alpha_{\beta\gamma} e^\beta \wedge e^\gamma.$$

The structure equations are

$$d^2 x^i = 0 \quad \text{and} \quad d^2 e^\alpha = 0.$$

Admissible maps and Morphisms

A bundle map Φ between E and E' is said to be admissible if

$$\Phi^\star df = d\Phi^\star f.$$

A bundle map Φ between E and E' is said to be a morphism of Lie algebroids if

$$\Phi^\star d\theta = d\Phi^\star \theta.$$

Obviously every morphism is an admissible map.

Admissible maps transform E-paths into E'-paths.

Variational description

Variational description

Consider the action functional

$$\mathcal{S}(a) = \int_{t_0}^{t_1} L(a(t))\, dt$$

defined on curves on E with fixed base endpoints, which are moreover constrained to be E-paths.

But we also have to constraint the variations to be of the form

$$\delta x^i = \rho^i_\alpha \sigma^\alpha \qquad \delta y^\alpha = \dot\sigma^\alpha + C^\alpha_{\beta\gamma} a^\beta \sigma^\gamma$$

for some curve $\sigma(t)$ such that $\tau(a(t)) = \tau(\sigma(t))$.

Variation vector fields are of the form

$$\Xi_a(\sigma) = \rho^i_\alpha \sigma^\alpha \frac{\partial}{\partial x^i} + [\dot\sigma^\alpha + C^\alpha_{\beta\gamma} a^\beta \sigma^\gamma] \frac{\partial}{\partial y^\alpha}.$$

E-Homotopy

(Crainic and Fernandes 2003)

Let $I = [0,1]$ and $J = [t_0, t_1]$, and (s,t) coordinates in $\mathbb{R}^2$.

Definition 1 *Two E-paths a_0 and a_1 are said to be E-homotopic if there exists a morphism of Lie algebroids $\Phi\colon TI \times TJ \to E$ such that*

$$\Phi\left(\left.\frac{\partial}{\partial t}\right|_{(0,t)}\right) = a_0(t) \qquad \Phi\left(\left.\frac{\partial}{\partial s}\right|_{(s,t_0)}\right) = 0$$

$$\Phi\left(\left.\frac{\partial}{\partial t}\right|_{(1,t)}\right) = a_1(t) \qquad \Phi\left(\left.\frac{\partial}{\partial s}\right|_{(s,t_1)}\right) = 0.$$

It follows that the base map is a homotopy (in the usual sense) with fixed endpoints between the base paths.

Homotopy foliation

The set of E-paths

$$\mathcal{A}(J,E) = \left\{ a\colon J \to E \;\middle|\; \rho \circ a = \frac{d}{dt}(\tau \circ a) \right\}$$

is a Banach submanifold of the Banach manifold of C^1-paths whose base path is C^2. Every E-homotopy class is a smooth Banach manifold and the partition into equivalence classes is a smooth foliation. The distribution tangent to that foliation is given by $a \in \mathcal{A}(J,E) \mapsto F_a$ where

$$F_a = \{\, \Xi_a(\sigma) \in T_a\mathcal{A}(J,E) \mid \sigma(t_0) = 0 \quad \text{and} \quad \sigma(t_1) = 0 \,\}.$$

and the codimension of F is equal to $\dim(E)$. The E-homotopy equivalence relation is regular if and only if the Lie algebroid is integrable (i.e. it is the Lie algebroid of a Lie groupoid).

Variational description

The E-**path space** with the appropriate differential structure is

$$\boxed{\mathcal{P}(J,E) = \mathcal{A}(J,E)_F}.$$

Fix $m_0, m_1 \in M$ and consider the set of E-paths with such base endpoints

$$\mathcal{P}(J,E)_{m_0}^{m_1} = \{\, a \in \mathcal{P}(J,E) \mid \tau(a(t_0)) = m_0 \quad \text{and} \quad \tau(a(t_1)) = m_1 \,\}$$

It is a Banach submanifold of $\mathcal{P}(J,E)$.

Theorem 1 *Let $L \in C^\infty(E)$ be a Lagrangian function on the Lie algebroid E and fix two points $m_0, m_1 \in M$. Consider the action functional $S\colon \mathcal{P}(J,E) \to \mathbb{R}$ given by $S(a) = \int_{t_0}^{t_1} L(a(t))dt$. The critical points of S on the Banach manifold $\mathcal{P}(J,E)_{m_0}^{m_1}$ are precisely those elements of that space which satisfy Lagrange's equations.*

Morphisms and reduction

Given a morphism of Lie algebroids $\Phi\colon E \to E'$ the induced map $\hat{\Phi}\colon \mathcal{P}(J,E) \to \mathcal{P}(J,E')$ given by $\hat{\Phi}(a) = \Phi \circ a$ is smooth and $T\hat{\Phi}(\Xi_a(\sigma)) = \Xi_{\Phi\circ a}(\Phi \circ \sigma)$.

- If Φ is fiberwise surjective then $\hat{\Phi}$ is a submersion.
- If Φ is fiberwise injective then $\hat{\Phi}$ is a immersion.

Consider two Lagrangians $L \in C^\infty(E)$, $L' \in C^\infty(E')$ and $\Phi\colon E \to E'$ a morphism of Lie algebroids such that $L' \circ \Phi = L$.

Then, the action functionals S on $\mathcal{P}(J,E)$ and S' on $\mathcal{P}(J,E')$ are related by $\hat{\Phi}$, that is

$$S' \circ \hat{\Phi} = S.$$

Theorem 2 (Reduction) *Let $\Phi\colon E \to E'$ be a fiberwise surjective morphism of Lie algebroids. Consider a Lagrangian L on E and a Lagrangian L' on E' such that $L = L' \circ \Phi$. If a is a solution of Lagrange's equations for L then $a' = \Phi \circ a$ is a solution of Lagrange's equations for L'.*

Proof. From $S' \circ \hat{\Phi} = S$ we get

$$\langle\, dS(a)\,, v\,\rangle = \langle\, dS'(\hat{\Phi}(a))\,, T_a\hat{\Phi}(v)\,\rangle = \langle\, dS'(a')\,, T_a\hat{\Phi}(v)\,\rangle.$$

Since $T_a\Phi(v)$ surjective, if $dS(a) = 0$ then $dS'(a') = 0$. $\square$

Theorem 3 (Reconstruction) *Let $\Phi\colon E \to E'$ be a morphism of Lie algebroids. Consider a Lagrangian L on E and a Lagrangian L' on E' such that $L = L' \circ \Phi$. If a is an E-path and $a' = \Phi \circ a$ is a solution of Lagrange's equations for L' then a itself is a solution of Lagrange's equations for L.*

Proof. We have

$$\langle\, dS(a)\,, v\,\rangle = \langle\, dS'(a')\,, T_a\hat{\Phi}(v)\,\rangle.$$

If $dS'(a') = 0$ then $dS(a) = 0$. $\square$

Theorem 4 (Reduction by stages) *Let $\Phi_1 \colon E \to E'$ and $\Phi_2 \colon E' \to E''$ be fiberwise surjective morphisms of Lie algebroids. Let L, L' and L'' be Lagrangian functions on E, E' and E'', respectively, such that $L' \circ \Phi_1 = L$ and $L'' \circ \Phi_2 = L'$. Then the result of reducing first by Φ_1 and later by Φ_2 coincides with the reduction by $\Phi = \Phi_2 \circ \Phi_1$.*

Examples.

Lie groups.

Consider a Lie group G and its Lie algebra $\mathfrak{g}$. The map $\Phi\colon TG \to \mathfrak{g}$ given by $\Phi(g,\dot{g}) = g^{-1}\dot{g}$ is a fiberwise bijective morphism of Lie algebroids.

For an invariant Lagrangian $L(g,\dot{g}) = L'(g^{-1}\dot{g})$, every solution $(g(t),\dot{g}(t))$ for L projects to a solution $g^{-1}\dot{g}$ for L'.

Conversely, if $\xi(t) = g(t)^{-1}\dot{g}(t)$ is a solution for L', then $(g(t),\dot{g}(t))$ is a solution for L.

Thus, the Euler-Lagrange equations on the group reduce to the Euler-Poincaré equations on the Lie algebra.

Lie groupoids.

Consider a Lie groupoid $\boldsymbol{G}$ over M with source $\boldsymbol{s}$ and target $\boldsymbol{t}$, and with Lie algebroid E. Denote by $T^{\boldsymbol{s}}\boldsymbol{G} \to \boldsymbol{G}$ the kernel of $T\boldsymbol{s}$ with the structure of Lie algebroid as integrable subbundle of $T\boldsymbol{G}$. Then the map $\Phi\colon T^{\boldsymbol{s}}\boldsymbol{G} \to E$ given by left translation to the identity, $\Phi(v_g) = TL_{g^{-1}}(v_g)$ is a morphism of Lie algebroids, which is moreover fiberwise surjective. As a consequence, if L is a Lagrangian function on E and $\boldsymbol{L}$ is the associated left invariant Lagrangian on $T^{\boldsymbol{s}}\boldsymbol{G}$, then the solutions of Lagrange's equations for $\boldsymbol{L}$ project by Φ to solutions of the Lagrange's equations.

Group actions.

G Lie group acting free and properly on a manifold Q, so that the quotient map $\pi\colon Q \to M$ is a principal bundle.

$E = TQ$ the standard Lie algebroid
$E' = TQ/G \to M$ Atiyah algebroid
$\Phi\colon E \to E'$, $\Phi(v) = [v]$ the quotient map

Φ is a fiberwise bijective Lie algebroid morphism.

Every G-invariant Lagrangian on TQ defines uniquely a Lagrangian L' on E' such that $L' \circ \Phi = L$.

Thus, the Euler-Lagrange equations on the principal bundle reduce to the Lagrange-Poincaré equations on the Atiyah algebroid.

Semidirect products.

Let G be a Lie group acting from the right on a manifold M.

$E = TG \times M \to G \times M$ where M is a parameter manifold
$E' = \mathfrak{g} \times M \to M$ transformation Lie algebroid
$\Phi(v_g, m) = (g^{-1}v_g, mg)$ is a fiberwise surjective morphism of Lie algebroids.

Consider a Lagrangian L on TG depending on the elements of M as parameters which is invariant by the joint action $L(g^{-1}\dot{g}, mg) = L(\dot{g}, m)$, and the reduced Lagrangian L' on E' by $L'(\xi, m) = L(\xi_G(e), m)$, so that $L' \circ \Phi = L$.

Euler-Lagrange equations on the group, with parameters, reduce to Euler-Poincaré equations with advected parameters.

Abelian Routh reduction.

A Lagrangian $L \in C^\infty(TQ)$ with cyclic coordinates θ and denote by q the other coordinates The Lagrangian L on TQ projects to a Lagrangian L' on TQ/G with the same coordinate expression. The solutions for L obviously project to solutions for L'.

The momentum $\mu = \frac{\partial L}{\partial \dot\theta}(q, \dot q, \dot\theta)$ is conserved and we can find $\dot\theta = \Theta(q, \dot q, \mu)$. The Routhian $R(q, \dot q, \mu) = L(q, \dot q, \Theta(q, \dot q, \mu) - \mu\dot\theta$ when restricted to a level set of the momentum $\mu = c$ defines a function L'' on $T(Q/G)$ which is just $L''(q, \dot q) = R(q, \dot q, c)$.

Thus $L''(q, \dot q) = L(q, \dot q, \Theta(q, \dot q, c)) - \frac{d}{dt}(c\theta)$, i.e. L and L'' differ on a total derivative. Lagrange equations reduce to $T(Q/G)$.

Symplectic formalism

Prolongation

Given a Lie algebroid $\tau\colon E \to M$ and a submersion $\mu\colon P \to M$ we can construct the E-tangent to P (the prolongation of P with respect to E). It is the vector bundle $\tau_P^E\colon \mathcal{T}^E P \to P$ where the fibre over $p \in P$ is

$$\mathcal{T}_p^E P = \{\, (b, v) \in E_m \times T_p P \mid T\mu(v) = \rho(b) \,\}$$

where $m = \mu(p)$.

Redundant notation: (p, b, v) for the element $(b, v) \in \mathcal{T}_p^E P$.

The bundle $\mathcal{T}^E P$ can be endowed with a structure of Lie algebroid. The anchor $\rho^1\colon \mathcal{T}^E P \to TP$ is just the projection onto the third factor $\rho^1(p, b, v) = v$. The bracket is given in terms of projectable sections (σ, X), (η, Y)

$$[(\sigma, X), (\eta, Y)] = ([\sigma, \eta], [X, Y]).$$

Prolongation of maps: If $\Psi\colon P \to P'$ is a bundle map over $\varphi\colon M \to M'$ and $\Phi\colon E \to E'$ is a morphism over the same map φ then we can define a morphism $\mathcal{T}^\Phi\Psi\colon \mathcal{T}^E P \to \mathcal{T}^{E'}P'$ by means of

$$\mathcal{T}^\Phi\Psi(p,b,v) = (\Psi(p), \Phi(b), T_p\Psi(v)).$$

In particular, for $P = E$ we have the E-tangent to E

$$\mathcal{T}^E_a E = \left\{\, (b,v) \in E_m \times T_a E \;\mid\; T\tau(v) = \rho(b) \,\right\}.$$

The structure of Lie algebroid in $\mathcal{T}^E E$ can be defined in terms of the brackets of vertical and complete lifts

$$[\eta^{\mathsf{C}}, \sigma^{\mathsf{C}}] = [\sigma, \eta]^{\mathsf{C}}, \qquad [\eta^{\mathsf{C}}, \sigma^{\mathsf{V}}] = [\sigma, \eta]^{\mathsf{V}} \quad \text{and} \quad [\eta^{\mathsf{V}}, \sigma^{\mathsf{V}}] = 0.$$

Geometric Lagrangian Mechanics

Associated to L there is a section θ_L of $(\mathcal{T}^E E)^*$,

$$\langle\, \theta_L\,, (a, b, V)\,\rangle = \frac{d}{ds} L(a + sb)\Big|_{s=0}.$$

Equivalent conditions:

$$i_\Gamma \omega_L = dE_L$$

with $\omega_L = -d\theta_L$ and $E_L = d_\Delta L - L$ the energy, or

$$d_\Gamma \theta_L = dL$$

with Γ a SODE-section. (Martínez 2001)

Poisson bracket

The dual E^* of a Lie algebroid carries a canonical Poisson structure. In terms of linear and basic functions, the Poisson bracket is defined by

$$\{\hat{\sigma}, \hat{\eta}\} = \widehat{[\sigma, \eta]}$$
$$\{\hat{\sigma}, \tilde{g}\} = \rho(\sigma) g$$
$$\{\tilde{f}, \tilde{g}\} = 0$$

for f, g functions on M and σ, η sections of E.

Basic and linear functions are defined by

$$\begin{aligned} \tilde{f}(\mu) &= f(m) \\ \hat{\sigma}(\mu) &= \langle\, \mu \,, \sigma(m) \,\rangle \end{aligned} \qquad \text{for } \mu \in E^*_m.$$

In coordinates

$$\{x^i, x^j\} = 0 \qquad \{\mu_\alpha, x^j\} = \rho^i_\alpha \qquad \{\mu_\alpha, \mu_\beta\} = C^\gamma_{\alpha\beta}\mu_\gamma.$$

Hamiltonian formalism

Consider the prolongation $\mathcal{T}^E E^*$ of the dual bundle $\pi\colon E^* \to M$:

$$\mathcal{T}^E E^* = \{\, (\mu, a, W) \in E^* \times E \times TE^* \mid \mu = \tau_{E^*}(W) \quad \rho(a) = T\pi(W) \,\}\,.$$

There is a **canonical symplectic structure** $\Omega = -d\Theta$, where the 1-form Θ is defined by

$$\langle\, \Theta_\mu\,, (\mu, a, W)\,\rangle = \langle\, \mu\,, a\,\rangle.$$

In coordinates

$$\Theta = \mu_\alpha \mathcal{X}^\alpha,$$

and

$$\Omega = \mathcal{X}^\alpha \wedge \mathcal{P}_\alpha + \frac{1}{2}\mu_\gamma C^\gamma_{\alpha\beta}\mathcal{X}^\alpha \wedge \mathcal{X}^\beta.$$

The Hamiltonian dynamics is given by the vector field $\rho(\Gamma_H)$ associated to the section Γ_H solution of the symplectic equation

$$i_{\Gamma_H}\Omega = dH.$$

In coordinates, **Hamilton equations** are

$$\frac{dx^i}{dt} = \rho^i_\alpha \frac{\partial H}{\partial \mu_\alpha} \qquad \frac{d\mu_\alpha}{dt} = -\left(\mu_\gamma C^\gamma_{\alpha\beta}\frac{\partial H}{\partial \mu_\beta} + \rho^i_\alpha \frac{\partial H}{\partial x^i}\right).$$

The canonical Poisson bracket on E^* can be re-obtained by means of

$$\Omega(dF, dG) = \{F, G\}$$

for $F, G \in C^\infty(E^*)$.

The equations of motion are Poisson

$$\dot{F} = \{F, H\}.$$

Hamilton-Jacobi theory

Let $H \in C^\infty(E^*)$ a Hamiltonian function and Γ_H the Hamiltonian section.

Theorem: Let α be a closed section of E^* and let $\sigma = \mathcal{F}_H \circ \alpha$. The following conditions are equivalent

- If $m(t)$ is an integral curve of $\rho(\sigma)$ then $\mu(t) = \alpha(m(t))$ is a solution of the Hamilton equations.
- α satisfies the equation $d(H \circ \alpha) = 0$.

We can try $\alpha = dS$, for $S \in C^\infty(M)$ (but notice that closed $\neq$ exact, even locally). In such case if $H \circ dS = 0$ then $\frac{d}{dt}(S \circ m) = L \circ \sigma \circ m$, or in other words

$$S(m(t_1)) - S(m(t_0)) = \int_{t_0}^{t_1} L(\sigma(m(t)))\, dt$$

Optimal control

Optimal Control

- **Data**
  - Control bundle $\pi \colon B \to M$
  - $L \in C^\infty(B)$ cost function
  - σ a section of E along π
- **Locally**
  - $B = M \times U$, $\pi : (x, u) \mapsto x$.
  - $L = L(x, u)$.
  - $\sigma = \sigma^\alpha(x, u) e_\alpha$.
- **Problem:** Given (t_0, m_0) and (t_1, m_1), minimize

$$\int_{t_0}^{t_1} L(x(t), u(t))\, dt$$

among the curves $(x(t), u(t))$ such that $x(t_0) = m_0$ and $x(t_1) = m_1$, the curves $\sigma(x(t), u(t))$ are admissible and in a fixed E-homotopy class.

Integral curves: $\dot{x}(t) = \rho(\sigma(x(t), u(t)))$.

Locally $\dot{x}^i = \rho^i_\alpha(x)\sigma^\alpha(x, u)$.

Pontryagin maximum principle

Pontryagin Hamiltonian: $H(x,\mu,u) = \langle\, \mu\,, \sigma(x,u)\,\rangle - L(x,u)$.

Look for σ_H, a section of $\mathcal{T}^E E^*$ along $\mathrm{pr}_1 \colon E^* \times_M B \to E^*$, satisfying the symplectic equation

$$i_{\sigma_H}\Omega = dH.$$

Critical trajectories: Integral curves of the vector field $\rho(\sigma_H)$.

In local coordinates,

$$\begin{aligned} \dot{x}^i &= \rho^i_\alpha \frac{\partial H}{\partial \mu_\alpha}, \\ \dot{\mu}_\alpha &= -\left[\rho^i_\alpha \frac{\partial H}{\partial x^i} + \mu_\gamma C^\gamma_{\alpha\beta} \frac{\partial H}{\partial \mu_\beta}\right], \\ 0 &= \frac{\partial H}{\partial u^A}. \end{aligned}$$

Reduction

Theorem 5 *Let $\psi\colon B \to B'$ and $\Phi\colon E \to E'$ be fibered maps over the same map $\varphi\colon M \to M'$, and assume that ψ is fiberwise submersive and Φ is a morphism of Lie algebroids which is fiberwise bijective. Let L be an index function on B' and L' be an index function on B' such that $L = L' \circ \psi$ and let σ_H and $\sigma_{H'}$ the corresponding critical sections. Then we have that $\Psi(S_H) \subset S_{H'}$ and*

$$\mathcal{T}^\Phi \Phi^c \circ \sigma_H = \sigma_{H'} \circ \Psi$$

on the subset S_H.

As a consequence, the image under Ψ of any critical trajectory for the index L is a critical trajectory for the index L'.

Here $\Psi = (\phi^c, \psi)$ and $\phi^c = \phi^{-1*}$.

Example: symmetry reduction

As an application of the above result we can consider the case of reduction by a symmetry group G with a free and proper action on the bundle B.

$E = TQ$, $B = B$, $M = Q$

$E' = TQ/G$, $B' = B/G$, $M' = Q/G$

$\psi(b) = [b]$, $\Phi(v) = [v]$, $\varphi(q) = [q]$, (quotient maps)

Index L, $L'([b]) = L(b)$ (so that $L = L' \circ \psi$)

Result: the projection of any critical trajectory for L in Q is a critical trajectory for L' in the reduced space Q/G.

Note:

In the above expression, the meaning of i_{σ_H} is as follows. Let $\Phi\colon E \to E'$ be a morphism over a map $\varphi\colon M \to M'$ and let η be a section of E' along φ. If ω is a section of $\bigwedge^p E'^*$ then $i_\eta\omega$ is the section of $\bigwedge^{p-1} E^*$ given by

$$(i_\eta\omega)_m(a_1, \ldots, a_{p-1}) = \omega_{\varphi(n)}(\eta(m), \Phi(a_1), \ldots, \Phi(a_{p-1}))$$

for every $m \in M$ and $a_1, \ldots, a_{p-1} \in E_m$. In our case, the map Φ is $\mathcal{T}\,\mathrm{pr}_1\colon \mathcal{T}^E(E^* \times_M B) \to \mathcal{T}^E E^*$, the prolongation of the map $\mathrm{pr}_1\colon E^* \times_M B \to E^*$ (this last map fibered over the identity in M), and σ_H is a section along pr_1. Therefore, $i_{\sigma_H}\Omega - dH$ is a section of the dual bundle to $\mathcal{T}^E(E^* \times_M B)$.

Discrete Mechanics

Lie groupoids

A **groupoid** over a set M is a set $\boldsymbol{G}$ together with the following structural maps:

- A pair of maps (source) $\boldsymbol{s}\colon \boldsymbol{G} \to M$ and (target) $\boldsymbol{t}\colon \boldsymbol{G} \to M$.
- A partial **multiplication** $\boldsymbol{m}$, defined on the set of composable pairs $\boldsymbol{G}_2 = \{\, (g,h) \in G \times G \mid \boldsymbol{t}(g) = \boldsymbol{s}(h) \,\}$.
  - ▷ $\boldsymbol{s}(gh) = \boldsymbol{s}(g)$ and $\boldsymbol{t}(gh) = \boldsymbol{t}(h)$.
  - ▷ $g(hk) = (gh)k$.
- An **identity** section $\boldsymbol{\epsilon} : M \to \boldsymbol{G}$ such that
  - ▷ $\boldsymbol{\epsilon}(\boldsymbol{s}(g))g = g$ and $g\boldsymbol{\epsilon}(\boldsymbol{t}(g)) = g$.
- An **inversion** map $\boldsymbol{i} : G \to G$, to be denoted simply by $\boldsymbol{i}(g) = g^{-1}$, such that
  - ▷ $g^{-1}g = \epsilon(\boldsymbol{t}(g))$ and $gg^{-1} = \boldsymbol{\epsilon}(\boldsymbol{s}(g))$.

$$\begin{tikzcd} \boldsymbol{s}(g) \arrow[rr, bend left, "g"] & & \boldsymbol{t}(g) \end{tikzcd}$$

$$\begin{tikzcd} \bullet \arrow[r, bend left, "g"'] \arrow[rr, bend left=40, "gh"] & \bullet \arrow[r, bend left, "h"'] & \bullet \end{tikzcd}$$

$$\begin{tikzcd} \bullet \arrow[r, bend left, "g"] & \bullet \arrow[l, bend left, "g^{-1}"] \end{tikzcd}$$

A groupoid is a **Lie groupoid** if $\boldsymbol{G}$ and M are manifolds, all maps (source, target, inversion, multiplication, identity) are smooth, $\boldsymbol{s}$ and $\boldsymbol{t}$ are submersions (then $\boldsymbol{m}$ is a submersion, $\boldsymbol{\epsilon}$ is an embedding and $\boldsymbol{i}$ is a diffeomorphism).

The Lie algebroid of a Lie groupoid

The Lie algebroid of a Lie groupoid $\boldsymbol{G}$ is the vector bundle $\tau \colon E \to M$ where $E_m = \operatorname{Ker}(T_{\boldsymbol{\epsilon}(m)}\boldsymbol{s})$ with $\rho_m = T_{\boldsymbol{\epsilon}(m)}\boldsymbol{t}$.

The bracket is defined in terms of left-invariant vector fields.

Left and right translation:
$g \in \boldsymbol{G}$ with $\boldsymbol{s}(g) = m$ and $\boldsymbol{t}(g) = n$

$$l_g \colon \boldsymbol{s}^{-1}(n) \to \boldsymbol{s}^{-1}(m), \qquad l_g(h) = gh$$
$$r_g \colon \boldsymbol{t}^{-1}(m) \to \boldsymbol{t}^{-1}(n), \qquad r_g(h) = hg$$

Every section σ of E can be extended to a left invariant vectorfield $\overleftarrow{\sigma} \in \mathfrak{X}(\boldsymbol{G})$. The bracket of two sections of E is defined by $\overleftarrow{[\sigma, \eta]} = [\overleftarrow{\sigma}, \overleftarrow{\eta}]$.

Examples

Pair groupoid.

$G = M \times M$ with $\boldsymbol{s}(m_1, m_2) = m_1$ and $\boldsymbol{t}(m_1, m_2) = m_2$.
Multiplication is $(m_1, m_2)(m_2, m_3) = (m_1, m_3)$
Identities $\boldsymbol{\epsilon}(m) = (m, m)$
Inversion $\boldsymbol{i}(m_1, m_2) = (m_2, m_1)$.
The Lie algebroid is $TM \to M$.

Lie group.

A Lie group is a Lie groupoid over one point $M = \{e\}$. Every pair of elements is composable.
The Lie algebroid is just the Lie algebra.

Transformation groupoid.

Consider a Lie group H acting on a manifold M on the right. The set $\boldsymbol{G} = M \times H$ is a groupoid over M with $\boldsymbol{s}(m,g) = m$ and $\boldsymbol{t}(m,g) = mg$.
Multiplication is $(m,h_1)(mh_1,h_2) = (m,h_1h_2)$.
Identity $\boldsymbol{\epsilon}(m) = (m,e)$
Inversion $\boldsymbol{i}(m,h) = (mh,h^{-1})$
The Lie algebroid is the transformation Lie algebroid $M \times \mathfrak{h} \to M$.

Atiyah or gauge groupoid.

If $\pi\colon Q \to M$ is a principal H-bundle, then $(Q \times Q)/H$ is a groupoid over M, with source $\boldsymbol{s}([q_1,q_2]) = \pi(q_1)$ and target $\boldsymbol{t}([q_1,q_2]) = \pi(q_2)$.
Multiplication is $[q_1,q_2][hq_2,q_3] = [hq_1,q_3]$.
Identity $\boldsymbol{\epsilon}(m) = [q,q]$
Inversion $\boldsymbol{i}([q_1,q_2]) = [q_2,q_1]$
(An element of $(Q \times Q)/G$ can be identified with an equivariant map between fibers)

Discrete Lagrangian Mechanics

A discrete Lagrangian on a Lie groupoid $\boldsymbol{G}$ is just a function $\boldsymbol{L}$ on $\boldsymbol{G}$. It defines a discrete dynamical system by mean of **discrete Hamilton principle**.

- **Action sum:** defined on composable sequences $(g_1, g_2, \cdots, g_n) \in \boldsymbol{G}_n$

$$S(g_1, g_2, \ldots, g_n) = \boldsymbol{L}(g_1) + \boldsymbol{L}(g_2) + \cdots + \boldsymbol{L}(g_n).$$

- **Discrete Hamilton principle:** Given $p \in \boldsymbol{G}$, a solution of a Lagrangian system is a critical point of the action sum on the set of composable sequences with product p, i.e. sequences $(g_1, g_2 \cdots, g_n) \in \boldsymbol{G}_n$ such that $g_1 g_2 \cdots g_n = p$

Discrete Euler-Lagrange equations

We can restrict to sequences of two elements (g, h). Since $gh = p$ is fixed, variations are of the form $g \mapsto g\eta(t)$ and $h \mapsto \eta(t)^{-1}h$, with $\eta(t)$ a curve thought the identity at $m = \boldsymbol{t}(g) = \boldsymbol{s}(h)$ with $\dot{\eta}(0) = a \in E_m$. Then the discrete Euler-Lagrange equations are:

$$\begin{aligned}\langle\, \mathrm{DEL}(g,h)\,, a\,\rangle &= \frac{d}{dt}[\boldsymbol{L}(g\eta(t)) + \boldsymbol{L}(\eta(t)^{-1}h)]\Big|_{t=0} \\ &= \langle\, d^0\big(\boldsymbol{L}\circ l_g + \boldsymbol{L}\circ r_h \circ \boldsymbol{i}\big)\,, a\,\rangle.\end{aligned}$$

Simplecticity

In the case of the pair groupoid, it is well known that the algorithm defined by the discrete Euler-Lagrange equations is symplectic.

In the general case of a Lagrangian system on a Lie groupoid one can also define a symplectic section on an appropriate Lie algebroid which is conserved by the discrete flow. From this it follows that the algorithm is Poisson (In the standard sense).

Such appropriate Lie algebroid is called the prolongation of the Lie groupoid $\mathcal{P}G \to \boldsymbol{G}$, where

$$\mathcal{P}_g\boldsymbol{G} = \mathrm{Ker}(T_g\boldsymbol{s}) \oplus \mathrm{Ker}(T_g\boldsymbol{t})$$

It can be seen isomorphic to

$$\mathcal{P}\boldsymbol{G} = \{\, (a, g, b) \in E \times \boldsymbol{G} \times E \mid \tau(a) = \boldsymbol{s}(g) \quad \text{and} \quad \tau(b) = \boldsymbol{t}(g) \,\}$$

where $\tau\colon E \to M$ is the Lie algebroid of $\boldsymbol{G}$.

Cartan forms

Given a discrete Lagrangian $\boldsymbol{L} \in C^\infty(\boldsymbol{G})$ we define the Cartan 1-sections $\Theta^-_{\boldsymbol{L}}$ and $\Theta^+_{\boldsymbol{L}}$ of $\mathcal{P}\boldsymbol{G}^*$ by

$$\Theta^-_{\boldsymbol{L}}(g)(X_g, Y_g) = -X_g(L), \qquad \text{and} \qquad \Theta^+_{\boldsymbol{L}}(g)(X_g, Y_g) = Y_g(L),$$

for each $g \in G$ and $(X_g, Y_g) \in V_g\beta \oplus V_g\alpha$.

The difference between them is

$$d\boldsymbol{L} = \Theta^+_{\boldsymbol{L}} - \Theta^-_{\boldsymbol{L}}.$$

The Cartan 2-section is

$$\Omega_{\boldsymbol{L}} = -d\Theta^+_{\boldsymbol{L}} = -d\Theta^-_{\boldsymbol{L}}$$

A Lagrangian is said to be regular if $\Omega_{\boldsymbol{L}}$ is a symplectic section.

Discrete evolution operator

For a regular Lagrangian there exists a locally unique map $\xi\colon \boldsymbol{G} \to \boldsymbol{G}$ such that it solves the discrete Euler-Lagrange equations

$$\mathrm{DEL}(g, \xi(g)) = 0 \qquad \text{for all } g \text{ in an open } \mathcal{U} \subset \boldsymbol{G}.$$

One of such maps is said to be a discrete Lagrangian evolution operator.

Given a map $\xi\colon \boldsymbol{G} \to \boldsymbol{G}$ such that $\boldsymbol{s} \circ \xi = \boldsymbol{t}$, there exists a unique vector bundle map $\mathcal{P}\xi\colon \mathcal{P}\boldsymbol{G} \to \mathcal{P}\boldsymbol{G}$, such that $\Phi = (\mathcal{P}\xi, \xi)$ is a morphism of Lie algebroids.

A map ξ is a discrete Lagrangian evolution operator if and only if

$$\Phi^*\Theta^-_{\boldsymbol{L}} - \Theta^-_{\boldsymbol{L}} = d\boldsymbol{L}.$$

If ξ is a discrete Lagrangian evolution operator then it is symplectic, that is, $\Phi^*\Omega_{\boldsymbol{L}} = \Omega_{\boldsymbol{L}}$.

Hamiltonian formalism

Define the **discrete Legendre transformations** $\mathcal{F}^-\boldsymbol{L} : \boldsymbol{G} \to E^*$ and $\mathcal{F}^+\boldsymbol{L} : \boldsymbol{G} \to E^*$ by

$$\begin{aligned}(\mathcal{F}^-L)(h)(a) &= -a(L \circ r_h \circ i), \qquad \text{for } a \in E_{\boldsymbol{s}(h)}\\ (\mathcal{F}^+L)(g)(b) &= \quad b(L \circ l_g), \qquad \text{for } b \in E_{\boldsymbol{t}(g)}\end{aligned}$$

The Lagrangian is regular if and only if $\mathcal{F}^\pm\boldsymbol{L}$ is a local diffeomorphism.

If Θ is the canonical 1-section on the prolongation of E^* then

$$(\mathcal{P}\mathcal{F}^\pm\boldsymbol{L})^*\Theta = \Theta_L^\pm,$$

and

$$(\mathcal{P}\mathcal{F}^\pm\boldsymbol{L})^*\Omega = \Omega_L.$$

We also have that

$$\mathrm{DEL}(g,h) = \mathcal{F}^+ L(g) - \mathcal{F}^- L(h)$$

so that the Hamiltonian evolution operator $\xi_{\boldsymbol{L}}$ is

$$\xi_{\boldsymbol{L}} = (\mathcal{F}^+ L) \circ (\mathcal{F}^- L)^{-1},$$

which is therefore symplectic

$$(\mathcal{P}\xi_L)^* \Omega = \Omega.$$

Morphisms and reduction

A morphism of Lie groupoids is a bundle map (ϕ, φ) between groupoids $\boldsymbol{G}$ over M and $\boldsymbol{G}'$ over M' such that $\Phi(gh) = \Phi(g)\Phi(h)$.

The prolongation $\mathcal{P}\phi$ of ϕ is the map $\mathcal{P}\phi(X,Y) = (T\phi(X), T\Phi(Y))$ from $\mathcal{P}\boldsymbol{G}$ to $\mathcal{P}\boldsymbol{G}'$.

Assume that we have a Lagrangian $\boldsymbol{L}$ on $\boldsymbol{G}$ and a Lagrangian $\boldsymbol{L}'$ on $\boldsymbol{G}'$ related by a morphism of Lie groupoids ϕ, that is $\boldsymbol{L}' \circ \phi = \boldsymbol{L}$. Then

- $\langle \mathrm{DEL}(g,h)\,, a \rangle = \langle D_{DEL}\boldsymbol{L}'(\phi(g), \phi(h))\,, \phi_*(a) \rangle$
- $\mathcal{P}\phi^*\Theta^{\pm}_{\boldsymbol{L}'} = \Theta^{\pm}_{\boldsymbol{L}}$
- $\mathcal{P}\phi^*\Omega_{\boldsymbol{L}'} = \Omega_{\boldsymbol{L}}$

As a consequence:

Let (ϕ, φ) be a morphism of Lie groupoids from $G \rightrightarrows M$ to $G' \rightrightarrows M'$ and suppose that $(g, h) \in G_2$.

1. If $(\phi(g), \phi(h))$ is a solution of the discrete Euler-Lagrange equations for $\boldsymbol{L}' = \boldsymbol{L} \circ \Phi$, then (g, h) is a solution of the discrete Euler-Lagrange equations for $\boldsymbol{L}$.
2. If ϕ is a submersion then (g, h) is a solution of the discrete Euler-Lagrange equations for $\boldsymbol{L}$ if and only if $(\phi(g), \phi(h))$ is a solution of the discrete Euler-Lagrange equations for $\boldsymbol{L}'$.

Thank you !

Example: Heavy top

Consider the transformation Lie algebroid $\tau : S^2 \times \mathfrak{so}(3) \to S^2$ and Lagrangian

$$L_c(\Gamma, \Omega) = \frac{1}{2}\Omega \cdot I\Omega - mgl\Gamma \cdot \mathrm{e} = \frac{1}{2}\operatorname{Tr}(\hat{\Omega}\mathbb{I}\hat{\Omega}^T) - mgl\Gamma \cdot \mathrm{e}.$$

where $\Omega \in \mathbb{R}^3 \simeq \mathfrak{so}(3)$ and $\mathbb{I} = \frac{1}{2}\operatorname{Tr}(I)I_3 - I$.

Discretize the action by the rule

$$\hat{\Omega} = R^T \dot{R} \approx \frac{1}{h} R_k^T (R_{k+1} - R_k) = \frac{1}{h}(W_k - I_3),$$

where $W_k = R_k^T R_{k+1}$ to obtain a discrete Lagrangian (an approximation of the continuous action) on the transformation Lie groupoid $L : S^2 \times SO(3) \to \mathbb{R}$

$$L(\Gamma_k, W_k) = -\frac{1}{h}\operatorname{Tr}(\mathbb{I}W_k) - hmgl\Gamma_k \cdot \mathrm{e}.$$

The value of the action on a variated sequence is

$$
\begin{aligned}
\lambda(t) &= L(\Gamma_k, W_k e^{tK}) + L(e^{-tK}\Gamma_{k+1}, e^{-tK}W_{k+1}) \\
&= -\frac{1}{h}\left[\mathrm{Tr}(\mathbb{I}W_k e^{tK}) + mglh^2\Gamma_k \cdot \mathrm{e} + \mathrm{Tr}(\mathbb{I}e^{-tK}W_{k+1}) + mglh^2(e^{-tK}\Gamma_{k+1}\right.
\end{aligned}
$$

where $\Gamma_{k+1} = W_k^T \Gamma_k$ (since the above pairs must be composable) and $K \in \mathfrak{so}(3)$ is arbitrary.

Taking the derivative at $t = 0$ and after some straightforward manipulations we get the DEL equations

$$
M_{k+1} - W_k^T M_k W_k - mglh^2(\widehat{\Gamma_{k+1} \times \mathrm{e}}) = 0
$$

where $M = W\mathbb{I} - \mathbb{I}W^T$.

In terms of the axial vector Π in $\mathbb{R}^3$ defined by $\hat{\Pi} = M$, we can write the equations in the form

$$
\Pi_{k+1} = W_k^T \Pi_k + mglh^2 \Gamma_{k+1} \times \mathrm{e}.
$$

Examples

- **Pair groupoid.**

  Lagrangian: $\boldsymbol{L}\colon M \times M \to \mathbb{R}$ Discrete Euler-Lagrange equations:

$$D_2\boldsymbol{L}(x,y) + D_1\boldsymbol{L}(y,z) = 0.$$

- **Lie group.**

  Lagrangian: $\boldsymbol{L}\colon G \to \mathbb{R}$ Discrete Euler-Lagrange equations:

$$\mu_{k+1} = Ad^*_{g_k}\mu_k, \qquad \text{discrete Lie-Poisson equations}$$

where $\mu_k = r^*_{g_k} d\boldsymbol{L}(e)$.

Action Lie groupoid.

Lagrangian: $\boldsymbol{L}\colon M \times H \to \mathbb{R}$ Discrete Euler-Lagrange equations: Defining $\mu_k(x, h_k) = d(\boldsymbol{L}_x \circ r_{h_k})(e)$, we have

$$\mu_{k+1}(xh_k, h_{k+1}) = Ad^*_{h_k}\mu_k(x, h_k) + d(L_{h_{k+1}} \circ ((xh_k)\cdot))(e),$$

where $(xh_k)\cdot : H \to M$ is the map defined by

$$(xh_k) \cdot (h) = x(h_k h).$$

These are the discrete Euler-Poincaré equations.

Atiyah groupoid.

Lagrangian: $\boldsymbol{L}\colon (Q \times Q)/H \to \mathbb{R}$. Discrete Euler-Lagrange equations: Locally $Q = M \times H$

$$\begin{aligned} &D_2 L((x,y),h_k) + D_1 L((y,z),h_{k+1}) = 0,\\ &\mu_{k+1}(y,z) = Ad^*_{h_k}\mu_k(x,y), \end{aligned} \tag{1}$$

where

$$\mu_k(\bar{x},\bar{y}) = d(r^*_{h_k} L_{(\bar{x},\bar{y},\ )})(e)$$

for $(\bar{x},\bar{y}) \in M \times M$.

One can find a global expression in terms of a discrete connection.

Examples

- Let G be a Lie group and consider the pair groupoid $G \times G$ over G. Consider also G as a groupoid over one point. Then we have that the map

$$\begin{array}{rccc} \Phi_l : & G \times G & \longrightarrow & G \\ & (g,h) & \mapsto & g^{-1}h \end{array}$$

is a Lie groupoid morphism, and a submersion. The discrete Euler-Lagrange equations for a left invariant discrete Lagrangian on $G \times G$ reduce to the discrete Lie-Poisson equations on G for the reduced Lagrangian.

- Let G be a Lie group acting on a manifold M by the left. We consider a discrete Lagrangian on $G \times G$ which depends on the variables of M as parameters $\boldsymbol{L}_m(g,h)$. The Lagrangian is invariant in the sense $\boldsymbol{L}_m(rg, rh) = \boldsymbol{L}_{r^{-1}m}(g,h)$.

  We consider the Lie groupoid $G \times G \times M$ over $G \times M$ where the elements in M as parameters, and thus $\boldsymbol{L} \in C^\infty(G \times G \times M)$ and then $\boldsymbol{L}(rg, rh, rm) = \boldsymbol{L}(g,h,m)$. Thus we define the reduction map (submersion)

$$\begin{array}{rccc} \Phi: & G \times G \times M & \longrightarrow & G \times M \\ & (g,h,m) & \mapsto & (g^{-1}h, g^{-1}m) \end{array}$$

  where on $G \times M$ we consider the transformation Lie groupoid defined by the right action $m \cdot g = g^{-1}m$.
  The Euler-Lagrange equations on $G \times G \times M$ reduces to the Euler-Lagrange equations on $G \times M$.

- A G-invariant Lagrangian L defined on the pair groupoid $L\colon Q \times Q \to \mathbb{R}$, where $p\colon Q \to M$ is a G-principal bundle. In this case we can reduce to the Atiyah gauge groupoid by means of the map

$$\begin{array}{cccc} \Phi : & Q \times Q & \longrightarrow & (Q \times Q)/G \\ & (q, q') & \mapsto & [(q, q')] \end{array}$$

Thus the discrete Euler-Lagrange equations reduce to the discrete Lagrange-Poincaré equations.

Thank you !